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ON UNITARIZABILITY AND REDUCIBILITY

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To the 65^{th} birthday of Rebecca A. Herb,
and the memory of Paul J. Sally, Jr.

The work of both Rebecca Herb and Paul Sally is in the area of harmonic analysis on locally compact groups, a theory which has its roots in the classical Fourier analysis. The classical theory is one of the most applied parts of math, in math as well as outside of math. The reason for this fact is certainly the power of the theory. But is is also related to the **simplicity** of basic principles of the classical theory. It is hard to expect such simplicity in the setting which we shall consider, since the groups with which we shall deal are much more complicated then the one of the classical theory (which deals with $(\mathbb{R}/\mathbb{Z})^n$ and $\mathbb{R}^n$). Nevertheless, at some directions we get remarkably simple answers.

In the Gelfand concept of harmonic analysis on a locally compact group G, roughly the role of sine and cosine functions is played by the set

$$\widehat{G}$$

of all equivalence classes of irreducible unitary representations of G, which is called the unitary dual of G. It is a topological space in a natural way.

Both Rebecca and Paul studied harmonic analysis on reductive groups over a locally compact non-discrete field F. Denote such a group by G. Here one defines the non-unitary dual

$$\widetilde{G}$$

of G similarly as the unitary dual, but drops the condition of the unitarity of the action. Then we have a natural embeding

$$\widehat{G} \hookrightarrow \widetilde{G}.$$

Harish-Chandra created a strategy of getting the unitary dual in two steps:

- classify $\widetilde{G}$;
- determine the subset $\widehat{G}$ of $\widetilde{G}$.

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The second step is called the

unitarizability problem.

This problem turned out to be quite hard to solve for a long time, even in the case when $\widetilde{G}$ was completely classified, like in the case of archimedean fields F.

Both Rebecca and Paul worked with real as well as p-adic groups, which is not too often. Talking about these two directions, it is natural to recall of the Harish-Chandra prediction called **Lefschetz principle**, which says:

"**Whatever** is true for real groups is true for p-adic groups."

Our primary interest in this lecture will be the **unitarizability problem**. We shall also discuss how much we can understand here the **Lefschetz principle**, and also, can we have still some reasonable amount of **simplicity**.

Our experience is that the unitarizability is understood in the simplest and the most natural way when we have also good understanding of the Lefschetz principle[1]. Also from our experience, the key for achieving this seems to be appropriate understanding of the **reducibility** questions.

Reducibility questions interested also Rebecca and Paul. For example, the thesis of their Ph.D. students N. Winarsky, D. Keys, C. Jantzen and D. Goldberg were on the reducibility.

We shall talk in our lecture only about classical groups, the most important subclass of the reductive groups. In a way as reductive groups are natural setting among algebraic groups[2] for the questions that interested Harish-Chandra, classical groups seem to have similar role inside reductive groups for some other questions.

Paul complete work is related to (particular cases of) classical groups, while smaller but important part of Rebecca work is also related to them (on automorphic induction or elliptic representations for example).

In what follows I shall try to have exposition as simple as possible, and I shall try to avoid technicalities as much as possible, trying not to oversimplify.

We shall start with a very simple solution of the

[1]A general problem regarding the Lefschetz principle is that we often relay in the two settings addressed by this principle on incompatible "group algebras" even in the cases when we can use more uniform tools. For example, (complex) representations of p-adic Lie algebras (or its enveloping algebras) do not give any non-trivial information. The similar situation is regarding representations of algebras of locally constant functions on Lie groups. When after using such different tools we get the same results, it is not surprising that the fact that they are the same may look sometimes pretty mystical.

[2]and groups close to the algebraic groups

1. Unitarizability in the case of $GL(n,F)$

We shall use well-known Bernstein-Zelevinsky notation $\times$ for parabolic induction of two representations π_i of $GL(n_i,F)$:

$$\pi_1 \times \pi_2 = \mathrm{Ind}_{P_{(n_1,n_2)}}^{GL(n_1+n_2,F)}(\pi_1 \otimes \pi_2),$$

where $P_{(n_1,n_2)}$ denotes the parabolic subgroup containing upper triangular matrices, whose Levi subgroup is naturally isomorphic to the direct product $GL(n_1,F)\times GL(n_2,F)$. Denote by

$$D_u = D_u(F)$$

the set of all equivalence classes of irreducible square integrable (modulo center) representations of all $GL(n,F)$, $n \geq 1$. Let

$$\nu = |\det|_F,$$

where $|\ |_F$ is the normalized absolute value. For $\delta \in D_u$ and $m \geq 1$ denote by

$$u(\delta, m)$$

the unique irreducible quotient of

$$\nu^{(m-1)/2}\delta \times \nu^{(m-1)/2-1}\delta \times \ldots \times \nu^{-(m-1)/2}\delta, \tag{1.1}$$

which is called a Speh representation. Let

$$B_{rigid}$$

be the set of all Speh representations, and

$$B = B(F) = B_{rigid} \cup \{\nu^{\alpha}\sigma \times \nu^{-a}\sigma; \sigma \in B_{rigid}, 0 < \alpha < 1/2\}.$$

Denote by $M(B)$ the set of all finite multisets in B. Then the following simple theorem solves the unitarizability for archimedean and non-archimedean general linear groups in the uniform way:

Theorem A. *A mapping*

$$(\sigma_1, \ldots, \sigma_k) \mapsto \sigma_1 \times \ldots \times \sigma_k$$

defined on $M(B)$ goes into $\cup_{n\geq 0} GL(n,F)\widehat{\ }$, and it is a bijection.

The main reason for the simplicity of the above answer, as well as of the proof of the theorem, is the following fundamental reducibility property of the representations theory of general linear groups:

- (U0) Unitary parabolic induction for $GL(n,F)$ is irreducible (i.e., it carries irreducible unitary representations to the irreducible ones).

Theorem A follows easily from five facts - kind of axioms, which are besides (U0):

- (U1) $u(\delta, m)$'s are unitarizable,

(U2) $\nu^{\alpha}u(\delta,m) \times \nu^{-\alpha}u(\delta,m)$'s are unitarizable (for $0 < \alpha < 1/2$),
(U3) $u(\delta,m)$'s are prime elements in appropriate factorial ring

and (U4), which is an easy consequence of a property of the Langlands classification. The proofs of (U2), (U3) and (U4) are straitforward.

Further, (U1) can be replaced by requirements that $u(\delta,m) \times u(\delta,m-2)$'s are irreducible, while for (U2), it is enough to show that $\nu^{\alpha}u(\delta,m) \times \nu^{-\alpha}u(\delta,m)$'s are irreducible. Also (U3) is a consequence of a reducibility fact. Therefore, in the above axioms are crucial irreducibility and reducibility facts.

One can find how above facts imply Theorem A in our short note from 1985[3] in the p-adic setting. But there is no any difference with the archimedean case.

The above uniform approach to the p-adic and real case is achieved by not going into the internal structure of representations (which is pretty different in these two cases). Because of this, we call this strategy the external approach to the unitarizability.

Let us return to

2. (U0)

The work on (U0) includes some of the greatest names of the representation theory, like I. M. Gelfand, A. A. Kirillov and J. Bernstein, while also Harish-Chandra is indirectly related to it. We shall say a few words about their relation to (U0).

Denote by P the (mirabolic) subgroup of $GL(n,F)$ consisting of all matrices in the group with bottom raw equal to $(0,\dots,0,1)$. In a very early stage of the development of the non-commutative harmonic analysis, I. M. Gelfand and M. A. Naimark realized[4] that for $F=\mathbb{C}$, the claim

(K) $\pi|P$ is irreducible for $\pi \in GL(n,F)\widehat{\ }$

implies (U0). More precisely, they proved this implication only related to the representations that they considered (but for which they expected to exhaust the whole unitary duals; they worked with $SL(n,\mathbb{C})$-groups). For them, (K) was almost obvious, since great majority of the representations by which they were inducing, were one-dimensional.

[3] M. Tadić, *Unitary dual of p-adic GL(n), Proof of Bernstein Conjectures*, Bulletin Amer. Math. Soc. 13 (1985), 39-42.

[4] in I. M. Gelfand and M. A. Naimark, *Unitäre Darstellungen der Klassischen Gruppen* (German translation of Russian publication from 1950), Akademie Verlag, Berlin, 1957.

In 1962[5], A. A. Kirillov completely changed the point of view, proposing a proof of (K) for a general $\pi \in GL(n,F)\widehat{\ }$ (he considered $F = \mathbb{R}$ and $\mathbb{C}$ cases). Kirillov is best known for his orbit method, and it is not surprising that he saw (K) as a consequence of facts about orbits.

Very roughly, he observed that for the adjoint action, in GL-orbits there are large P-orbits (having the full measure in the group). After this observation, Kirillov considered "twisted" characters

$$ch_{\pi,A} : f \mapsto \mathrm{Trace}(\pi(f)A), \quad A \in End_P(\pi).$$

If these P-invariant distributions are given by functions, then the above observation about orbits gives that they are GL-invariant. Then A must be a scalar, and the Schur lemma implies the irreducibility of $\pi|K$.

Kirillov discovered this important way to prove (K), but in his four page long paper, which is kind of announcement, he did not supply an argument that $ch_{\pi,A}$'s are functions.

It is interesting to note that about that time was available a technology to complete Kirillov argument. Namely, $ch_{\pi,A}$ are P-invariant eigendistributions, and we have very big P-conjugacy classes in GL-conjugacy classes. Therefore, it is natural to try to apply Harish-Chandra type strategy, how he proved his regularity theorem for invariant eigendistributions, that they are given by functions. Along this strategy E. M. Baruch completed the proof of (K)[6].

J. Bernstein proved[7] (K) for p-adic F following the Kirillov strategy.

3. Division algebra case

The above unitarizability story is roughly three decades old[8], except Baruch proof (which follows Harish-Chandra strategy from the beginning of 1960-es). But this is not the end of this unitarizability story. In the p-adic case, this is just a beginning. The field case is just a tip of an iceberg, level one of infinitely many levels (recall that the Brauer group of p-adic F is $\mathbb{Q}/\mathbb{Z}$).

[5]in A. A. Kirillov, *Infinite dimensional representations of the general linear group*, Dokl. Akad. Nauk SSSR 114 (1962), 37–39; Soviet Math. Dokl. **3** (1962), 652–655.

[6]in E. M. Baruch, *A proof of Kirillov's conjecture,* Ann. of Math. (2) 158 no. 1 (2003), 207–252.

[7]in J. Bernstein, *P-invariant distributions on $GL(N)$ and the classification of unitary representations of $GL(N)$ (non-archimedean case),* Lie Group Representations II, Lecture Notes in Math. 1041, Springer-Verlag, Berlin, 1984, 50-102.

[8]It is in the papers M. Tadić, *Classification of unitary representations in irreducible representations of general linear group (non-archimedean case)*, Ann. Sci. Ecole Norm. Sup. 19 (1986), 335-382, and

Tadić, M., *Unitary representations of general linear group over real and complex field,* preprint MPI/SFB 85-22 Bonn (1985) (`http://www.mpim-bonn.mpg.de/preblob/5395`), published in slightly modified version much later as M. Tadić, *$GL(n,\mathbb{C})\widehat{\ }$ and $GL(n,\mathbb{R})\widehat{\ }$*, in "Automorphic Forms and L-functions II, Local Aspects", Contemp. Math. 489 (2009), 285-313.

Namely, if we for division algebra A change in the definition of B_{rigid} and B

$$D_u(F) \qquad \text{by} \qquad D_u(A)$$

and

$$\nu \qquad \text{by} \qquad \nu_\sigma := \nu^{\min\{\alpha>0; \sigma\times\nu^\alpha\sigma \text{ reduces}\}},$$

then after these simple modifications, Theorem A is also the solution of the unitarizability problem for $GL(n, A)$-groups.

The proof in this generalisation follows the same strategy, i. e., it follows also using axioms (U0) - (U4), but actually one needs to prove only (U0) and (U1) (the other axioms follow more or less as in the filed case, and they were known to hold already at the end of 1980-es[9].). The proof of these two facts was achieved in the last decade, and it was not easy.

I. Badulescu and D. Renard proved[10] (U1) using previous work of Badulescu on the Jacquet-Langlands correspondence (which is using a simple form of the Arthur trace formula).

The proof of (U0) was a particular problem, since (K) does not hold here, so one needed to find a new strategy. V. Sécherre proved[11] (U0) using his classification of simple types, and using a number of other very non-trivial results, including Bernstein proof of (U0) in the field case (which is based on (K)).

Very recently, very simple proofs of these two facts were obtained. The first is a local proof of (U1) of I. Badulescu[12] (he proved irreducibility of $u(\delta, m) \times u(\delta, m-2)$ using the Mœglin-Waldspurger algorithm for the Zelevinsky involution).

The second is an announcement of E. Lapid and A. Mínguez of a proof of (U0). Their proof is essentially non-unitary. It is a proof of a fact about unique irreducible subrepresentation. This fact implies a weaker form of (U0), which can be used instead (U0) to get Theorem A (which obviously implies (U0)). Their proof, based on Jacquet modules, is pretty simple (and elementary). It is even interesting in the field case, since it is simpler then the delicate Bernstein argument.

So finally, we have not only simple description of the unitarizability for GL-groups over p-adic division algebras, but also simple proof.

[9] M. Tadić, *Induced representations of $GL(n, A)$ for p-adic division algebras A*, J. reine angew. Math. 405 (1990), 48-77.

[10] in A. I. Badulescu, and D. A. Renard, *Sur une conjecture de Tadić*, Glasnik Mat. 39 no. 1 (2004), 49-54.

[11] in V. Sécherre, *Proof of the Tadić conjecture (U0) on the unitary dual of $GL_m(D)$*, J. reine angew. Math. 626 (2009), 187-203.

[12] in A. I. Badulescu, *On p-adic Speh representations*, Bulletin de la SMF 142, 2 (2014)

Let us observe that not only the unitarizability can be simple. We shall give a following example where remarkable simplicity related to the characters of the irreducible unitary representations shows up. It is an example how one can express

4. Irreducible unitary characters

of general linear groups by characters of irreducible square integrable representations. Theorem A implies that for this, it is enough to have such expression for Speh representations. We consider bellow the case of p-adic field F (the story below holds also for division algebras, after one replaces ν by ν_ρ).

Recall that by the Bernstein-Zelevinsky theory, we have bijection

$$\Delta \leftrightarrow \delta(\Delta)$$

between segments $\Delta = [\nu^a\rho, \nu^b\rho]$ of irreducible cuspidal GL-representations and irreducible essentially square integrable representations ($\delta(\Delta)$ is a unique irreducible subrepresentation of $\nu^a\rho \times \nu^{a+1}\rho \times \ldots \times \nu^b\rho$).

Parameterize now by segments essentially square integrable representations entering the definition (1.1) of Speh representation:

$$\nu^{-(m-1)/2+i-1}\delta = \delta([\nu^{a_i}\rho, \nu^{b_i}\rho]), \qquad i = 1, \ldots, m.$$

Then in the Grothendieck group we have the following simple formula

$$u(\delta, m) = \det \left[\delta([\nu^{a_i}\rho, \nu^{b_j}\rho])\right]_{1\leq i,j\leq m}, \tag{4.2}$$

expressing the Speh representation by irreducible square integrable representations (the multiplication in above formula is parabolic induction). In the above formula we use a convention that for $b_j < a_i$ we take $\delta([\nu^{a_i}\rho, \nu^{b_j}\rho]) = 0$, except if $a_i = b_j + 1$ when we take it to be 1.

We got the above simple formula in 1995[13], while the above obvious but very important determinant interpretation of it belongs to G. Chenevier and D. Renard[14]. Thanks to the determinant interpretation, they got a significant simplification of our proof (replacing one our very complicated technical argument by a simple determinate identity of the nineteen century mathematics). I. Badulescu gave further simplification of the proof in his recent paper which we have mentioned (with a simple local proof of (U1)). E. Lapid and A.

[13] in M. Tadić, *On characters of irreducible unitary representations of general linear groups*, Abh. Math. Sem. Univ. Hamburg 65 (1995), 341-363.

[14] G. Chenevier and D. Renard, *Characters of Speh representations and Lewis Caroll identity*, Represent. Theory 12 (2008), 447-452.

Mínguez generalized the above formula to a non-unitary setting[15]. Curiously, their non-unitary generalization has interesting unitary applications.

One may ask if the above simple formula, which resembles a little bit to high school math, can find a place in contemporary math full of complicated and powerful theories. It can, at least in one case. Namely, J. Arthur in his recent fundamental book[16] on endoscopic classification of representations says that this formula, and such formula in the archimedean case, are used "in the critical final stages of the global classification".

Let us return to the unitarizability. Up to now, we have covered all except one local division algebra, the algebra of Hamiltonian quaternions $\mathbb{H}$. For this we need

5. Vogan classification

D. Vogan has classified $GL(n,F)\widehat{}$ for $F = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$[17]. Formulation of his classification looks quite different from the classification Theorem A, together with the proof (compare Theorem 6.18 of Vogan Inventiones paper with Theorem A). The main toll in his classification is (besides parabolic) the cohomological induction (and K types). Clearly, for $F = \mathbb{R}$ and $\mathbb{C}$, his classification can be shown to be equivalent to Theorem A[18]. For $F = \mathbb{H}$, Vogan classification implies that Theorem A holds also in this case. In the moment, his classification is the only way to conclude (U0) in this case.

Remark: *Recall that the classification theorem A is* **field insensitive** *completely. Presently, we have the proofs of Theorem A for real and p-adic fields using the same strategy (and they are pretty close). It would be interesting to find also a proof which is completely field insensitive. Such a proof would give a new insight in our understanding of unitarity (and in our opinion it would be a part of a future general theory).*

Now we go to the other

[15] in E. Lapid and A. Mínguez, *On a determinantal formula of Tadić*, Amer. J. Math. 136 (2014), no. 1, 111-142.

[16] J. Arthur, *The endoscopic classification of representations. Orthogonal and symplectic groups*, American Mathematical Society Colloquium Publications, 61. American Mathematical Society, Providence, RI, 2013.

[17] in D. A. Vogan, *The unitary dual of $GL(n)$ over an archimedean field*, Invent. Math. 82 (1986), 449-505.

[18] Written proof of this fact can be found in A. I. Badulescu and D. A. Renard, *Unitary dual of GL_n at archimedean places and global Jacquet-Langlands correspondence*, Compositio Math. 146 no. 5 (2010), 1115-1164

6. Classical groups ($\neq GL$)

After Theorem A for the case of general linear groups, where the nature of the field is completely irrelevant, it is natural to expect that the most natural solution of the unitarizability for the (other) classical groups will be again the one which is independent of the nature of the field. The problem is that (U0), which was crucial in the GL-case (in formulation of the classification, but also in the proofs) does not hold here in general (already for $SL(2) = Sp(2)$).

Among classical groups, for simplicity, we shall restrict below to the symplectic groups $Sp(2n, F)$ and the split odd-orthogonal groups $SO(2n + 1, F)$. For these groups, as far as we know, we do not have explicit solution of the unitarizability, i. e., the one which is not kind of algorithm. We have explicit classification of unitarizability for two important[19] subclasses.

The first is explicit classification of unitarizability for generic irreducible representations. We shall not go here into details of the classification of the generic unitary dual[20], except mentioning that the classification is the same for both non-archimedean and archimedean case.

From the point of view of the unitarizability, more interesting is the classification of

7. Spherical unitary dual (p-adic case)

In this case, explicit classification[21] reduces the spherical unitary dual to some reducibilities, which are then explicitly described.

We shall not present here the classification, but rather present an example[22] which illustrates the difference between classical groups and the groups of type A. The example addresses the most delicate parts of the unitary duals, the isolated representations.

Example: *(a) The number of isolated spherical representations in* $SL(n, F)\hat{}$ *is 1 for any* $n \geq 3$ *.*

(b) The number of isolated spherical representations in $SO(451, F)\hat{}$ *is*

$$1\ 289\ 535\ 202\ 500. \tag{7.3}$$

[19] In particular for the theory of automorphic forms.

[20] obtained in E. Lapid, G. Muić and M. Tadić, *On the generic unitary dual of quasisplit classical groups,* Int. Math. Res. Not. no. 26 (2004), 1335–1354.

[21] obtained in G. Muić and M. Tadić, *Unramified unitary duals for split classical p–adic groups; the topology and isolated representations*, in "On Certain L-functions", Clay Math. Proc. vol. 13, 2011, 375-438.

[22] from M. Tadić, *On automorphic duals and isolated representations; new phenomena*, J. Ramanujan Math. Soc. 25 no. 3 (2010), 295-328.

From this example we can drew two conclusions:

• The unitarizability is much more complicated for classical groups then for SL and GL-groups.

• Despite the complexity of the unitary duals for classical groups, we can still have full explicit control of the most delicate part of the unitary dual in the spherical case.

To get an idea of the spherical unitary dual, and which information is important there, we shall say few words about the proof of (b). Isolated representation can not be a subrepresentation of a representation parabolically induced by unitary representation of a proper Levi subgroup. It also cannot be a member of a complementary series. When we count all the representations in the spherical part of $SO(451, F)\widehat{\ }$ which remain after we remove these two groups, we have

$$140\ 630\ 679\ 543\ 940 \tag{7.4}$$

representations (these representations seem to be important in the automorphic dual). Observe that a very huge amount of the above representations, more then 99% of them, is not isolated. One gets that they are not isolated from the fact that to each of these more then 99% representations, there is a complementary series approaching it.

From the other side, still a huge number of representations is isolated. To prove this, one needs roughly to show that no complementary series reach any of the isolated representations (recall that we have indicated above that there are really very many complementary series representations).

Existence of complementary series is related to the irreducibility of the corresponding unitarily induced representations (when a symmetry condition is satisfied). The above discussion shows that we must have a very explicit understanding of the above (unitary) reducibility questions (not only on some algorithmic level). Also, we need to know lengths of complementary series (and therefore understand related non-unitary reducibility questions). Such explicit understanding was obtained by G. Muić for this case[23].

Instead of explaining the Muić classification, we shall give an idea of the way how one can control such type of reducibilities in the dual setting, which is more essential for the unitarizability. We shall consider the first representation which one considers in the harmonic analysis, the regular representation $L^2(G)$ (and to which a considerable part of the Harish-Chandra work is related). The support of this representations consist of

[23]in G. Muić, *On the Non-Unitary Unramified Dual for Classical p-adic Groups,* Trans. Amer. Math. Soc. 358 (2006), 4653–4687.

8. Irreducible tempered representations

We continue to assume F to be p-adic. For $Sp(2n, F)$, maximal Levi subgroups are of the form

$$GL(k, F) \times Sp(2(n-k), F).$$

Now for a representation π of $GL(k, F)$ and σ of $Sp(2(n-k), F)$, we denote by

$$\pi \rtimes \sigma$$

the parabolically induced representation $\mathrm{Ind}^{Sp(2n,F)}(\pi \otimes \sigma)$. Analogously, we define $\pi \rtimes \sigma$ in the case of $SO(2n+1, F)$. This is a natural extension of the Bernstein-Zelevinsky notation $\times$ from the GL-case.

We shall assume below always that inducing representations are irreducible. Further, we shall denote always by ρ an irreducible cuspidal representation of a general linear group. For the reducibility questions of the parabolic induction that we shall consider, interesting case is when ρ is self dual (otherwise, we have always irreducibility). Because of that, we shall assume below always that ρ is self dual.

We get tempered representations in two stages:

(i) Classifying irreducible square integrable representations of Levi subgroups.
(ii) Classifying irreducible subrepresentations of the representations parabolically induced by representations from (i).

The first problem seems harder, but these two problems are related in a nice way, and in a sense, they are not very far from being equivalent as we shall indicate soon.

Regarding understanding of the square integrable induction (and (ii)), the first step may be to try to understand the simplest case, the question of reducibility of representations

$$\delta \rtimes \pi, \tag{8.5}$$

where δ and π are square integrable. Actually, D. Goldberg has shown[24] that understanding of this case is enough for understanding of (ii).

We shall illustrate on example how one can handle the problem of understanding of the reducibility of representations (8.5). For this, it will be useful to recall our old, natural result[25] (proved using Jacquet modules):

[24] in D. Goldberg, *Reducibility of induced representations for $Sp(2n)$ and $SO(n)$*, Amer. J. Math. 116 (1994), 1101-1151.

[25] from M. Tadić, *On reducibility of parabolic induction,* Israel J. Math. 107 (1998), 29–91.

Theorem B. *Let $\delta(\Delta)$ be an irreducible essentially square integrable representation of a general linear group and let σ be an irreducible cuspidal representation of a classical group. Then*

$$\delta(\Delta) \times \sigma \text{ reduces } \iff \mu \rtimes \sigma \text{ reduces for some } \mu \in \Delta.$$

Denote

$$\delta(\rho, n) := \delta([\nu^{-(n-1)/2}\rho, \nu^{(n-1)/2}\rho]).$$

One gets each irreducible square integrable representation of a general linear group as some $\delta(\rho, n)$.

Example: *(1) We shall consider now odd-orthogonal groups, and the simplest possible square integrable representation π there, the trivial representation $1_{SO(1,F)}$ of the trivial group $SO(1,F)$. F. Shahidi has shown*[26] *that either $\rho \rtimes 1_{SO(1,F)}$ or $\nu^{1/2}\rho \rtimes 1_{SO(1,F)}$ reduces (this is the only reducibility point ≥ 0 along ρ). Now this and Theorem B obviously imply that*

$$\delta(\rho, n) \rtimes 1_{SO(1,F)} \text{ is } \begin{cases} \text{reducible for one parity of } n; \\ \text{irreducible for the other parity of } n. \end{cases}$$

So, we have a very regular and simple picture here.

(2) Move now to the symplectic groups and consider the same question. For $\rho \neq 1_{F^\times}$, we have the same answer, except that the parities are switched (the switching is a consequence of the Shahidi duality). For $\rho = 1_{F^\times}$, as it is well-known, the reducibility point is at 1 (i.e. $\nu 1_{F^\times} \rtimes 1_{Sp(0,F)}$ reduces). Now Theorem B obviously implies

$$\delta(1_{F^\times}, n) \rtimes 1_{Sp(0,F)} \text{ is } \begin{cases} \text{reducible for all odd } n, \textbf{ except } \text{for } n = 1; \\ \text{irreducible all even } n. \end{cases}$$

Therefore, the only exception in the above even/odd pattern reducibility story is

$$\delta(1_{F^\times}, 1) = 1_{F^\times}.$$

In general, for a general irreducible square integrable representation π of a classical group, there can be more then one exception as above (from the above even/odd pattern) for fixed ρ, but there can be at most finitely many exceptions.

Definition: *The union of all such exceptions $\delta(\rho, n)$'s (for all ρ's) is denoted by*

$$Jord(\pi).$$

In previous example we have obviously

$$Jord(1_{SO(1,F)}) = \emptyset,$$
$$Jord(1_{Sp(0,F)}) = \{1_{F^\times}\}.$$

[26]in F. Shahidi, *Twisted endoscopy and reducibility of induced representations for p-adic groups*, Duke Math. J. 66 (1992), 1-41.

A more non-trivial example of Jordan blocks we get for $n \geq 1$:

$$Jord(St_{SO(2n+1,F)}) = \{\delta(1_{F^\times}, 2n)\} = \{St_{GL(2n,F)}\},$$
$$Jord(St_{Sp(2n,F)}) = \{\delta(1_{F^\times}, 2n+1)\} = \{St_{GL(2n+1,F)}\},$$

where St_G denotes the Steinberg representation of a reductive group G.

Let us give one more example. Denote by ψ any character of order two of $F^\times$. Then $\delta([\psi, \nu\psi]) \rtimes 1_{Sp(0,F)}$ has precisely two irreducible square integrable subquotients. Denote them by $\pi_\pm$. Then

$$Jord(\pi_\pm) = \{1_{F^\times}, \psi, \delta(\psi, 3)\} = \{1_{F^\times}, \psi, \psi St_{GL(3,F)}\}.$$

We may say that $Jord(\pi)$ is the set of all singularities regarding the above even/odd pattern regarding reducibility. In this way, $Jord(\pi)$ is the most delicate part of the tempered induction related to an irreducible square integrable representation π of a classical group.

Let us recall that J. Arthur classified in his recent fundamental book mentioned earlier irreducible tempered representations of classical groups by pairs consisting of an admissible homomorphism of the Weil-Deligne group, and a characters of the corresponding component group. Clearly, it would be important to know Jordan blocks in terms of his parameters. This fundamental result is done[27] by C. Mœglin. The answer is very niece:

Theorem C. (C. Mœglin) *The admissible homomorphism attached by J. Arthur to an irreducible square integrable representation π of a classical group is*

$$\oplus_{\delta \in Jord(\pi)} \Phi_{GL}(\delta),$$

where Φ_{GL} denotes the local Langlands correspondence for general linear groups.

After this, the use of admissible homomorphisms or Jordan blocks is equivalent. This indicates that the very basic level of the harmonic analysis (considering just the regular representations) determines in natural way L-packets for classical groups.

The Mœglin theorem implies that Arthur classification gives a very direct control of the most delicate part of the tempered induction. Recall that Jordan blocks are defined entirely in terms of the unitary (tempered) induction. But what to do regarding the non-unitary parabolic induction (which is important for example for construction of complementary series). We shall illustrate on an example of cuspidal representations how one can control such reducibilities. This reducibility is very important in the GL-case, but for the difference with the GL-case, here we can have infinitely many possibilities for cuspidal reducibility points.

Let σ be an irreducible cuspidal representation of a classical group. Then

$$\nu^{\alpha_{\rho,\sigma}} \rho \rtimes \sigma$$

[27]in C. Mœglin, *Multiplicité 1 dans les paquets d'Arthur aux places p-adiques*, in "On certain L-functions", Clay Math. Proc. 13 (2011), 333-374.

reduces for a unique $\alpha_{\rho,\sigma} \geq 0$ (and $\alpha_{\rho,\sigma}$ is in $(1/2)\mathbb{Z}$). Observe that Theorem B obviously implies, what Mœglin has observed, that $Jord(\sigma)$ is without gaps, i.e.

$$\delta(\rho, k) \in Jord(\pi) \text{ and } k \geq 3 \implies \delta(\rho, k-2) \in Jord(\pi).$$

Actually, Mœglin has classified cuspidal representations in the Arthur classification. Her classification is simple, but we shall not go into details here. For Jordan blocks the condition is that there are no gaps.

Theorem B (and some most basic elementary arithmetic) obviously imply that

$$\alpha_{\rho,\sigma} = \frac{1 + \max\{k; \delta(\rho, k) \in Jord(\sigma)\}}{2}.$$

for reducibility $\alpha_{\rho,\sigma} \geq 1$ (this is called the basic assumption). Obviously for fixed ρ, this type of reducibility can happen only for finitely many ρ's. Remaining reducibilities (i. e., those < 1) does not depend on σ, but only on the series of the groups which is considered, and they can be determined using the local Langlands correspondence Φ_{GL} for general linear groups.

Now the question of more general non-unitary reducibility

$$\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$$

becomes trivial. Namely, one just needs to check if $\nu^{\alpha_{\rho,\sigma}}\rho$ or $\nu^{-\alpha_{\rho,\sigma}}\rho$ is in $[\nu^a\rho, \nu^b\rho]$, or not.

These were some kind of very initial examples of handling reducibilities. G. Muić has done much more[28] regarding non-unitary reducibility of representations

$$\delta(\Delta) \rtimes \pi,$$

for general square integrable π (he also considered the case of tempered π). But this requires understanding of square integrable representations in terms of cuspidal representations (which is given in works of C. Mœglin and me, but we shall not talk about it here).

At the end of this section, let us observe that similarly as it was easy to conclude from Theorem B the condition on Jordan blocks of cuspidal representations, it is easy to conclude how look Jordan blocks of generic cuspidal representations using the Shahidi result that $\alpha_{\rho,\sigma}$ in this case is in $\{0, 1/2, 1\}$. Using this, Theorem B obviously implies (well-known fact) that the Jordan blocks of a generic cuspidal representations of classical groups consist only of cuspidal representations (and it is equal to to the set of all ρ such that $\nu\rho \rtimes \sigma$ reduces).

[28]in G. Muić, *Composition Series of Generalized Principal Series; The Case of Strongly Positive Discrete Series*, Israel J. Math 149 (2004), pp. 157-202, and
G. Muić, *Reducibility of Generalized Principal Series*, Canad. J. Math. 57 (2005), pp. 616-647.

9. Thinking about general case of unitarizability for classical groups (p-adic case mostly)

Very oversimplified, we expect that we shall have similar shape of (explicit) classification of unitary duals of classical groups as in the spherical (and generic) case, but certainly much more technically complicated. In a sense, the existing classifications of spherical and generic unitary duals should be two opposite poles (or specializations) of the general classification (which we do not know). Our feeling is that we know rough strategy how to attack the problem. But on the way to the general classification, there are some not very easy questions to solve, and the expected answer will certainly be much more complicated then in the case of general linear groups. Despite this, our feeling is that we are in better position now then we have been at the beginning of 1980-es, before the very fast development which resulted with the solution of the unitarizability for general linear groups started. We shall discuss this shortly later.

Recall that we have denoted before the Speh representations of general linear groups by B_{rigid}. In the sequel we shall denote this set by

$$B_{rigid}^{GL}.$$

One of these hard questions is: what would for classical groups play the role of B_{rigid}^{GL}, i. e. what would be

$$B_{rigid}^{cl.\,gr.} = ?.$$

The general classification should be based on these two sets. An important part of the problem would be to find natural (explicit) parameterization of this hypothetical set, such that the parameters in a natural way contains essential information related to the reducibility (relevant for the unitarizability). Our hope is that the Arthur classification of irreducible square integrable representations and Muić relatively simple classification of strongly negative irreducible spherical representations belong roughly to two opposite poles of a such hypothetic general story. It is natural to expect that representations in hypothetic $B_{rigid}^{cl.\,gr.}$ are automorphic. Actually, probably some important parts of this story are already solved in the works of C. Mœglin and J. Arthur.

Let us now compare briefly the present moment of the unitarizability for the classical groups with the period before 1984, when we did not know how unitarizability looks for the general linear groups.

The crucial Bernstein paper for our work (mentioned before, where (U0) was proved in the p-adic case) was published in 1984. There Bernstein gave also an algorithm for testing unitarizability of an irreducible representation of p-adic $GL(n)$ (in terms of Zelevinsky polynomials analogous to the Kazhdan-Lusztig polynomials, conjecturally describing multiplicities). There Bernstein discusses if the unitarizability is solvable explicitly (but he does not rule out the possibility that that unitarizable representations can be described explicitly by "simple-minded methods").

It is interesting to note that Paul Sally in a sense was the one who brought me to the the story of unitarizability from the first part of the lecture. Namely, he organized p-adic summer period at the University of Chicago in 1983, and gave me a preprint of the Bernstein paper. Then I started to work on the unitarizability problem from the Bernstein paper, and my work progressed very fast. Namely, my paper with Theorem A (in the p-adic case) was submitted to Ann. Sci. Ecole Norm. Sup. in the January of 1984, before the Bernstein paper was published (in the fall of 1983 we reported about the progress on our work on the unitarizability in the Jussieu automorphic forms seminar).

In the moment, the unitarizability problem for classical groups seems quite hard, and the solution will be definitely not simple. Despite this, it seems that regarding this problem we are now in better position then we have been regarding GL-unitarizability just before that problem was solved (at the beginning of 1980-es), although the solution there turner out to be very simple. The biggest problem in that time was that a simple answer (as in Theorem A) was not expected. The expectations were quite opposite (and even a possibility of an explicit solution of the problem was in question). Observe that for example, after Gelfand - Naimark book from 1950 and the E. M. Stein paper on complementary series from 1967[29], all essential ingredients of Theorem A in the complex case were present (here B_{rigid} is simply $(\mathbb{C}^\times)\widehat{}$), while after B. Speh paper from 1983[30], the whole B_{rigid}^{GL} was available in the real case (B_{rigid}^{GL} was the biggest problem in the p-adic case; this set is much bigger in this case then in the archimedean).

The second big problem for us was the exhaustion, which usually makes the papers on the unitarizability messy even in the low ranks (to solve GL-groups, we needed to handle groups of arbitrary high split ranks). Surprisingly, this problem was solved in a relatively simple way for general linear groups. In this solution (U0) played a crucial role.

Regarding the first problem in the case of classical groups, now we do not have such problem as we had for general linear groups before 1984. Let us just quote Bernstein opinion in the case if we will have explicit description of the unitarizability for GL-groups (what we have obtained): "Then we can suppose that the classification of irreducible unitary representations for any reductive group (p-adic or real) can be given by reasonably explicit formulae". We completely agree with his opinion (more precisely, our opinion is that it will be given in two steps).

The second big problem that we mention above, how to handle exhaustion, looked hopeless for a long time in the case of classical groups (since (UO), which was crucial in the GL-case, is very far from being true here). But now we have reasonable proofs of exhaustions in the generic and spherical case. For these proofs we expect that they can be extended to the

[29]E. M. Stein, *Analysis in matrix spaces and some new representations of* $SL(N,\mathbb{C})$, Ann. of Math. 86 (1967), 461-490.

[30]B. Speh, *Unitary representations of* $GL(n,\mathbb{R})$ *with non-trivial* (g,K)*- cohomology*, Invent. Math. 71 (1983), 443-465.

general case (actually, that there are specialisations of a general proof of the exhaustion). But for such an extension, a number of not so easy questions will be necessary to solve.

Let us end with a joke of Paul Sally related to the the above unitarizability story in the case of GL-groups. Lefschetz principle usually functions in a way that one gets first a result in the real (often simpler) case, which gives an idea to extend it to the p-adic case. Since Theorem A was first proved in the p-adic case, and the idea that it holds in the real case came from the p-adic case, he called this fact an example of "Scheffetz principle"[31].

[31] This is our free interpretation how to write Paul pronunciation.