ON UNITARIZABILITY AND ARTHUR PACKETS

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ABSTRACT. In this paper we begin to explore relation between the question of unitarizability of classical p-adic groups, and Arthur packets, starting from [Tad20].

Contents

1. Introduction

We proposed in [Tad18] a strategy to approach unitarizability of classical groups over a p-adic field F of characteristic 0. In that strategy, the only relevant informations are cuspidal reducibility exponents, which are the elements of

$$\frac{1}{2}\mathbb{Z}_{\geq 0}$$

(therefore, they are the parameters with which we work). We applied this strategy in [Tad20] to classify unitarizability in coranks ≤ 3. The key to control unitarizability in [Tad20] is to understand it in the case of so-called critical points (see Definition 8.1). These are the places where the most important irreducible unitarizable representations show up (we expect that all isolated representations show up in these points). Non-unitarizable representations there give also key information for proving exhaustion. Majority of irreducible

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subquotients there are non-unitarizable. Still, approximately 100 types of irreducible subquotients there are unitarizable. Unitarizability of that representations was proved using standard methods of representation theory, except in the case of the representations given in the Langlands classification by

$$L(\nu^{\alpha-1}\rho, \nu^{\alpha}\rho; \delta([\nu^{\alpha}\rho]; \sigma))^1, \tag{1.1}$$

where ρ and σ are irreducible cuspidal representations of general linear group and classical group respectively such that ρ is selfcontragredient, α is (exceptional) reducibility between them which is $\geq \frac{3}{2}$ (see 2.6 and 2.9 for description of the notation). C. Mœglin proved its unitarizability using her explicit characterisation of Arthur packets (see her Appendix A in [Tad20]). This is the single place in [Tad20] where Arthur packets interacts (explicitely) to questions of unitarizability.

We expect that the role of Arthur packets is much deeper in the unitarizability. This paper is a step in trying to understand (and attempt to formulate more precisely) this interplay. We consider in this paper symplectic and split special odd-orthogonal groups over F (we expect that results of this paper hold also for other classical groups).

The first starting point of this paper in the direction of the Arthur packets are the Mœglin representations (1.1) which we considered in [Tad20]. In this paper we extend Mœglin construction to a two-parameter family

$$\pi_{m,n} := L([\nu^{\alpha-1}\rho], [\nu^{\alpha}\rho], \dots, [\nu^{\alpha+m}\rho]; \delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho]; \sigma)), \quad m, n \geq 0$$

(see Theorem 4.2). For representation $\pi_{0.0}$ (i.e. (1.1)) we have seen in [Tad20] that they are isolated in the unitary duals. We expect that all the representations $\pi_{n,m}$ are isolated also. Further, these representations satisfy the following very simple formula for the Aubert duality

$$\pi_{m,n}^{\mathbf{t}} = \pi_{n,m}. \tag{1.2}$$

Recall that for the Speh representations we have analogous formula for duality

$$u_\rho(m,n)^{\mathbf{t}} = u_\rho(n,m)$$

(see 2.3 for explanation of the notation).

Further, we describe in Theorems 5.1 and 6.2 analogous two-parameter families also in the case of non-exceptional reducibilities 0 and $\frac{1}{2}$ (they show up always, excluding finitely many cases), and also handle the case of exceptional reducibility at 1 (Theorem 7.3). We also compute the Aubert duals of these representations, and get formulas similar to the formula (1.2) (see Propositions 5.2, 6.1 and 7.2). We expect also for these representations to be isolated, excluding few of them (with very low indexes).

The second starting point of this paper in the direction of the Arthur packets is the list of unitarizable irreducible subquotients at critical points in [Tad20]. As we already mentioned, this is a list of around 100 types of them. Their unitarizability is obtained by a mixture of a different representation-theoretic methods, and for one case, using the Arthur packets. After we got their list (essentially case by case considerations - it took about

[1] The simplest cases when such representations can show up are $\mathrm{Sp}(10, F)$ and split $\mathrm{SO}(11, F)$.

half of [Tad20] to get that list), it was not easy to see a pattern (at the first glance, the list seems a little bit random). In this paper we prove that we have the following simple pattern (this is Theorem 8.4 in the paper):

Theorem 1.1. *Let π be an irreducible subquotient at a critical point in corank ≤ 3. Then the following two claims are equivalent:*

(1) π is unitarizable.
(2) π is a member of some Arthur packet[2].

The proof of above theorem gives also a much simpler proof then in [Tad20] of the unitarizability of irreducible subquotients in corank ≤ 3.

We conjecture that the above theorem holds in general (Conjecture 8.3 in the paper):

Conjecture 1.2. *Theorem 1.1 holds without assumption on the rank (i.e. for any rank).*

In other words, the above conjecture expects that in the case of the critical points, which is the most delicate part of the unitarizability, the mysterious line between unitarizability and non-unitarizability is drawn by Arthur packets. It may easily happen that the above conjecture is not true, but still we expect that this approach of thinking is in a good direction. This conjecture may be very hard to prove (if it is true).

The above conjecture could give a very precise (and in a sense a quantitative) relation between the unitarizability problem and the Arthur packets. It relates a highly mysterious and very hard question of unitarizability/non-unitarizability at the critical points to (at least a little bit) less mysterious and more combinatorial question of belonging to Arthur packets (a problem which seems easier to handle).

Let us recall that there is an expectation which dates from the very beginning of work on Arthur packets, that these representations will provide interesting (non-tempered) elements of unitary duals (see for example the comment at end of the introduction of the paper [Art89], the paper in which the existence of Arthur packets was conjectured). The most interesting representations in unitary duals are isolated representations. Our (pretty limited) experience so far, shows that from Mœglin construction of Arthur packets one can get rather easily isolated representations.

From the other side, if the above conjecture is true, it could provide also upper bounds for unitarizability in general (and therefore, be useful for exhaustion questions). Namely, the exhaustion is obtained in [Tad20] (as well as in other papers on unitarizability in corank two) by reducing the questions of non-unitarizability to known non-unitarizability at the critical points.

The key for handling Arthur packets in our paper is Mœglin explicit construction of Arthur packets. Let us note that techniques of Mœglin seem to fit well with approach to unitarizability based only on reducibility points.

We are very thankful to C. Mœglin for a series of discussions and sharing her results with us. Discussions with E. Lapid helped us to understand better some ideas on which this paper is based.

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[2]We know always (2) $\implies$ (1).

In section 4 we consider the case of reducibility > 1, where details of all the proofs are presented. In sections 5, 6 and 7 the case of reducibility points 0, $\frac{1}{2}$ and 1 is considered. The proofs of the claims in sections 5, 6 and 7 are obtained using the similar ideas and techniques as the proofs of the corresponding claims in the section 4. Because of this, we completely omit proofs in sections 5, 6 and 7.

The structure of the paper is following. The section 2 introduces the notation of the representation theory of classical p-adic groups, while the section 3 collects the notation and some facts of Mœglin construction of Arthur packets. The sections 4, 5, 6 and 7 bring constructions of families of Arthur representations corresponding to reducibility points > 1, 0, $\frac{1}{2}$ and 1 respectively. In the last section we show that each irreducible unitarizable subquotient of a critical point in corank ≤ 3 is of Arthur class. The fact that intermediate representations of simplest complementary series can be of Arthur class when reducibility exponent is > 1 is shown in the Appendix (9).

2. Notation

We first recall briefly the well-known notation for p-adic general linear groups established by J. Bernstein and A. V. Zelevinsky ([Zel80]; see also [Rod82]), and its natural extension to classical p-adic groups.

A p-adic field F of characteristic zero will be fixed. All representations considered in this paper will be smooth. The Grothendieck group of the category $\mathrm{Alg}_{\text{f.l.}}(G)$ of all finite length representations of a connected reductive p-adic group G is denoted by

$$\mathfrak{R}(G).$$

We have a natural map s.s. : $\mathrm{Alg}_{\text{f.l.}}(G) \to \mathfrak{R}(G)$ called semi simplification. There is a natural partial order on $\mathfrak{R}(G)$. For two finite length representations π_1 and π_2 of G, the fact $\mathrm{s.s.}(\pi_1) \leq \mathrm{s.s.}(\pi_2)$ we will write simply as $\pi_1 \leq \pi_2$. The contragredient representation of π will be denoted by $\tilde{\pi}$. We can lift the mapping $\pi \to \tilde{\pi}$ to an additive groups homomorphism $\sim: \mathfrak{R}(G) \to \mathfrak{R}(G)$.

If Π (resp. π) is a representation (resp. an irreducible representation) of G , then

$$\pi \underset{\text{u.i.sub.}}{\hookrightarrow} \Pi$$

will mean that π is a unique irreducible subrepresentation of Π.

2.1. **Hopf algebra for general linear groups.** The modulus character on F is denoted by $|\ |_F$, and character $|\det|_F$ of $\mathrm{GL}(n, F)$ by ν. Let $n = n_1 + n_2, n_i \geq 0$. Denote by $P_{(n_1,n_2)} = M_{(n_1,n_2)}N_{(n_1,n_2)}$ the parabolic subgroup of $\mathrm{GL}(n, F)$ which is standard with respect to the minimal parabolic subgroup of upper triangular matrices, such that $M_{(n_1,n_2)}$ is naturally isomorphic to $\mathrm{GL}(n_1, F) \times \mathrm{GL}(n_2, F)$. For representations π_i, $i = 1, 2$, of $\mathrm{GL}(n_i, F)$, denote

$$\pi_1 \times \pi_2 := \mathrm{Ind}_{P_{(n_1,n_2)}}^{\mathrm{GL}(n,F)}(\pi_1 \otimes \pi_2).$$

Let $R := \underset{n\geq 0}{\oplus} \mathfrak{R}(\mathrm{GL}(n,F)$. Then $\times$ lifts naturally to a multiplication on R, and we get in this way on R structure of commutative graded $\mathbb{Z}$-algebra. We can factorise $\times : R\times R \to R$ through $m : R\otimes R \to R$.

The normalised Jacquet module with respect to $P_{(n_1,n_2)}$ of a representation π of $\mathrm{GL}(n,F)$ is denoted by $r_{(n_1,n_2)}(\pi)$. If π is of finite length, then we can view

$$m^*(\pi) := \sum_{k=0}^{n} \mathrm{s.s.}(r_{(k,n-k)}(\pi)) \in R\otimes R.$$

One extends additively m^* on whole R, and gets a structure of graded Hopf algebra on R.

2.2. **Segments and corresponding irreducible subrepresentations.** Denote by $\mathcal{C}$ (resp. $\mathcal{D}$) the set of all equivalence classes of irreducible cuspidal (resp. essentially square integrable) representations of all $\mathrm{GL}(n,F)$, $n\geq 1$. For $\delta\in\mathcal{D}$, there exists unique $e(\delta)\in\mathbb{R}$ and unitarizable $\delta^u\in\mathcal{D}$ such that

$$\delta = \nu^{e(\delta)}\delta^u.$$

For $\rho\in\mathcal{C}$ and $x,y\in\mathbb{R}$ such that $y-x\in\mathbb{Z}_{\geq 0}$, the set $[\nu^x\rho,\nu^y\rho] := \{\nu^x\rho,\nu^{x+1}\rho,\dots,\nu^y\rho\}$ is called a segment of cuspidal representations of general linear groups (one-point segment $[\nu^x\rho,\nu^x\rho]$ will be denoted simply by $[\nu^x\rho]$).We denote it also by

$$[x,y]^{(\rho)},$$

or simply by $[x,y]$ when we will work with fixed ρ (usually this will be the case later). The set of all segments of cuspidal representations is be denoted by $\mathcal{S}(\mathcal{C})$. The representation $\nu^x\rho\times\nu^{x+1}\rho\times\cdots\times\nu^y\rho$ (resp. $\nu^y\rho\times\nu^{y-1}\rho\times\cdots\times\nu^x\rho$) contains a unique irreducible subrepresentation which will be denoted by

$$\langle\nu^x\rho,\nu^{x+1}\rho,\dots,\nu^y\rho\rangle \text{ (resp. } \langle\nu^y\rho,\nu^{y-1}\rho,\dots,\nu^x\rho\rangle).$$

When we deal with fixed ρ, these representations will be denoted simply by $\langle x,x+1,\dots,y\rangle$, (resp. $\langle y,y-1,\dots,x\rangle$). For a segment $[x,y]^{(\rho)}\in\mathcal{S}(\mathcal{C})$ denote $\delta([x,y]^{(\rho)}) := \langle\nu^y\rho,\nu^{y-1}\rho,\dots,\nu^x\rho\rangle$. Then $\delta([x,y]^{(\rho)})\in\mathcal{D}$. For $n\geq 1$ set $\delta(\rho,n) := \delta([-\frac{n-1}{2},\frac{n-1}{2}]^{(\rho)})$.S

Let π be an irreducible representation of a general linear group. Then there exist $\rho_1,\dots,\rho_k$ such that $\pi\hookrightarrow\rho_1\times\cdots\times\rho_k$. The multiset $(\rho_1,\dots,\rho_k)$ is called the (cuspidal) support of π, and is denoted by $\mathrm{supp}(\pi)$.

2.3. **Langlands classification for general linear groups.** For a set X, denote by $M(X)$ the set of all finite multisets in X. For $d=(\delta_1,\dots,\delta_k)\in M(\mathcal{C})$ chose permutation p of $\{1,\dots,k\}$ such that $e(\delta_{p(1)})\geq\dots e(\delta_{p(k)})$. Then the representation $\lambda(d) := \delta_{p(1)}\times\cdots\times\delta_{p(k)}$ has a unique irreducible quotient, denoted by $L(d)$. This defines a bijection from $M(\mathcal{D})$ onto the set of equivalence classes of irreducible representations of all groups $\mathrm{GL}(n,F)$, $n\geq 0$ (Langlands classification). Another way to express this classification is by $M(\mathcal{S}(\mathcal{C}))$. To $a=(\Delta_1,\dots,\Delta_k)\in M(\mathcal{S}(\mathcal{C}))$ attach

$$L(a) := L(\delta(\Delta_1),\dots,\delta(\Delta_k)).$$

This is the version of Langlands classification with which will be used in the paper. For $n, m \geq 1$ and $\rho \in \mathcal{C}$ we denote by

$$u_\rho(n,m) := L(\nu^{\frac{m-1}{2}}\delta(\rho,n), \nu^{\frac{m-1}{2}-1}\delta(\rho,n), \dots, \nu^{-\frac{m-1}{2}}\delta(\rho,n)),$$

and call a Speh representation.

2.4. **Module and comodule structures for classical groups.** In this paper we consider classical groups $\mathrm{Sp}(2n,F)$ and split $\mathrm{SO}(2n+1,F)$, $n \geq 0$. We will use their matrix realisation from [Tad95]. Such group of rank n will be denoted by S_n (we will always work with fixed series of groups). We fix in S_n minimal parabolic subgroup consisting of all upper triangular matrices in the group. Now for each $0 \leq k \leq n$, there is standard parabolic subgroup $P_{(k)} = M_{(k)}N_{(k)}$ such that $M_{(k)}$ is naturally isomorphic to the direct product $\mathrm{GL}(k,F) \times S_{n-k}$. For a representations π and σ of $\mathrm{GL}(k,F)$ and S_{n-k} respectively, one defines $\pi \rtimes \sigma := \mathrm{Ind}_{P_{(k)}}^{S_n}(\pi \otimes \sigma)$. Denote $R(S) := \underset{n\geq 0}{\oplus} \mathfrak{R}(S_n)$. Then $\rtimes$ lifts in a natural way to $\rtimes : R \times R(S) \to R(S)$, and in this was $R(S)$ becomes R-module. The normalised Jacquet module with respect to $P_{(k)}$ of a representation π of S_n is denoted by $s_{(k)}(\pi)$. Let π be of finite length. Then we can view

$$\mu^*(\pi) := \sum_{k=0}^{n} \mathrm{s.s.}(s_{(k)}(\pi)) \in R \otimes R(S),$$

and extend μ^* additively to $\mu^* : R(S) \to R \otimes R(S)$. In this way, $R(S)$ becomes R-comodule.

2.5. **Twisted Hopf algebra.** Denote by $\kappa : R \otimes R \to R \otimes R$ the transposition map $\sum_i x_i \otimes y_i \mapsto \sum_i y_i \otimes y_i$, and by

$$M^* := (m \otimes \mathrm{id}_R) \circ (\sim \otimes m^*) \circ \kappa \circ m^* : R \to R \otimes R. \tag{2.1}$$

Then for finite length representations π and σ of $\mathrm{GL}(n,F)$ and S_k respectively, by [Tad95] holds

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\pi).$$

Denote by $M^*_{GL}(\pi) \otimes 1$ the component of $M^*(\pi)$ which is in $\mathfrak{R}(\mathrm{GL}(n,F)) \otimes \mathfrak{R}(\mathrm{GL}(0,F))$. We calculate $M^*_{GL}(\pi)$ by the following simple formula: if $m^*(\pi) = \sum_i x_i \otimes y_i$, then $M^*_{GL}(\pi) = \sum_i x_i \times \tilde{y}_i$. The component of $M^*(\pi)$ which is in $\mathfrak{R}(\mathrm{GL}(0,F)) \otimes \mathfrak{R}(\mathrm{GL}(n,F))$ is $1 \otimes \pi$.

Let additionally σ be an irreducible cuspidal representation of a classical group, and τ a subquotient of $\pi \rtimes \sigma$. Then we denote $s_{\mathrm{GL}}(\tau) := s_{(n)}(\tau)$. Now for a finite length representation π' of $\mathrm{GL}(n,F)$ holds $\mathrm{s.\,s.}(s_{\mathrm{GL}}(\pi' \rtimes \tau)) = M^*_{GL}(\pi') \times \mathrm{s.\,s.}(s_{GL}(\tau))$.

2.6. **Langlands classification for classical groups.** Denote by Irr^{cl} the set of equivalence classes of all irreducible representations of groups $S_n, n \geq 0$, and by $\mathcal{T}^{cl}$ the subset of the tempered representations in Irr^{cl}. Let $\mathcal{D}_+ = \{\delta \in \mathcal{D} : e(\delta) > 0\}$. Take $t = ((\delta_1, \delta_2, \dots, \delta_k), \tau) \in M(\mathcal{D}_+) \times \mathcal{T}^{cl}$. Chose a permutation p of $\{1, \dots, k\}$ such that $e(\delta_{p(1)}) \geq e(\delta_{p(2)}) \geq \dots \geq e(\delta_{p(k)})$. Then, the representation $\lambda(t) := \delta_{p(1)} \times \delta_{p(2)} \times \dots \times \delta_{p(k)} \rtimes \tau$ has a unique irreducible irreducible quotient, denoted by $L(t)$. The mapping $t \mapsto L(t)$ defines a bijection between $M(\mathcal{D}_+) \times \mathcal{T}^{cl}$ and Irr^{cl}, and it is the Langlands classification for classical groups (the multiplicity of $L(t)$ in $\lambda(t)$ is one). If $t = (d;\tau)$, then $L(d;\tau)\tilde{} \cong L(d;\tilde{\tau})$ and $L(d;\tau)^{-} \cong L(\bar{d};\bar{\tau})$, where $\bar{\pi}$ denotes the complex conjugate representation of π.

Introducing $\mathcal{S}(\mathcal{C})_+ = \{\Delta \in \mathcal{S}(\mathcal{C}); e(\delta(\Delta)) > 0\}$, we can define in a natural way the Langlands classification $(a,\tau) \mapsto L(a;\tau)$ using parameter $M(\mathcal{S}(\mathcal{C})_+) \times \mathcal{T}^{cl}$. We will use this classification in the paper.

2.7. **Duality.** There is a natural involution D_G on the Grothendieck group of the representations of any connected reductive p-adic group G ([Aub95] and [SS97], see also [BBK18]). It takes any irreducible representation to an irreducible representation up to a sign. For any irreducible representation π, let π^t be the irreducible representation such that $D_G(\pi) = \pm\pi^t$. We call π^t the Aubert involution of π, or DL dual of π. This involution is compatible with parabolic induction in the sense that $(\pi \rtimes \tau)^t = \pi^t \rtimes \tau^t$ (on the level of Grothendieck groups). Furthermore, regarding Jacquet modules, the mapping

$$\pi_1 \otimes \dots \pi_l \otimes \mu \mapsto \tilde{\pi}_1^t \otimes \dots \tilde{\pi}_l^t \otimes \mu^t,$$

is a bijection from the semi simplification of $s_\beta(\pi)$ onto the semi simplification of $s_\beta(\pi^t)$ (β is the partition which parameterises the corresponding parabolic subgroup).

We will use the following simple but useful result proved in Theorem 13.2[3] of [Tad98]. For $\Delta \in \mathcal{S}(\mathcal{C})$ holds:

$$\delta(\Delta) \rtimes \sigma \text{ is reducible } \iff \theta \rtimes \sigma \text{ is reducible for some } \theta \in \Delta.$$

An irreducible representation will be called cotempered, if it is Aubert involution of a tempered representation.

2.8. **Some formulas for M^*.** Let $\rho \in \mathcal{C}$ be selfcontragredient and $[x,y]^{(\rho)} \in \mathcal{S}(\mathcal{C})$. Then, one easily gets

$$M^*\big(\delta([x,y]^{(\rho)})\big) = \sum_{i=x-1}^{y} \sum_{j=i}^{y} \delta([-i,-x]^{(\rho)}) \times \delta([j+1,y]^{(\rho)}) \otimes \delta([i+1,j]^{(\rho)}), \tag{2.2}$$

where $y-i, y-j \in \mathbb{Z}_{\geq 0}$ in the above sums[4]. In particular

$$M^*_{\mathrm{GL}}\big(\delta([x,y]^{(\rho)})\big) = \sum_{i=x-1}^{y} \delta([-i,-x]^{(\rho)}) \times \delta([i+1,y]^{(\rho)}). \tag{2.3}$$

We denote $[x]^{(\rho)}, [x+1]^{(\rho)}, \dots, [y]^{(\rho)}$ by

$$([x,y]^{(\rho)})^{\mathbf{t}}.$$

[3]To apply this theorem, we need to know that the cuspidal reducibility exponents are in $\frac{1}{2}$, which is implied by the basic assumption from [MT02], and this assumption follows from (ii) in Remarks 4.5.2 of [MW06] and Theorem 1.5.1 in [Art13].

[4]In the above formula and the formulas below, we take terms of the form $[t,t-1]^{(\rho)}$ to be identity of R (i.e. to be $L(\emptyset)$).

Then $\langle \nu^x\rho, \nu^{x+1}\rho, \dots, \nu^y\rho\rangle = L(([x,y]^{(\rho)})^{\mathbf{t}})$. Now

$$M^*(L(([x,y]^{(\rho)})^{\mathbf{t}})) = \sum_{x-1\leq i\leq y}\sum_{x-1\leq j\leq i} L(([-y,-i-1]^{(\rho)})^{\mathbf{t}}) \times L(([x,j]^{(\rho)})^{\mathbf{t}}) \otimes L(([j+1,i]^{(\rho)})^{\mathbf{t}}), \tag{2.4}$$

$$M^*_{GL}(L(([x,y]^{(\rho)})^{\mathbf{t}})) = \sum_{i=x-1}^{y} L(([-y,-i-1]^{(\rho)})^{\mathbf{t}}) \times L(([x,i]^{(\rho)})^{\mathbf{t}}). \tag{2.5}$$

2.9. **Some very simple irreducible square integrable and tempered representations of classical groups.** Let ρ and σ be irreducible cuspidal representations of a general linear and a classical group respectively, and suppose that ρ is selfcontragredient (i.e. $\rho \cong \tilde\rho$). Then there exists unique $\alpha_{\rho,\sigma} \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ such that

$$\nu^{\alpha_{\rho,\sigma}}\rho \rtimes \sigma$$

reduces. We denote $\alpha_{\rho,\sigma}$ simply by α once when we deal all the time with the same α and σ.

Suppose $\alpha > 0$ and $n \geq 0$. Then the representation $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho]) \rtimes \sigma$ contains a unique irreducible representation, which is denoted by

$$\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho]; \sigma).$$

This representation is square integrable, and it is called generalised Steinberg representation. Further, $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho]; \sigma)$ is unique irreducible subrepresentation of $\nu^{\alpha+n}\rho \rtimes \delta([\nu^\alpha\rho, \nu^{\alpha+n-1}\rho]; \sigma)$.

Starting from generalised Steinberg representations, one can construct further (strongly positive) square integrable representations (in [Mœg02] and [MT02] is general construction such representations; see also section 34 of [Tad12]). We will here describe only the first step of the construction. Let $\alpha \geq \frac{3}{2}$. Take $m \in \mathbb{Z}_{\geq 0}$ such that $m \leq n$. Then the representation $\delta([\nu^{\alpha-1}\rho, \nu^{\alpha-1+m}\rho]) \rtimes \delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho]; \sigma)$ has a unique irreducible subrepresentation, denoted by

$$\delta_{\text{s.p.}}([\nu^{\alpha-1}\rho, \nu^{\alpha-1+m}\rho], [\nu^\alpha\rho, \nu^{\alpha+n}\rho]; \sigma).$$

This representation is square integrable.

Suppose $\alpha > 0$. Take $x, y \in \mathbb{R}$ such that $x \leq y$ and $x-\alpha,\ y-\alpha \in \mathbb{Z}_{\geq 0}$. Then the representation $\delta([\nu^{-x}\rho, \nu^y\rho]) \rtimes \sigma$ contains precisely two irreducible subrepresentations. If $\alpha \in \mathbb{Z}$ (resp. $\frac{1}{2} + \mathbb{Z}$), then precisely one of these subrepresentations contains in its Jacquet module the term $\delta([\nu\rho, \nu^x\rho]) \times \delta([\rho, \nu^y\rho]) \otimes \sigma$ (resp. $\delta([\nu^{\frac{1}{2}}\rho, \nu^x\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^y\rho]) \otimes \sigma$). We denote this irreducible subrepresentation by

$$\delta([\nu^{-x}\rho, \nu^y\rho]_+; \sigma)$$

and the other irreducible subrepresentation by $\delta([\nu^{-x}\rho, \nu^y\rho]_-; \sigma)$. Both subrepresentations are square integrable if $x < y$, and tempered (but not square integrable) otherwise (see [Tad99] for more details).

Let now $\alpha = 0$ and $n \geq 0$. Take irreducible tempered representations $\delta([\rho]_{\pm};\sigma)$[5] such that

$$\rho \rtimes \sigma := \delta([\rho]_{+};\sigma) \oplus \delta([\rho]_{-};\sigma). \tag{2.6}$$

Then the representation $\delta([\nu\rho, \nu^{\alpha+n}\rho]) \rtimes \delta(\rho_{\pm};\sigma)$ contains a unique irreducible subrepresentation, which is denoted by $\delta([\rho, \nu^{n}\rho]_{\pm};\sigma)$. These representations are square integrable for $n \geq 1$. Further for $n \geq 1$, $\delta([\rho, \nu^{\alpha+n}\rho]_{\pm};\sigma)$ is unique irreducible subrepresentation of $\nu^{\alpha+n}\rho \rtimes \delta([\rho, \nu^{\alpha+n-1}\rho]_{\pm};\sigma)$.

Let now $\alpha = 1$ and $n \geq 1$. Then $\rho \rtimes \delta([\nu\rho, \nu^{n}\rho];\sigma)$ decompose into a direct sum of two irreducible (tempered) representations, which we denote by

$$\tau([\rho]_{\pm}; \delta([\nu\rho, \nu^{n}\rho];\sigma)).$$

The representation $\tau([\rho]_{+}; \delta([\nu\rho, \nu^{n}\rho];\sigma))$ is characterised by the fact that $\delta([\rho, \nu^{n}\rho]) \otimes \sigma$ is in its Jacquet module.

2.10. **Jantzen lemma.**

Definition 2.1. *Let π be an irreducible representation of some S_n and $\rho \in \mathcal{C}$.*

(1) We let $\mu^*_\rho(\pi)$ be the sum of all irreducible terms in $\mu^*(\pi)$ of the form $\rho \otimes \tau$.
(2) We let

$$\mathrm{Jac}_\rho(\pi)$$

be the sum of all τ when irreducible $\rho \otimes \tau$ runs over $\mu^*_\rho(\pi)$.

Observe that $\mu^*_\rho(\pi) = \rho \otimes \mathrm{Jac}_\rho(\pi)$. By Lemma 5.6 of [Xu17b] we have $\mathrm{Jac}_{\rho_1} \circ \mathrm{Jac}_{\rho_2} = \mathrm{Jac}_{\rho_2} \circ \mathrm{Jac}_{\rho_1}$ if $\rho_1 \notin \{\nu\rho_2, \nu^{-1}\rho_2\}$. Below we recall of Lemma 3.1.3 from [Jan14] (in a slightly less general form).

Definition 2.2. *Let π be an irreducible representation of some S_n and $\rho \in \mathcal{C}$. Denote by $f_\pi(\rho)$ to be the largest value of f such that some Jacquet module of π contains an irreducible subquotient of the form $\rho \otimes \cdots \otimes \rho \otimes \tau$, where ρ shows up f times. We let*

$$\mu^*_{\{\rho\}}(\pi)$$

be the sum of all irreducible terms in $\mu^*(\pi)$ of the form $\rho \times \cdots \times \rho \otimes \tau$, where ρ shows up f times in the last formula and τ is irreducible.

Lemma 2.3. *Let π be an irreducible representation of some S_n and $\rho \in \mathcal{C}$. Then there is unique irreducible representation θ and unique $f \geq 0$ such that the following are all satisfied:*

(1) $\pi \hookrightarrow \lambda \rtimes \theta$, where $\lambda := \rho \times \cdots \times \rho$ and ρ shows up f times in the last formula.
(2) $\mu^*_\rho(\theta) = 0$.

[5] If σ is generic, then we take $\delta([\rho]_{+};\sigma)$ to be generic.

Furthermore, $f = f_\pi(\rho)$ and $\lambda\otimes\theta$ is the only irreducible subquotient of $\mu^*_{\{\rho\}}(\pi)$.

If $\rho\not\cong\tilde\rho$, then $\mu^*_{\{\rho\}}(\pi)$ is irreducible[6] *and $\pi\hookrightarrow\lambda\rtimes\theta$ is the unique irreducible subrepresentation. In particular, if π' is an irreducible representation with $\mu^*_{\{\rho\}}(\pi)=\mu^*_{\{\rho\}}(\pi')$, then $\pi\cong\pi'$.*

Remark 2.4. *(1) If $\mu^*_\rho(\pi)=\rho\otimes\theta$, then $\mu^*_{\tilde\rho}(\pi^{\mathbf{t}})=\tilde\rho\otimes\theta^{\mathbf{t}}$.*

(2) $\mathrm{Jac}_\rho(\pi)^{\mathbf{t}}=\mathrm{Jac}_{\tilde\rho}(\pi^{\mathbf{t}})$.

(3) If $\rho\not\cong\tilde\rho$ and $\mu^*_{\{\rho\}}(\pi)=\lambda\otimes\theta$, then $\mu^*_{\tilde\rho}(\pi^{\mathbf{t}})=\tilde\lambda\otimes\theta^{\mathbf{t}}$ and $\pi^{\mathbf{t}}\hookrightarrow\tilde\lambda\rtimes\theta^{\mathbf{t}}$ as the unique irreducible subrepresentation.

(4) We recall of Remark in 2.3 of [Mœg06] *which she proved by the same type of arguments as the above lemma is proved. Let $\rho\in\mathcal{C}$, $x\in\mathbb{R}$, $x\neq 0$, and let π,π' be irreducible representations of S_n and S_m. Suppose $\pi\hookrightarrow\nu^x\rho\rtimes\pi'$,* $\mathrm{Jac}_{\nu^{-x}\rho}(\pi)\neq 0$ *and* $\mathrm{Jac}_{\nu^{-x}\rho}(\pi')=0$. *Then $\nu^x\rho\rtimes\pi'$ is irreducible.*

3. Parameters of A-packets

In this section we recall of well-known terminology related to A-packets following mainly C. Mœglin.

3.1. **A-parameters.** For an irreducible cuspidal representation ρ of $GL(n_\rho,F)$ (this defines n_ρ), by ρ will be denoted also the corresponding irreducible representation of the Weil group W_F under the local Langlands correspondence for general linear groups. The irreducible algebraic representation of $SL(2,\mathbb{C})$ of dimension a over $\mathbb{C}$ is denoted by E_a. A triple (ρ,a,b), $\rho\in\mathcal{C}$, $a,b\in\mathbb{Z}_{>0}$ is called Jordan block. To shorten notation in the paper, we denote

$$E^\rho_{a,b}:=\rho\otimes E_a\otimes E_b.$$

For a connected reductive group G over F, the connected component of the dual group ${}^LG$ is denoted by ${}^LG^0$, and called the complex dual group. Then ${}^L\mathrm{Sp}(2n,F)^0=\mathrm{SO}(2n+1,\mathbb{C})$ and ${}^L\mathrm{SO}(2n+1,F)^0=\mathrm{Sp}(2n,\mathbb{C})$. Set $n^*=2n+1$ (resp. $n^*=2n$) if $G=\mathrm{Sp}(2n,F)$) (resp. $G=\mathrm{SO}(2n+1,F)$).

Definition 3.1. *An A-parameter for group S_n is a continuous homomorphism $\psi: W_F\times SL(2,\mathbb{C})\times SL(2,\mathbb{C})\to {}^LS_n^0$, which is bounded on W_F and (complex) algebraic on $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$.*

We can decompose ψ as above into a the sum of irreducible representations

$$\psi=\bigoplus_{(\rho,a,b)\in\mathrm{Jord}(\psi)}E^\rho_{a,b} \tag{3.1}$$

where $Jord(\psi)$ is a finite multiset, which is called the Jordan blocks of ψ. Then we have $\sum_{(\rho,a,b)\in\mathrm{Jord}(\psi)}n_\rho ab=n^*$. Clearly, $\mathrm{Jord}_\rho(\psi)$ determines ψ (up to an equivalence). We can work with $\mathrm{Jord}(\psi)$ as A-parameter instead of ψ For a finite multiset of Jordan blocks

[6] I.e. the multiplicity of $\lambda\otimes\theta$ in $\mu^*_{\{\rho\}}(\pi)$ is one.

(ρ, a, b)'s, some additional conditions may be needed to be satisfied that this multiset is the set of Jordan blocks of an A-parameter (we will not discuss these conditions here). Denote

$$\mathrm{Jord}_\rho(\psi) = \{(a, b); (\rho, a, b) \in \mathrm{Jord}(\psi)\}.$$

The set of all equivalence classes of A-parameters of S_n will be denoted by $\Psi(S_n)$, and $\Psi = \cup_{n\geq 0}\Psi(S_n)$ (we will indicate the series of the groups with which we work when this will be necessary).

One says that $(\rho, a, b) \in Jord(\psi)$ has good parity (with respect to S_n) if there exist $z \in \mathbb{Z}$ such that $\nu^{\frac{a+b}{2}+z}\rho \rtimes 1_{S_0}$ reduces. We say that $\mathrm{Jord}(\psi)$ has good parity if each its element has good parity. The subset of A-parameters of good parity will be denoted by

$$\Psi_{\text{g.p.}}.$$

In this paper we will work only with A-parameters of good parity. This implies that cuspidal representation ρ will be always selfcontragredient.

To an A-parameter ψ is associated an irreducible unitarizable representation

$$\pi_\psi := \underset{(\rho,a,b)\in Jord(\psi)}{\times} u_\rho(a, b)$$

of a general linear groups over F (we can work with π_ψ as A-parameter instead of ψ).

3.2. **Another parameters of Jordan blocks.**

Let $(\rho, a, b) \in Jord(\psi)$, $\psi \in \Psi_{\text{g.p.}}$. Put

$$A = \tfrac{a+b}{2} - 1, \quad B = \tfrac{|a-b|}{2}$$

and $\zeta_{a,b} = sign(a-b)$ if $a \neq b$, and $\zeta_{a,b} = 1$ arbitrary element of $\{\pm 1\}$ otherwise. Obviously either $A, B \in \mathbb{Z}_{\geq 0}$ or $A, B \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$. Observe that

$$a = A + 1 + \zeta_{a,b}B, \qquad b = A + 1 - \zeta_{a,b}B.$$

Jordan block (ρ, a, b) will be denoted also by

$$(\rho, A, B, \zeta_{a,b}).$$

3.3. **Modifying A-parameters.**

Fix a series of classical groups that we take and $\psi \in \Psi_{\text{g.p.}}$. Let (ρ, a, b) and (ρ, a', b') be two Jordan blocks which satisfy

$$\tilde{\rho} \cong \rho, \qquad a \equiv a' (\mathrm{mod}\ 2\mathbb{Z}), \qquad b \equiv b' (\mathrm{mod}\ 2\mathbb{Z}). \tag{3.2}$$

Then:

1. If $(\rho, a, b) \in \mathrm{Jord}_\rho(\psi)$, and if we define a new parameter ψ' by replacing (ρ, a, b) with (ρ, a', b') in $\mathrm{Jord}(\psi)$, then $\psi' \in \Psi_{\text{g.p.}}$.
2. If (ρ, a, b) has good parity, then $\psi \oplus E^\rho_{a,b} \oplus E^\rho_{a',b'} \in \Psi_{\text{g.p.}}$
3. If (ρ, a, b) has good parity and $ab \in 2\mathbb{Z}$, the $\psi \oplus E^\rho_{a,b} \in \Psi_{\text{g.p.}}$.

3.4. **Elementary A-parameters.** A Jordan block (ρ, a, b) will be called elementary if $1 \in \{a, b\}$. An A-parameter ψ will be called elementary if it has good parity and if each $(\rho, a, b) \in Jord(\psi)$ is elementary. The last condition can be expressed that for each $(\rho, A, B, \zeta) \in Jord(\psi)$, we have $A = B$. The subset of elementary A-parameters in Ψ (and $\Psi_{\text{g.p.}}$) is denoted by

$$\Psi_{\text{ele.}}.$$

Let ψ be elementary. Then using parameterisation introduced in 3.2, each element in the Jordan block can be written as $(\rho, \frac{c-1}{2}, \frac{c-1}{2}, \delta_c)$, where $c \in \mathbb{Z}_{>0}$, $\delta_c \in \{-1, 1\}$, and we denote this Jordan block simply by

$$(\rho, c, \delta_c)$$

In the case of elementary A-parameters, we take $\delta_1 = 1$. Observe that if $\delta_c = 1$ (resp. $\delta_c = -1$), the corresponding Speh representation is square integrable (resp. Aubert dual of a square integrable representation).

Definition 3.2. *If for $\psi \in \Psi$ holds $b = 1$ (resp. $a = 1$) for each $(\rho, a, b) \in \mathrm{Jord}(\psi)$, then ψ will be called tempered (resp. cotempered) A-parameter.*

3.5. **Discrete A-parameters.** Denote by $\Phi(S_n)$ the set of equivalence classes of admissible homomorphisms $W_F \times SL(2, \mathbb{C}) \to {}^L S_n^0$, and $\Phi = \cup_{i \geq 0} \Phi(S_n)$. Let Φ_2 be the subset corresponding to the square integrable L-packets. For $\phi \in \Phi$, one defines $\mathrm{Jord}(\phi)$ and $\mathrm{Jord}_\rho(\phi)$ analogously as in the case of A-packets.

Denote by $\Delta : SL(2, \mathbb{C}) \to SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ the diagonal map. Let ψ be an A-parameter. Then the composition $\psi \circ \Delta$ carries $(w, g) \mapsto \psi(w, g, g)$, and this element of Φ denoted by ψ_d. Then

$$\psi_d = \bigoplus_{(\rho,a,b) \in \mathrm{Jord}(\psi)} \bigoplus_{j \in [B,A]} \rho \otimes E_{2j+1}, \tag{3.3}$$

where B and A are defined in 3.2 (i.e. $B = \frac{|a-b|}{2}$ and $A = \frac{a+b}{2} - 1$). One says that an A-parameter ψ is discrete (or that has discrete diagonal restriction) if $\psi_{\mathrm{d}} \in \Phi_2$. It is equivalent to the fact that ψ has good parity and that ψ_d is a multiplicity one representation (in particular, then ψ is multiplicity one representation). The subset of all $\psi \in \Psi$ which are discrete is denoted by

$$\Psi_{\text{d.d.r.}}.$$

By 3.3, an A-parameter $\psi \in \Psi$ is in $\Psi_{\text{d.d.r.}}$ if and only if ψ has good parity and for each fixed ρ, all the segments $[B, A]$ when (ρ, A, B, ζ) runs over $\mathrm{Jord}(\psi)$, are disjoint.

3.6. **Characters of component group - good parity case.** Let $\psi \in \Psi_{\text{g.p.}}$. Then characters of the component group can be identified with functions ϵ on the multiset $\mathrm{Jord}(\psi)$ into $\{\pm 1\}$ which satisfy $\prod_{(\rho,a,b) \in \mathrm{Jord}(\psi)} \epsilon(\rho, a, b) = 1$ and

$$\epsilon(\rho, a, b) = \epsilon(\rho', a', b')), \text{ whenever } (\rho, a, b) = (\rho', a', b').$$

Sometimes we will look at characters of the component group of ψ as function on irreducible constituents of ψ in a natural way.

Definition 3.3. *Let $\psi \in \Psi_{\text{g.p.}}$, ϵ be a character of the component group of ψ, $(\rho, a, b) \in \mathrm{Jord}_\rho(\psi)$ and let (ρ, a', b') be a Jordan block such that $(\rho, a', b') \notin \mathrm{Jord}_\rho(\psi)$, $a \equiv a' (mod\ 2\mathbb{Z})$ and $b \equiv b'(mod\ 2\mathbb{Z})$. Denote by ψ' the A-parameter obtained from ψ by replacing (ρ, a, b) with (ρ, a', b') in $\mathrm{Jord}_\rho(\psi)$ (then $\psi' \in \Psi_{\text{g.p.}}$). Denote by ϵ' the character of the component group of ψ' such that $\epsilon'(\rho, a', b') = \epsilon(\rho, a, b)$, and that ϵ' and ϵ coincide on the remaining elements. We say that (ψ', ϵ') is obtained from (ψ, ϵ) deforming (ρ, a, b) to (ρ, a', b') (or deforming $E^\rho_{a,b}$ to $E^\rho_{a',b'}$).*

3.7. **A-packets.** To each A-parameter ψ of S_n, J. Arthur has attached in [Art13, Theorem 2.2.1] a finite multiset Π_ψ of irreducible unitarizable representations of S_n, called the A-packet of ψ, such that endoscopic distribution properties are satisfied. We will not recall that properties, but we will follow C. Mœglin explicit representation-theoretic construction of A-packets ([Mœg06], [Mœg09] and [Mœg11]). For the difference of L-packets, A-packets does not need to be disjoint for different conjugacy classes of ψ[7], and for the difference of L-packets, A-packets consists always of the unitarizable representations only.

More precisely, Arthur has attached to each character ϵ of the component group of ψ a multiset $\pi(\psi, \epsilon)$ of irreducible representations. Their sum is Π_ψ. Mœglin has proved that $\pi(\psi, \epsilon)$ are multiplicity one ([Mœg11]), and that for fixed ψ, $\pi(\psi, \epsilon)$'s are disjoint for different ϵ's. Therefore, she has proved that A-packets Π_ψ's have also multiplicity one[8].

Mœglin has also proved that in the case of elementary discrete A-parameters, $\pi(\psi, \epsilon)$ are always irreducible representations ([Mœg06]). Note that in the case of elementary discrete A-parameter ψ, the number of characters of the component group of ψ is the same as the number of characters of the component group of ψ_d, and we can identify them in obvious way.

Remark 3.4. *For $\psi_0 \in \Psi$ and $\psi_1 = \rho \otimes E_a \otimes E_b$ denote $\psi = \psi_1 \oplus \psi_o \oplus \tilde{\psi}_1$. Then there exists a canonical injection $\Pi_{\psi_0} \hookrightarrow \Pi_\psi$ and all irreducible constituents of $u_\rho(a, b) \rtimes \pi_0$, $\pi_0 \in \Pi_{\psi_0}$, are contained in the image of this injection (*[AM20, section 5]*; there is a more precise statement regarding $u_\rho(a, b) \rtimes \pi_0$).*

3.8. **Notation $b_{\rho,\psi,\epsilon}$ and $a_{\rho,\psi,\epsilon}$.** Fix $\psi \in \Psi_{\text{ele.}} \cap \Psi_{\text{g.p.}}$, a character ϵ of the component group of ψ and selfcontragredient $\rho \in \mathcal{C}$. Let X be a subset of $\mathrm{Jord}(\psi)$ of the form $X = \{(\rho, c_1, \delta_{c_1}), \ldots, (\rho, c_k, \delta_{c_k})\}$, and chose enumeration such that $c_1 < c_2 < \cdots < c_k$. We say that ϵ is cuspidal on X if

(1) $c_1 \in \{1, 2\}$.
(2) $c_{i+1} - c_i = 2$ for $1 \leq i \leq k - 1$.
(3) $\epsilon(\rho, c_{i+1}, \delta_{c_{i+1}}) = -\epsilon(\rho, c_i, \delta_{c_i})$ for $1 \leq i \leq k - 1$.
(4) $\epsilon(\rho, 2, \delta_2) = -1$ if $c_1 = 2$.

Denote by

$$b_{\rho,\psi,\epsilon}$$

[7] See Corollary 4.2 of [MW06] for more information in that direction.

[8] Mœglin and Arthur definitions of $\pi(\psi, \epsilon)$ are not the same, but they are simply related (see [Xu17a]). We will in this paper follow Mœglin definition of $\pi(\psi, \epsilon)$.

the maximal positive integer (if exists) such that ϵ on $\{(\rho, c, \delta_c) \in \text{Jord}(\psi); c \leq b_{\rho,\psi,\epsilon}\}$ is cuspidal subset of $\text{Jord}(\psi)$. If there is no integer as above, we take $b_{\rho,\psi,\epsilon} = -1$ if elements of $\text{Jord}_\rho(\psi_d)$ are odd, and $b_{\rho,\psi,\epsilon} = 0$ if elements of $\text{Jord}_\rho(\psi_d)$ are even. Further, let

$$a_{\rho,\psi,\epsilon}$$

be the minimum of the set $\{c; (\rho, c, \delta_c) \in \text{Jord}(\psi), c > b_{\rho,\psi,\epsilon}\}$ if the above set is non-empty. Otherwise, put $a_{\rho,\psi,\epsilon} = \infty$. Note that $a_{\rho,\psi,\epsilon} \geq 3$ if $\text{Jord}_\rho(\psi_d) \subseteq 1 + 2\mathbb{Z}$. Since always $a_{\rho,\psi,\epsilon} \geq b_{\rho,\psi,\epsilon} + 2$, we have the following definition:

Definition 3.5. *If $a_{\rho,\psi,\epsilon} = b_{\rho,\psi,\epsilon} + 2$, then we say that we are in the boundary case.*

We will use in the paper Mœglin construction of A-packets, but we will not recall the construction. We will recall only of the following simple step which we will use most often:

3.9. **Simple reduction step: case of $a_{\rho,\psi,\epsilon} > b_{\rho,\psi,\epsilon} + 2$ or $b_{\rho,\psi,\epsilon} = 0$** ([Mœg06, section 2.4, 1] or [Xu17a, Definition 6.3, (2)])**.** We consider two possibilities.

If $a_{\rho,\psi,\epsilon} > 2$, then the pair (ψ', ϵ') is is obtained from (ψ, ϵ) deforming $(\rho, a_{\rho,\psi,\epsilon}, \delta_{a_{\rho,\psi,\epsilon}})$ to $(\rho, a_{\rho,\psi,\epsilon} - 2, \delta_{a_{\rho,\psi,\epsilon}})$. If $a_{\rho,\psi,\epsilon} = 2$, then the pair (ψ', ϵ') is defined by deleting $(\rho, a_{\rho,\psi,\epsilon}, \delta_{a_{\rho,\psi,\epsilon}})$ from $\text{Jord}_\rho(\psi)$, and taking ϵ' to be the restriction of ϵ.

Definition 3.6. *If $a_{\rho,\psi,\epsilon} > b_{\rho,\psi,\epsilon} + 2$ or $b_{\rho,\psi,\epsilon} = 0$, one defines*

$$\pi(\psi, \epsilon) \underset{u.i.sub.}{\hookrightarrow} \nu^{\delta_{a_{\rho,\psi,\epsilon}} \frac{a_{\rho,\psi,\epsilon}-1}{2}} \rho \rtimes \pi(\psi', \epsilon')$$

to be a unique irreducible subrepresentation of the right hand side.

3.10. **Irreducible square integrable representations.** These representations of S_n decompose into disjoint union $\bigsqcup \Pi_\Psi$ when ψ runs over all tempered discrete A-parameters of S_n. For any character ψ of the component group, $\pi(\psi, \epsilon)$ is an irreducible representation. In this situation one usually works with Weil-Deligne group (and drops b's which are always one in this case). Then we are in the case of local Langlands correspondence for square integrable representations.

In [MT02] is completed classification of irreducible square integral representations of groups S_n modulo cuspidal data. To an irreducible square integral representation is attached an admissible triples $(\text{Jord}(\pi), \epsilon_\pi, \pi_{cusp})$ consisting of Jordan blocks, partially defined function and partial cuspidal support of π. Such triples classify irreducible square integral representations (see [MT02] for details). Then $\text{Jord}(\pi) = \text{Jord}(\psi)$ if and only if $\pi \in \Pi_\psi$ ([Mœg11, Theorem1.3.1] or Theorem 10.1 of [Xu17b]). Further, ϵ_π is a restriction of the character of the component group of ψ which is attached to π by Arthur (Theorem 10.1 of [Xu17b], see also Proposition 8.1 there).

3.11. **A consequence of involution.** Let (ψ, ϵ) be a pair of $\psi \in \Psi$ and a character ϵ of the component group of ψ. One defines a pair

$$(\psi^{\mathbf{t}}, \epsilon^{\mathbf{t}})$$

of $\psi^{\mathbf{t}} \in \Psi$ and a character ϵ of the component group of $\psi^{\mathbf{t}}$ by the requirement that $\mathrm{Jord}(\psi^{\mathbf{t}})$ consists of all (ρ, b, a) when (ρ, a, b) runs over $\mathrm{Jord}(\psi)$, and $\epsilon^{\mathbf{t}}$ is defined using natural bijection between $\mathrm{Jord}(\psi^{\mathbf{t}})$ and $\mathrm{Jord}(\psi)$.

Let $\psi \in \Psi_{\text{ele.}} \cap \Psi_{\text{d.d.r.}}$. Obviously $\psi_d = (\psi^{\mathbf{t}})_d$. Using this, we identify characters of component groups of ψ and ψ_d. Therefore, if $\psi', \psi'' \in \Psi_{\text{ele.}} \cap \Psi_{\text{d.d.r.}}$ such that $(\psi')_d = (\psi'')_d$, their characters of component groups can be identified in a natural way.

C. Mœglin defined in [Mœg06] involutions on irreducible representations, which generalises Aubert involution, and showed that each element $\pi(\psi, \epsilon)$ of elementary discrete A-packet can be obtained from square integrable representation corresponding to ϵ in the L-packet of ψ_d applying the involution (see [Mœg06, Theorem 5]). A consequence of it for classical Aubert involution is that

$$\pi(\psi, \epsilon)^{\mathbf{t}} = \pi(\psi^{\mathbf{t}}, \epsilon) \quad \text{for} \quad \psi \in \Psi_{\text{ele.}} \cap \Psi_{\text{d.d.r.}} \tag{3.4}$$

([Mœg06, Theorem 5], Theorem 6.10 of [Xu17a]).

3.12. **Cuspidal representations in elementary discrete A-packets.** Let $\psi \in \Psi_{\text{ele.}} \cap \Psi_{\text{d.d.r.}}$. From our observations in 3.7 (or in 3.11) follows that the cardinality of the L-packet of ψ_d is equal to the cardinality of the A-packet of ψ.

Partial Aubert involutions (defined in section 4 of [Mœg06]) carry irreducible non-cuspidal representations to non-cuspidal ones, and they cannot carry non-cuspidal to cuspidal ones. This implies that for an irreducible cuspidal representation σ of a classical group, σ belongs to the L-packet of ψ_d if and only if it belongs to the A-packet of $\psi \in \Pi_\psi$. Moreover, they determine the same character of the component groups (after we identify them).

3.13. **Orders on Jordan blocks.** Let $\psi \in \Psi$. Any total order $>_\psi$ on $\mathrm{Jord}_\rho(\psi)$ which satisfies for any $(\rho, a, b), (\rho, a', b') \in \mathrm{Jord}_\rho(\psi)$ the following condition

$$(\mathcal{P}) \qquad a + b > a' + b', \quad |a - b| > |a' - b'|, \quad \zeta_{a,b} = \zeta_{a',b'} \quad \Longrightarrow (\rho, a, b) >_\psi (\rho, a', b'),$$

will be called **admissible order**.

We will always fix some total order $>'$ on the set $\{\rho; \mathrm{Jord}_\rho(\psi) \neq \emptyset\}$, and assume for each $(\rho, a, b), (\rho', a', b') \in \mathrm{Jord}_\rho(\psi)$ that if $\rho >' \rho'$, then $(\rho, a, b) >_\psi (\rho', a', b')$[9]. Therefore, for describing admissible order on $\mathrm{Jord}_\rho(\psi)$, it is enough to describe it on each $\mathrm{Jord}_\rho(\psi)$.

Suppose $\psi \in \Psi_{\text{d.d.r.}}$. Then in $(\mathcal{P})$ holds $a + b > a' + b' \iff |a - b| > |a' - b'|$ (and the condition $|a - b| > |a' - b'|$ is redundant in $(\mathcal{P})$ in this case). Actually, in this case we can find an admissible order $>_\psi$ satisfying

$$(\rho, a, b) >_\psi (\rho, a', b') \iff a + b > a' + b'.$$

Such an order will be called **natural**.

One says that $\psi_\gg \in \Psi_{\text{d.d.r.}}$ with a natural order $>_{\psi_\gg}$ **dominates** $\psi \in \Psi_{\text{g.p.}}$ **with respect to an admissible order** $>_\psi$ on $\mathrm{Jord}_\rho(\psi)$ if $\{\rho; \mathrm{Jord}_\rho(\psi_\gg) \neq \emptyset\} = \{\rho; \mathrm{Jord}_\rho(\psi) \neq \emptyset\}$, and

[9] We do not need to assume this, but it simplifies descriptions of admissible orders.

if for each ρ from the last set we have order preserving bijection $(a_\gg, b_\gg) \mapsto (a, b)$ from $\mathrm{Jord}_\rho(\psi_\gg)$ onto $\mathrm{Jord}_\rho(\psi)$ which satisfies

$$A_\gg - A = B_\gg - B \geq 0 \quad \text{and} \quad \zeta_{a,b} = \zeta_{a_\gg, b_\gg}.$$

Define the function $\boldsymbol{T} : \mathrm{Jord}(\psi) \to \mathbb{Z}_{\geq 0}$ by $\boldsymbol{T}(\rho, a, b) = A_\gg - A = B_\gg - B$. Observe that

$$(\rho, a_\gg, b_\gg) = \big(\rho, \quad a + (1 + \zeta_{(\rho,a,b)})\boldsymbol{T}(\rho, a, b), \quad b + (1 - \zeta_{(\rho,a,b)})\boldsymbol{T}(\rho, a, b)\big)$$

(the bigger of numbers a and b is increased for $2\,\boldsymbol{T}(\rho, a, b)$, and the smaller is unchanged; in the case $a = b$, we increase the first a or the second a for $2\,\boldsymbol{T}(\rho, a, a)$ and leave the other one unchanged, depending if we took $\zeta_{a,a}$ to be 1 or -1).

3.13.1. *Orders on elementary packets.* Let $\psi \in \Psi_{\text{ele.}}$. Any total order $>$ satisfying for any ρ and any $(a, b), (a', b') \in \mathrm{Jord}_\rho(\psi)$ the condition $a + b > a' + b' \implies (a, b) > (a', b')$ will be called standard. Any standard order is obviously admissible. Let $\psi \in \Psi_{\text{ele.}} \cap \Psi_{\text{d.d.r.}}$. Since we have fixed total order on $\{\rho; \mathrm{Jord}_\rho(\psi) \neq \emptyset\}$, there is only one natural order on $\mathrm{Jord}_\rho(\psi)$ (the standard one).

Let $\psi, \psi' \in \Psi_{\text{ele.}}$ and assume $\psi' \in \Psi_{\text{d.d.r.}}$. Suppose that we have a bijection $\varphi : \mathrm{Jord}(\psi') \to \mathrm{Jord}(\psi)$ which for any ρ induces bijection $(\rho, a', b') \mapsto (\rho, a, b)$ from $\mathrm{Jord}_\rho(\psi')$ onto $\mathrm{Jord}_\rho(\psi)$ which satisfies

$$\max(k', l') > \max(k, l) \implies \max \varphi(k', l') \geq \max \varphi(k, l).$$

Then any such bijection will be called **standard**.

3.14. **Cuspidal representations and reducibility exponent.** Fix an irreducible cuspidal selfcontragredient representation ρ of a general linear group and an irreducible cuspidal representation σ of a classical group. The $\nu^{\alpha_{\rho,\sigma}}\rho \rtimes \sigma$ reduces for unique $\alpha_{\rho,\sigma} \geq$ (this defines $\alpha_{\rho,\sigma}$). Further, $\alpha_{\rho,\sigma} \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, and we denote $\alpha_{\rho,\sigma}$ simply by

$$\alpha.$$

Denote by $(\phi_\sigma, \epsilon_\sigma)$ a pair of an admissible homomorphism of the Weil-Deligne group and a character of the component group of ϕ_σ, corresponding to σ under the local Langlands correspondence. Let

$$\psi_\sigma := \phi_\sigma \otimes E_1,$$

and lift ϵ_σ to a character of the component group of ψ in natural way, and denote it again by ϵ_σ. Then $a_{\rho', \psi_\sigma, \epsilon_\sigma} = \infty$ for all selfcontragredient $\rho' \in \mathcal{C}$. Further:

(1) Suppose $\alpha \geq 1$. This is equivalent to $\mathrm{Jord}_\rho(\phi) \neq \emptyset$. Then $\alpha = \frac{\max(\mathrm{Jord}_\rho((\psi_\sigma)_d))+1}{2}$.
(2) Suppose $\alpha < 1$. Then $\alpha = 0$ (resp. $\alpha = \frac{1}{2}$) if and only if $\nu^{\frac{1}{2}}\rho \rtimes 1_{S_0}$[10] is irreducible (resp. reducible).

4. Case of reducibility > 1

In this section we assume that ρ, σ and α are as in 3.14, and we assume that $\alpha > 1$.

[10] 1_G denotes the trivial (one-dimensional) representation of a group G.

4.1. **Involution.** The proof of the following proposition and other claims in this paper that compute Aubert involution, is based on the basic idea of [Jan18] (another possibility is to apply [AM20]).

Proposition 4.1. *Let* $\alpha \geq \frac{3}{2}$ *and* $m, n \in \mathbb{Z}_{\geq 0}$*. Denote*

$$\pi_{m,n} := L([\alpha - 1, \alpha + m]^{\mathbf{t}}; \delta([\alpha, \alpha + n]; \sigma)).$$

Then

$$\pi_{n,m}^{\mathbf{t}} = \pi_{m,n}.$$

Proof. We will first prove by induction that $\pi_{0,n}^{\mathbf{t}} = \pi_{n,0}$ for $n \geq 0$. For $n = 0$, $\pi_{0,0}^{\mathbf{t}} = \pi_{0,0}$ by [Tad20, Proposition 4.6, (2)]. Suppose that the claim holds for some $n \geq 0$. Observe that

$$\begin{aligned} (4.1)\quad \pi_{0,n+1} &\hookrightarrow [-\alpha] \times [-(\alpha-1)] \rtimes \delta([\alpha, \alpha+n+1]; \sigma) \\ &\hookrightarrow [-\alpha] \times [-(\alpha-1)] \times [\alpha+n+1] \rtimes \delta([\alpha, \alpha+n]; \sigma) \\ &\cong [\alpha+n+1] \times [-\alpha] \times [-(\alpha-1)] \rtimes \delta([\alpha, \alpha+n]; \sigma). \end{aligned}$$

Further, observe that

$$[\alpha+n+1] \rtimes \pi_{0,n} \hookrightarrow [\alpha+n+1] \times [-\alpha] \times [-(\alpha-1)] \rtimes \delta([\alpha, \alpha+n]; \sigma).$$

One directly checks that the right hand side has unique irreducible subrepresentation, which implies $\pi_{0,n+1} \hookrightarrow [\alpha+n+1] \rtimes \pi_{0,n}$ From this easily follows $\mu^*_{\{[\alpha+n+1]\}}(\pi_{0,n+1}) = [\alpha+n+1] \otimes \pi_{0,n}$ (see Definition 2.2 for notation). Now

$$\pi_{0,n+1}^{\mathbf{t}} \hookrightarrow [-(\alpha+n+1)] \rtimes \pi_{0,n}^{\mathbf{t}} = [-(\alpha+n+1)] \rtimes \pi_{n,0}$$

(see (iii) of Remark 2.4). This directly implies $\pi_{0,n+1}^{\mathbf{t}} = \pi_{n+1,0}$.

We prove now the general formula $\pi_{m,n}^{\mathbf{t}} = \pi_{n,m}$ for $m \geq n$ by induction with respect to n, which will complete the proof of the proposition. We have proved the claim for $n = 0$. Suppose $0 \leq n < m$ and $\pi_{m,n}^{\mathbf{t}} = \pi_{n,m}$. Observe that

$$\begin{aligned} (4.2)\quad \pi_{m,n+1} &\hookrightarrow L([-(\alpha+m), -(\alpha-1)]^{\mathbf{t}}) \rtimes \delta([\alpha, \alpha+n+1]; \sigma) \\ &\hookrightarrow L([-(\alpha+m), -(\alpha-1)]^{\mathbf{t}}) \times [\alpha+n+1] \rtimes \delta([\alpha, \alpha+n]; \sigma) \\ &\cong [\alpha+n+1] \times L([-(\alpha+m), -(\alpha-1)]^{\mathbf{t}}) \rtimes \delta([\alpha, \alpha+n]; \sigma). \end{aligned}$$

Further, obviously

$$[\alpha+n+1] \rtimes \pi_{m,n} \hookrightarrow [\alpha+n+1] \times L([-(\alpha+m), -(\alpha-1)]^{\mathbf{t}}) \rtimes \delta([\alpha, \alpha+n]; \sigma).$$

We will show that the representation on the right hand side has unique irreducible subrepresentation by showing that $\gamma := [\alpha+n+1] \times L([-(\alpha+m), -(\alpha-1)]^{\mathbf{t}}) \otimes \delta([\alpha, \alpha+n]; \sigma)$ has multiplicity one in the whole Jacquet module (using formula (2.1)). Suppose first that $n+1 < m$. Then (2.4) (actually (2.5) is enough) tells that we can get the term $[\alpha+n+1]$ on the left hand side of $\otimes$ only from $M^*([\alpha+n+1])$ and term $[\alpha+n+1] \otimes 1$ there. This directly implies the multiplicity one.

Suppose now that $n+1 = m$, and suppose that $[\alpha+n+1]$ on the left hand side of $\otimes$ is not coming from $M^*([\alpha+n+1])$ (and term $[\alpha+n+1] \otimes 1$ there). Obviously, it cannot come from $\mu^*(\delta([\alpha,\alpha+n];\sigma))$. Therefore it comes from the M^* of the second term. Formula (2.4) ((2.5) is enough) implies that we must take from it the term $L([\alpha-1,\alpha+m]^{\mathbf{t}})\otimes\dots$. But then we cannot get γ for a subquotient (consider cuspidal support). Therefore, we have proved multiplicity one also in this case.

This implies $\pi_{m,n+1}\hookrightarrow[\alpha+n+1]\rtimes\pi_{m,n}$. Observe that $\pi_{m,n}\leq L([\alpha-1,\alpha+m]^{\mathbf{t}})\rtimes\delta([\alpha,\alpha+n];\sigma))$, which implies

$$s_{GL}(\pi_{m,n})\leq M^*_{GL}(L([\alpha-1,\alpha+m]^{\mathbf{t}}))\times\delta([\alpha,\alpha+n])\otimes\sigma.$$

From this and formula (2.5) follows that $\mu^*_{\{[\alpha+n+1]\}}(\pi_{m,n+1})=[\alpha+n+1]\otimes\pi_{m,n}$. Therefore $\pi^{\mathbf{t}}_{m,n+1}\hookrightarrow[-(\alpha+n+1)]\otimes\pi_{n,m}$, which implies $\pi^{\mathbf{t}}_{m,n+1}=\pi_{n+1,m}$. □

4.2. **Notation for A-parameters in the case $\alpha>1$.** The assumption $\alpha>1$ and 3.14 imply that $\psi_\sigma=\psi_-\oplus E^\rho_{2\alpha-3,1}\oplus E^\rho_{2\alpha-1,1}$ for some $\psi_-\in\Psi_{\text{ele.}}\cap\Psi_{\text{d.d.r.}}$ (this defines ψ_-). Denote in the sequel

$$\psi_{k,l}:=\psi_-\oplus E^\rho_{k,1}\oplus E^\rho_{1,l},$$

where $k,l\geq0$ will be always chosen to be the same parity as $2\alpha-1$ (and therefore $\psi_{k,l}\in\Psi_{\text{g.p.}}$, which implies $\psi_{k,l}\in\Psi_{\text{ele.}}$). In the sequel, we will always chose k and l such that $\psi_{k,l}$ is a multiplicity one representation. Clearly, $(\psi_{2\alpha-1,2\alpha-3})_d=(\psi_\sigma)_d$.

Further, denote by $\epsilon_{k,l}$ the character of the component group of $\psi_{k,l}$ which extends ϵ_σ on ψ_-, and which satisfies

$$\epsilon_{k,l}(E^\rho_{k,1})=\epsilon_\sigma(E^\rho_{2\alpha-1,1}),$$
$$\epsilon_{k,l}(E^\rho_{1,l})=\epsilon_\sigma(E^\rho_{2\alpha-3,1})\text{ if }\alpha>\tfrac{3}{2},\text{ and }\epsilon_{k,l}(E^\rho_{1,l})=1\text{ if }\alpha=\tfrac{3}{2}.$$

We will work all the time in this section with pairs $(\psi_{k,l},\epsilon_{k,l})$. Therefore, to shorten notation in the sequel, we denote such pair by $(\psi,\epsilon)_{k,l}$.

Theorem 4.2. *Let $\alpha\geq\frac{3}{2}$ and $m,n\in\mathbb{Z}_{\geq0}$. Using the notation for A-parameters introduced above, we have*

(1) $L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+n];\sigma))\in\Pi_{\psi_{2\alpha+1+2n,2\alpha+1+2m}}$. In particular, $L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+n];\sigma))$ is unitarizable.

(2) For $m\neq n$ we have

$$\pi((\psi,\epsilon)_{2\alpha+1+2n,2\alpha+1+2m})=L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+n];\sigma)).\tag{4.3}$$

Proof. The proof goes through several steps.

4.2.1. *Case of $m=-2$.* Using induction, we first prove the (well-known) simple fact that

$$\pi((\psi,\epsilon)_{2\alpha+1+2n,2\alpha-3}))=\delta([\alpha,\alpha+n];\sigma),\quad n\geq-1.$$

Observe first that $\sigma=\pi((\psi,\epsilon)_{2\alpha-1,\,2\alpha-3})$. Therefore, we have a basis of induction. Suppose $n\geq0$ and that the above formula holds for $n-1$. Consider now $(\psi,\epsilon)_{2\alpha+1+2n,2\alpha-3}$. Then

$$b_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha-3}}=2\alpha-3,\quad a_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha-3}}=2\alpha+1+2n,$$
$$\delta_{a_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha-3}}}=1.$$

Now by 3.9 we know that $\pi((\psi,\epsilon)_{2\alpha+1+2n,2\alpha-3})$ is a unique irreducible subrepresentation of

$$[(\delta_{(\psi,\epsilon)_{2\alpha+1+2n,2\alpha-3}})^{\frac{(2\alpha+1+2n)-1}{2}}] \rtimes \pi((\psi,\epsilon)_{2\alpha-1+2(n-1),2\alpha-3}) = [\alpha+n] \rtimes \delta([\alpha,\alpha+n-1];\sigma),$$

which easily implies $\pi((\psi,\epsilon)_{2\alpha+1+2n,2\alpha-3}) = \delta([\alpha,\alpha+n];\sigma)$, and completes the proof of the inductive step.

4.2.2. *Proof of* (4.3) *for* $-2 \leq m < n$. We have proved above that the claim holds for $m = -2$. We now fix some $m \geq -1$ (together with $n > m$), and assume that the formula (4.3) holds for $m-1$. We will prove by induction that it holds for m. Now

$$b_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha+1+2m}} = 2\alpha-5, \quad a_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha+1+2m}} = 2\alpha+1+2m,$$
$$\delta_{a_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha+1+2m}}} = -1,$$

with exception that for $\alpha = \frac{3}{2}$ we take $b_{(\psi,\epsilon)_{2\alpha+1+2n,2\alpha-1}} = 0$. By 3.9 we know that $\pi((\psi,\epsilon)_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha+1+2m}})$ is a unique irreducible subrepresentation of

$$\begin{aligned}(4.4)\quad [(\delta_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha+1+2m}})^{\frac{(2\alpha+1+2m)-1}{2}}] &\rtimes \pi((\psi,\epsilon)_{(\psi,\epsilon)_{2\alpha+1+2n,2\alpha-1+2m}}) \\ &= [-(\alpha+m)] \rtimes L([\alpha-1,\alpha+m-1]^{\mathbf{t}};\delta([\alpha,\alpha+n];\sigma)),\end{aligned}$$

which implies the formula (4.3), and complete the proof of the inductive step.

4.2.3. *Reverse setting,* $n = 0$. We repeat now the previous construction in reversed setting. Denote by $\epsilon'_{k,l}$ the character of the component group of $\psi_{k,l}$ which extends ϵ_σ on ψ_-, and which satisfies

$$\epsilon'_{k,l}(E^\rho_{k,1}) = \epsilon_\sigma(E^\rho_{2\alpha-3,1}) \text{ if } \alpha > \tfrac{3}{2}, \text{ and } \epsilon'_{k,l}(E^\rho_{1,l}) = 1 \text{ if } \alpha = \tfrac{3}{2},$$
$$\epsilon'_{k,l}(E^\rho_{1,l}) = \epsilon_\sigma(E^\rho_{2\alpha-1,1}).$$

We claim that

$$(4.5)\qquad \pi((\psi,\epsilon')_{2\alpha-3,2\alpha+1+2m}) = L([\alpha,\alpha+m]^{\mathbf{t}};\sigma), \quad m \geq -1.$$

The proof goes by induction. Observe that $\sigma = \pi((\psi,\epsilon')_{2\alpha-3,2\alpha-1})$, which is the basis of the induction. Suppose $m \geq 0$ and that the formula holds for $m-1$. Then

$$b_{\rho,(\psi,\epsilon')_{2\alpha-3,2\alpha+1}} = 2\alpha-3, \quad a_{\rho,(\psi,\epsilon')_{\rho,2\alpha-3,2\alpha+1+2m}} = 2\alpha+1+2m,$$
$$\delta_{a_{\rho,(\psi,\epsilon')_{2\alpha-3,2\alpha+1+2m}}} = -1.$$

By 3.9 we know that $\pi((\psi,\epsilon')_{2\alpha-3,2\alpha+1+2m})$ is a unique irreducible subrepresentation of

$$[(\delta_{(\psi,\epsilon')_{2\alpha-3,2\alpha+1+2m}})^{\frac{(2\alpha+1+2m)-1}{2}}] \rtimes \pi((\psi,\epsilon')_{2\alpha-3,2\alpha-1+2(m-1)}) = [-\alpha-m] \rtimes L([\alpha,\alpha+m-1]^{\mathbf{t}};\sigma),$$

which directly implies the formula 4.5. This completes the proof of the inductive step.

4.2.4. *Reverse setting, $-1 \leq n < m$.* With $\epsilon'_{k,l}$ introduced in 4.2.3, we now prove the formula

$$\pi((\psi,\epsilon')_{2\alpha+1+2n,2\alpha+1+2m}) = L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+n];\sigma)), \quad m > n \geq -1 \tag{4.6}$$

by induction with respect to n. To have basis of induction for $n=-1$, we need to consider $(\psi,\epsilon')_{2\alpha-1,2\alpha+1+2m}$. Then

$$b_{\rho,(\psi,\epsilon')_{2\alpha-1,2\alpha+1+2m}} = 2\alpha-5, \quad a_{\rho,(\psi,\epsilon')_{2\alpha-1,2\alpha+1+2m}} = 2\alpha-1,$$
$$\delta_{a_{\rho(\psi,\epsilon')_{2\alpha-1,2\alpha+1+2m}}} = 1,$$

with exception that for $\alpha = \frac{3}{2}$ we take $b_{(\psi,\epsilon')_{2\alpha-1,2\alpha+1+2m}} = 0$. By 3.9 we know that $\pi((\psi,\epsilon')_{2\alpha-1,2\alpha+1+2m})$ is a unique irreducible subrepresentation of

$$[(\delta_{(\psi,\epsilon')_{2\alpha-1,2\alpha+1+2m}})\tfrac{(2\alpha-1)-1}{2}] \rtimes \pi((\psi,\epsilon')_{2\alpha-3,2\alpha+1+2m}) = [\alpha-1] \rtimes L([\alpha,\alpha+m]^{\mathbf{t}};\sigma)$$

(the above equality follows from 4.2.3). Therefore

$$\pi((\psi,\epsilon')_{2\alpha-1,2\alpha+1+2m}) \hookrightarrow [\alpha-1] \times L([-(\alpha+m),-\alpha]^{\mathbf{t}}) \rtimes \sigma \cong L([-(\alpha+m),-\alpha]^{\mathbf{t}}) \times [\alpha-1] \rtimes \sigma$$
$$\cong L([-(\alpha+m),-\alpha]^{\mathbf{t}}) \times [-(\alpha-1)] \rtimes \sigma.$$

This obviously implies (4.6) for $n=-1$.

We go now to the inductive step. Suppose $n \geq 0$ and that the formula (4.6) holds for $n-1$. Here

$$b_{\rho,(\psi,\epsilon')_{2\alpha+1+2n,2\alpha+1+2m}} = 2\alpha-5, \quad a_{\rho,(\psi,\epsilon')_{2\alpha+1+2n,2\alpha+1+2m}} = 2\alpha+1+2n,$$
$$\delta_{a_{\rho,(\psi,\epsilon')_{2\alpha+1+2n,2\alpha+1+2m}}} = 1.$$

Then $\pi((\psi,\epsilon')_{2\alpha+1+2n,2\alpha+1+2m})$ is a unique irreducible subrepresentation of

$$[(\delta_{\rho,(\psi,\epsilon')_{2\alpha+1+2n,2\alpha+1+2m}})\tfrac{(2\alpha+1+2n)-1}{2}] \rtimes \pi((\psi,\epsilon')_{2\alpha+1+2(n-1),2\alpha+1+2m})$$
$$= [\alpha+n] \rtimes L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+n-1];\sigma)).$$

The last representation embeds into

$$\Gamma := [\alpha+n] \times L([-(\alpha+m),-(\alpha-1)]^{\mathbf{t}}) \rtimes \delta([\alpha,\alpha+n-1];\sigma) \tag{4.7}$$
$$\cong L([-(\alpha+m),-(\alpha-1)]^{\mathbf{t}}) \times [\alpha+n] \rtimes \delta([\alpha,\alpha+n-1];\sigma)$$

We will show that the multiplicity of

$$\gamma := [\alpha+n] \times L([-(\alpha+m),-(\alpha-1)]^{\mathbf{t}}) \otimes \delta([\alpha,\alpha+n-1];\sigma)$$

in $\mu^*(\Gamma)$ is one. To get γ as a subquotient, from the formula

$$\mu^*(\Gamma) = M^*([\alpha+n]) \times M^*(L([-(\alpha+m),-(\alpha-1)]^{\mathbf{t}})) \rtimes \mu^*(\delta([\alpha,\alpha+n-1];\sigma))$$

we see that from the last term on the right hand side we must take $1 \otimes \delta([\alpha,\alpha+n-1];\sigma)$ (consider cuspidal supports). If we do not take $[\alpha+n] \otimes 1$ from $M^*([\alpha+n])$, the formula (2.5) implies that we would have on the left hand side of $\otimes$ positive exponents different from $\alpha+n$. Therefore, we must take $[\alpha+n] \otimes 1$, which further implies that from

$M^*(L([-(\alpha+m),-(\alpha-1)]^{\mathbf{t}}))$ we must take $L([-(\alpha+m),-(\alpha-1)]^{\mathbf{t}})\otimes 1$. This implies the multiplicity one.

Therefore Γ has unique irreducible subrepresentation. Since $L([-(\alpha+m),-(\alpha-1)]^{\mathbf{t}})\rtimes \delta([\alpha,\alpha+n];\sigma)\gamma\Gamma$, we get that (4.6) holds. This completes the proof of the inductive step.

4.2.5. *Case* $m=n\geq 0$. Denote $\psi_{\gg}:=\psi_{2\alpha+3+2m,2\alpha+1+2m}$, $\psi:=\psi_{2\alpha+1+2m,2\alpha+1+2m}$. We fix on $\mathrm{Jord}_\rho(\psi_{\gg})$ any standard order, and denote it by $>_{\psi_{\gg}}$. Then this is a natural order. Define a bijection $\mathrm{Jord}_\rho(\psi_{\gg})\to\mathrm{Jord}_\rho(\psi)$ which carries

$$\varphi:(\rho,2\alpha+3+2m,1)\mapsto(\rho,2\alpha+1+2m,1),$$

and on the remaining elements it is identity. Using bijection φ, we define total order $>_\psi$ on $\mathrm{Jord}_\rho(\psi)$ (i.e. $\varphi(u)>_\psi\varphi(v)\iff u>_{\psi_{\gg}}v$). This is an admissible order on $\mathrm{Jord}_\rho(\psi)$ and φ preserves the order (by definition of $>_\psi$). In this way $\mathrm{Jord}(\psi_{\gg})$ dominates $\mathrm{Jord}(\psi)$ and by [Mœg11, 3.1.2] or [Xu17a, section 8], we can get all elements of Π_ψ from elements of $\Pi_{\psi_{\gg}}$ applying $\mathrm{Jac}_{\alpha+m+1}$ (each application of the operator $\mathrm{Jac}_{\alpha+m+1}$ will result with either irreducible representation or 0). Observe that

$$\begin{aligned}(4.8)\quad L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+m+1];\sigma))&\hookrightarrow\\ &L([-(\alpha+m),-(\alpha-1)]^{\mathbf{t}})\rtimes\delta([\alpha,\alpha+m+1];\sigma)\\ &\hookrightarrow L([-(\alpha+m),-(\alpha-1)]^{\mathbf{t}})\times[\alpha+m+1]\rtimes\delta([\alpha,\alpha+m];\sigma)\\ &\cong[\alpha+m+1]\times L([-(\alpha+m),-(\alpha-1)]^{\mathbf{t}})\rtimes\delta([\alpha,\alpha+m];\sigma).\end{aligned}$$

Obviously

$$\begin{aligned}(4.9)\quad [\alpha+m+1]\rtimes L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+m];\sigma))&\\ \hookrightarrow[\alpha+m+1]\times L([-(\alpha+m),-(\alpha-1)]^{\mathbf{t}})&\rtimes\delta([\alpha,\alpha+m];\sigma).\end{aligned}$$

One directly sees that the last representation has unique irreducible subrepresentation (showing that the multiplicity of $[\alpha+m+1]\otimes L([-(\alpha+m),-(\alpha-1)]^{\mathbf{t}})\otimes\delta([\alpha,\alpha+m];\sigma)$ in the Jacquet module is one). This implies

$$\begin{aligned}(4.10)\quad L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+m+1];\sigma))&\\ \hookrightarrow[\alpha+m+1]\rtimes L([\alpha-1,\alpha+m]^{\mathbf{t}};&\delta([\alpha,\alpha+m];\sigma)).\end{aligned}$$

Now Frobenius reciprocity implies that

$$\mathrm{Jac}_{\alpha+m+1}(L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+m+1];\sigma)))=L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+m];\sigma)),$$

and therefore, $L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+m];\sigma))$ is in the A-packet of $\psi_{2\alpha+1+2m,2\alpha+1+2m}$.

4.2.6. *Case* $0\leq m<n$. Observe that if we take instead of ψ_- in 4.2 any $\psi'\in\Psi_{\text{ele.}}\cap\Psi_{\text{d.d.r.}}$ such that $(\psi')_d=(\psi_-)_d$, and use ψ' (instead of ψ_-) to define $\psi_{k,l}$, $\epsilon_{k,l}$ and $\psi'_{k,l}$, we get exactly the same results as we have obtained in the proof up to now.

Assume below $0 \leq m < n$. Now in 4.2.2 put $\psi' = (\psi_-)^{\mathbf{t}}$ and denote objects that correspond to $\psi_{k,l}$ and $\epsilon_{k,l}$ for this ψ' by $\psi''_{k,l}$ and $\epsilon''_{k,l}$ (recall $\psi''_{2\alpha+1+2n,2\alpha+1+2m} = L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+n];\sigma)))$. Then

$$(\psi''_{2\alpha+1+2n,2\alpha+1+2m})^{\mathbf{t}} = \psi_{2\alpha+1+2m,2\alpha+1+2n},$$

and $\epsilon''_{2\alpha+1+2n,2\alpha+1+2m}$ gives the same diagonal restriction as $\psi'_{2\alpha+1+2m,2\alpha+1+2n}$ (defined in 4.2.4). Now 3.4 and Proposition 4.1 imply

$$\begin{aligned}(4.11)\quad \pi((\psi,\epsilon)_{2\alpha+1+2m,2\alpha+1+2n}) &= \pi((\psi''_{2\alpha+1+2n,2\alpha+1+2m})^{\mathbf{t}},\epsilon''_{2\alpha+1+2n,2\alpha+1+2m}) \\ &= \pi((\psi,\epsilon)_{2\alpha+1+2n,2\alpha+1+2m})^{\mathbf{t}} = L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+n];\sigma))^{\mathbf{t}} \\ &= L([\alpha-1,\alpha+n]^{\mathbf{t}};\delta([\alpha,\alpha+m];\sigma)).\end{aligned}$$

□

Note that in the proof of the above theorem we have also proved what happens with few additional cases for $m \geq -2$ and $n \geq -1$. We comment these mostly well known cases briefly in the following

Corollary 4.3. *Let $m \geq -2$ and $n \geq -1$.*

1. *Representation $\delta([\alpha,\alpha+n];\sigma)$ (resp. $L([\alpha,\alpha+n]^{\mathbf{t}};\sigma)$) is in Π_ψ for $\psi = \psi_- \oplus E^\rho_{2\alpha-3,1} \oplus E^\rho_{2\alpha+1+2n,1}$ (resp. $\psi = \psi_- \oplus E^\rho_{1,2\alpha-3} \oplus E^\rho_{1,2\alpha+1+2n}$).*
2. *For $m,n \geq -1$, representations $L([\alpha-1];\delta([\alpha,\alpha+n];\sigma))$ and $L([\alpha-1,\alpha+m]^{\mathbf{t}};\sigma)$ are in A-packets. These representations are at the ends of of complementary series if $n \geq 0$ (resp $m \geq 0$).*
3. *$\delta([\alpha,\alpha+n];\sigma)^{\mathbf{t}} = L([\alpha,\alpha+n]^{\mathbf{t}};\sigma)$.*
4. *$L([\alpha-1];\delta([\alpha,\alpha+n];\sigma))^{\mathbf{t}} = L([\alpha-1,\alpha+n]^{\mathbf{t}};\sigma)$.*

Proof. The first three claims are proved in the previous theorem. It remains to consider only (4). This is very simple to prove by methods used is the proof of Proposition 4.1, and therefore we omit it. □

5. Case of reducibility 0

In this and the following two sections we will handle remaining reducibilities, i.e. $\alpha = 0, \frac{1}{2}$ and 1, and write down A-packets and representations in them which can be considered as analogous cases for these reducibilities. It is very easy to get that they are in A-packets. We will give also some additional information about them (formulas for Aubert involution, and to which characters of the component groups they correspond in the case of discrete parameter).

In this section ρ, σ and α are as in 3.14, and we assume that $\alpha = 0$. We fix a decomposition of $\rho \rtimes \sigma$ in (2.6). We denote by ψ_σ tempered elementary discrete parameter such that $\sigma \in \Pi_{\psi_\sigma}$. Applying [Mœg11, Proposition 6.0.3] to the parameter $\psi_\sigma \oplus E^\rho_{2n+1,1} \oplus E^\rho_{1,2m+1}$ (for which we know by 3.3 that it is again an A-parameter) we get directly that if $m,n \geq 0$, then

$$L([1,m]^{\mathbf{t}};\delta([0,n]_\pm;\sigma)) \in \Pi_{\psi_\sigma \oplus E^\rho_{2n+1,1} \oplus E^\rho_{1,2m+1}}.$$

In the following theorem, we give additional information about elements of these packets in the case $m \neq n$.

5.1. **Notation for A-parameters in the case $\alpha = 0$.** Denote in this section

$$\psi := \psi_\sigma, \qquad \psi_{k,l} := E^\rho_{k,1} \oplus E^\rho_{1,l},$$

where $k, l \geq 0$ will be always chosen to be of odd parity, and denote by $\epsilon^{\pm}_{k,l}$ the character of the component group of $\psi_{k,l}$ which extends ϵ_σ on ψ, and which is on remaining two elements equal to $\pm$. Similarly as before, we denote a pair $(\psi_{k,l}, \epsilon^{\pm}_{k,l})$ by $(\psi, \epsilon^{\pm})_{k,l}$.

Theorem 5.1. *Let $n, m \geq 0$. Then*

(1) $L([1,m]^{\mathbf{t}}; \delta([0,n]_{\pm}; \sigma)) \in \Pi_{\psi_{2n+1,2m+1}}$. In particular, $L([1,m]^{\mathbf{t}}; \delta([0,n]_{\pm}; \sigma))$ are unitarizable.

(2) For $m \neq n$ we have $\pi((\psi, \epsilon^{\xi})_{2n+1,2m+1}) = L([1,m]^{\mathbf{t}}; \delta([0,n]_{sign(n-m)\xi}; \sigma))$, $\xi \in \{\pm\}$.

Proposition 5.2. *Let $\alpha = 0$ and $m, n \in \mathbb{Z}_{\geq 0}$. Denote*

$$\pi^{\pm}_{m,n} := L([1,m]^{\mathbf{t}}; \delta([0,n]_{\pm}; \sigma)).$$

Then

$$(\pi^{\pm}_{n,m})^{\mathbf{t}} = \pi^{\mp}_{m,n}.$$

Remark 5.3. *Elements of an A-packet ψ of good parity which are not discrete are obtained from some suitable discrete A-packet $\psi_{\gg}$ of a bigger group by procedure described in* [Mœg11, 3.1.2] *(see also* [Xu17a, section 8]*). Here one applies Jacquet module operators to representations $\pi(\psi, \boldsymbol{t}, \boldsymbol{\eta})$ to get elements of Π_ψ (the result can be also 0). The result can depend on an admissible order that one fixes on $\mathrm{Jord}_\rho(\psi)$. We comment below an example where one gets different results for A-packets corresponding to $m = n > 1$ for admissible order which satisfies $(\rho, m, 1) >_\psi (\rho, 1, m)$ and admissible order which satisfies $(\rho, 1, m) >_\psi (\rho, m, 1)$*[11]*. A reason is that on $\mathrm{Jord}(\psi_{\gg})$ we need to take natural order. Therefore, in the case of the first admissible order, one needs to take (for example)*

$$\mathrm{Jac}_{m+1}(L([1,m]^{\mathbf{t}}; \delta([0,m+1]_{\pm}; \sigma))) = L([1,m]^{\mathbf{t}}; \delta([0,m]_{\pm}; \sigma)),$$

while in the case of the second admissible order, one needs to take (for example)

$$\mathrm{Jac}_{-(m+1)}(L([1,m+1]^{\mathbf{t}}; \delta([0,m]_{\mp}; \sigma))) = L([1,m]^{\mathbf{t}}; \delta([0,m]_{\mp}; \sigma)).$$

Note that if we change admissible orders as above in the setting of Theorem 4.2 for the case $m = n \geq 0$, we get there the same results. The same holds for settings of Theorem 6.2.

[11] In our case $\boldsymbol{t}$ is zero function, and $\boldsymbol{\eta} = \epsilon^{\pm}$.

6. Case of reducibility $\frac{1}{2}$

As in previous sections, ρ, σ and α are as in 3.14, and we assume in this section that $\alpha = \frac{1}{2}$. By ψ_σ is denoted the tempered elementary discrete A-parameter such that $\sigma \in \Pi_{\psi_\sigma}$. Applying [Mœg11, Proposition 6.0.3] to $\psi_\sigma \oplus E^\rho_{2n,1} \oplus E^\rho_{1,2m}$ (which is also A-parameter by 3.3), we get immediately that for $m, n \geq 0$,

$$L([\tfrac{1}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([\tfrac{1}{2}, \tfrac{2n-1}{2}]; \sigma)) \in \Pi_{\psi_\sigma \oplus E^\rho_{2n,1} \oplus E^\rho_{1,2m}}.$$

Later we will give additional information regarding these packets. First we will see how these representations transforms under Aubert involution.

Proposition 6.1. *For $m, n \geq 1$ denote*

$$\pi^+_{m,n} := L([\tfrac{1}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([\tfrac{1}{2}, \tfrac{2n-1}{2}]; \sigma)), \quad \pi^-_{m,n} := L([\tfrac{3}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([-\tfrac{1}{2}, \tfrac{2n-1}{2}]_-; \sigma)). \tag{6.1}$$

Then

$$(\pi^+_{m,n})^{\mathbf{t}} = \pi^-_{n,m}. \tag{6.2}$$

6.1. **Notation for A-parameters in the case $\alpha = \frac{1}{2}$.** Denote

$$\psi := \psi_\sigma, \qquad \psi_{k,l} := E^\rho_{k,1} \oplus E^\rho_{1,l},$$

where $k, l \geq 0$ will be always chosen to be of even parity, and denote by $\epsilon^\pm_{k,l}$ the character of the component group of $\psi_{k,l}$ which extends ϵ_σ on ψ, and which is on the remaining two elements equal to ± 1. As before, we denote a pair $(\psi_{k,l}, \epsilon^\pm_{k,l})$ by $(\psi, \epsilon^\pm)_{k,l}$. With this notation (and $\pi^\pm_{m,n}$ introduced in Proposition 6.1), we have the following

Theorem 6.2. *Let $m, n \geq 1$. Then the following holds:*

(1)

$$L([\tfrac{1}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([\tfrac{1}{2}, \tfrac{2n-1}{2}]; \sigma)), \ L([\tfrac{3}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([-\tfrac{1}{2}, \tfrac{2n-1}{2}]_-; \sigma)) \in \Pi_{\psi_{2n,2m}}.$$

In other words, $\pi^\pm_{m,n} \in \Pi_{\psi_{2n,2m}}$. In particular, representations $\pi^\pm_{m,n}$ are unitarizable.

(2)

$$\pi((\psi, \epsilon^+)_{2n,2m}) = \begin{cases} L([\tfrac{1}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([\tfrac{1}{2}, \tfrac{2n-1}{2}]; \sigma)) = \pi^+_{m,n}, & m < n, \\ L([\tfrac{3}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([-\tfrac{1}{2}, \tfrac{2n-1}{2}]_-; \sigma)) = \pi^-_{m,n}, & n < m. \end{cases} \tag{6.3}$$

(3)

$$\pi((\psi, \epsilon^-)_{2n,2m}) = \begin{cases} L([\tfrac{3}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([-\tfrac{1}{2}, \tfrac{2n-1}{2}]_-; \sigma)) = \pi^-_{m,n}, & m < n. \\ L([\tfrac{1}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}, \delta([\tfrac{1}{2}, \tfrac{2n-1}{2}]; \sigma)) = \pi^+_{m,n}, & n < m. \end{cases} \tag{6.4}$$

7. Case of reducibility at 1

Again in this section ρ, σ and α are as in 3.14, and we assume that $\alpha = 1$. Denote by ψ_σ the tempered elementary discrete parameter such that $\sigma \in \Pi_{\psi_\sigma}$. Applying Proposition 6.0.3 of [Mœg11] to $\psi_\sigma \oplus E^\rho_{2n+1,1} \oplus E^\rho_{1,2m+1}$ we get that for $m, n \geq 1$,

$$L([1,m]^{\mathbf{t}}; \tau([0]_\pm; \delta([1,n];\sigma))) \in \Pi_{\psi_\sigma \oplus E^\rho_{2n+1,1} \oplus E^\rho_{1,2m+1}}.$$

Before we give more information about these packets, we calculate some Aubert involutions of above representations. We start with the following

Lemma 7.1. *For $n \geq 1$ holds*

$$\tau([0]_x; \delta([1,n];\sigma))^{\mathbf{t}} = \begin{cases} L([1,n]^{\mathbf{t}}; [0] \rtimes \sigma), & x = +, \\ L([0,1],[2,n]^{\mathbf{t}}; \sigma), & x = -. \end{cases} \tag{7.1}$$

Above representations are unitarizable.

Proposition 7.2. *Let $m, n \geq 1$. Denote*

$$\pi^\pm_{m,n} := L([1,m]^{\mathbf{t}}; \tau([0]_\pm; \delta([1,n];\sigma))), \quad \tau^-_{m,n} := L([2,m]^{\mathbf{t}}; \delta([-1,n]_-;\sigma)).$$

Then

$$(\pi^+_{m,n})^{\mathbf{t}} = \pi^+_{n,m}, \qquad (\pi^-_{m,n})^{\mathbf{t}} = \tau^-_{n,m}. \tag{7.2}$$

7.1. **Notation for A-parameters in the case $\alpha = 1$.** Denote in the rest of this section

$$\psi_{k,l} := \psi \oplus E^\rho_{k,1} \oplus E^\rho_{1,l},$$

where $k, l \geq 0$ will be always chosen to be of odd parity. Set

$$\xi = \epsilon_\sigma(\rho, 1, 1).$$

Next we define a characters $\epsilon^\pm_{k,l}$ of the component group of $\psi^\pm_{k,l}$ when k and l are different odd integers > 1. It coincides with ψ_σ on $\mathrm{Jord}(\psi_\sigma) - ((\rho,1,1))$ and satisfies

$$\epsilon^\pm_{k,l}(\rho,1,1) = \epsilon^\pm_{k,l}(\rho, \min(k,l), \delta_{\min(k,l)}) = \pm\xi,$$
$$\epsilon^\pm_{k,l}(\rho, \max(k,l), \delta_{\min(k,l)}) = \xi$$

(we need to assume that $\epsilon^\pm_{k,l}$ is equal on pair of blocks for which $E^\rho_{k',l'} = E^\rho_{k'',l''}$). As before, we denote a pair $(\psi_{k,l}, \epsilon^\pm_{k,l})$ by $(\psi, \epsilon^\pm)_{k,l}$. Denote

$$\epsilon^{+,-,-}_{k,l}$$

$$\epsilon^{+,-,-}_{k,l}(\rho,1,1) = \xi,$$
$$\epsilon^{+,-,-}_{k,l}(\rho, \min(k,l), \delta_{\min(k,l)}) = \epsilon^{+,-,-}_{k,l}(\rho, \max(k,l), \delta_{\min(k,l)}) = -\xi.$$

With this notation we have the following

Theorem 7.3. *Let $m, n \geq 1$. Then the following holds:*

(1)

$$L([1,m]^{\mathbf{t}};\tau([0]_{\pm};\delta([1,n];\sigma))), L([2,m]^{\mathbf{t}};\delta([-1,n]_{-};\sigma)) \in \Pi_{\psi_\sigma\oplus E^{\rho}_{2n+1,1}\oplus E^{\rho}_{1,2m+1}}.$$

In other words, $\pi^{\pm}_{m,n},\tau^{-}_{m,n} \in \Pi_{\psi_{2n+1,2m+1}}$. *In particular, representations* $\pi^{\pm}_{m,n}$ *and* $\tau^{-}_{m,n}$ *are unitarizable.*

(2)

$$\pi((\psi,\epsilon^{+})_{2n+1,2m+1}) = \begin{cases} L([1,m]^{\mathbf{t}};\tau([0]_{-};\delta([1,n];\sigma))) = \pi^{-}_{m,n}, & m<n,\\ L([2,m]^{\mathbf{t}};\delta([-1,n]_{-};\sigma)) = \tau^{-}_{m,n}, & n<m.\end{cases} \tag{7.3}$$

(3)

$$\pi((\psi,\epsilon^{-})_{2n+1,2m+1}) = L([1,m]^{\mathbf{t}};\tau([0]_{+};\delta([1,n];\sigma))) = \pi^{+}_{m,n}, m\neq n. \tag{7.4}$$

(4)

$$\pi((\psi,\epsilon^{+--})_{2n+1,2m+1}) = \begin{cases} L([2,m]^{\mathbf{t}};\delta([-1,n]_{-};\sigma)) = \tau^{-}_{m,n}, & m<n,\\ L([1,m]^{\mathbf{t}};\tau([0]_{-};\delta([1,n];\sigma))) = \pi^{-}_{m,n}, & n<m.\end{cases} \tag{7.5}$$

8. On irreducible unitarizable subquotients at critical points

Definition 8.1. *Let* $\rho_1,\dots,\rho_k \in \mathcal{C}$ *and let* σ *be an irreducible cuspidal representation of a classical group. Assume that for any* i *holds*

(1) $\rho_i^u \cong (\rho_i^u)\check{}$;

(2) the set $\{e(\rho_j) : \rho_j^u \cong \rho_i^u\}$ *is a* $\mathbb{Z}$*-segment in* $\frac{1}{2}\mathbb{Z}$ *(possibly with multiplicities);*

(3) the $\mathbb{Z}$*-segment in (2) contains the reducibility exponent* $\alpha_{\rho_i^u,\sigma}$.

Then, we say that the representation $\rho_1\times\cdots\times\rho_k\rtimes\sigma$ *is of critical type. If additionally* π *is an irreducible subquotient of* $\rho_1\times\cdots\times\rho_k\rtimes\sigma$*, then we say also that* π *is of critical type.*

We recall below of Conjecture 8.16 from [Tad20]:

Conjecture 8.2. *Suppose that* π *is an isolated representation in the unitary dual. Then,* π *is a representation of critical type.*

The above conjecture holds if π is unramified representation (see [MT11]). Further we add a new

Conjecture 8.3. *Any irreducible representation of critical type of a split classical group which is unitarizable is in some A-packet.*

The above conjecture holds if π is unramified or generic representation (see [MT11] and [LMT04]).

For a more serious support of the last conjecture we prove below that it holds for corank ≤ 3 (this is for approximately 100 types of unitarizable representations[12] of critical type considered in [Tad20]).

[12] More precisely, for 103 types, but precise number depends on what one considers to be different types.

Theorem 8.4. *Let π be an irreducible unitarizable subquotient of a representation*

$$\rho_1 \times \cdots \times \rho_k \rtimes \sigma, \qquad k \leq 3$$

of critical type. Then π is contained in an A-packet.

Proof. First we recall some simple general facts which will shorten considerably proving of the theorem.

8.0.1. *Some simple remarks about A-packets.*

(1) Each irreducible tempered representation is an element of some A-packet (with tempered A-parameter).
(2) If π is an element of an elementary discrete A-packet, then $\pi^{\mathbf{t}}$ is also an element of an elementary discrete A-packet.
(3) Each irreducible cotempered representation is contained in an A-packet (with cotempered A-parameter).

For coranks 0 and 1 the theorem follows directly from remarks in 8.0.1. It remains to consider coranks 2 and 3. We will consider below only cases not covered by remarks 8.0.1. We will also prove the theorem in the case when all ρ_i^u are the same, denoted by ρ (the proof in the other case is very simple, and we omit it here). We fix an irreducible cuspidal representation σ of a classical group. We assume that $\sigma = \pi(\psi_\sigma, \epsilon_\sigma)$ for some $\psi_\sigma \in \Psi_{\text{ele.}} \cap \Psi_{\text{d.d.r.}}$. Denote $\alpha = \alpha_{\rho,\sigma}$ (as usually). If we have some $\psi \in \Psi_{\text{ele.}}$, and write $\text{Jord}_\rho(\psi) = ((a_1, b_1), \ldots, (a_k, b_k))$, then we will always assume that the enumeration satisfies $\max(a_1, b_1) \leq \cdots \leq \max(a_k, b_k)$.

Below we will consider exponents $(x_1, \ldots, x_k)$, $k = 2$ or 3, the representation $\nu^{x_k}\rho \times \cdots \times \nu^{x_1}\rho \rtimes \sigma$ of critical type, and irreducible unitarizable subquotients of it. We will give precise reference where that representations were considered in [Tad20], and denote irreducible unitarizable subquotients in the same way as in [Tad20] (therefore, we will not recall here of this notation).

The arguments below are usually simple (and we have already used them in the previous part of the paper). Therefore, we will only sketch them very briefly below.

When we will have parameter (ψ', ϵ') below, and when we will get a new parameter (ψ'', ϵ'') by replacing $(\rho, a', b') \in \psi'$ by (ρ, a'', b''), then we will always assume that $\epsilon'(\rho, a, b)' = \epsilon''(\rho, a'', b'')$ and that ϵ' and ϵ'' coincide on remaining blocks.

Also if we will get (ψ'', ϵ'') from (ψ', ϵ') by replacing some elements (ρ, a, b) with (ρ, b, a), then we will assume $\epsilon''(\rho, a, b) = \epsilon'(\rho, c, d)$ if $\max(a, b) = \max(c, d)$.

8.1. **Corank 2.**

8.1.1. *Case* $(\alpha - 1, \alpha), \alpha > 1$ (3.4.3 of [Tad20]). Here all 4 irreducible subquotients are unitarizable. One is square integrable, and another is its Aubert involution. Therefore, we need to consider only representations

$$\pi_2 := L([\alpha - 1]; \delta([\alpha]; \sigma)), \quad \pi_3 := L([\alpha - 1], [\alpha]; \sigma),$$

where $\pi_2^{\mathbf{t}} = \pi_3$. We have $\sigma \in \Pi_{\psi'_\sigma}$, where we get ψ'_σ from ψ_σ by replacing $(2\alpha - 3, 1)$ with $(1, 2\alpha - 3)$ in $\text{Jord}_\rho(\psi)$, i.e. $\text{Jord}_\rho(\psi'_\sigma) = \{\ldots, (1, 2\alpha - 3), (2\alpha - 1, 1)\}$. One defines new A-parameter ψ by increasing the last block for 2, and then previous block also for 2, and gets $\pi_2 \in \Pi_\psi$ (actually, $\pi_2 = \pi(\psi, \epsilon)$ for ϵ naturally deformed from ϵ_σ)[13]. Since $\psi \in \Psi_{\text{ele.}} \cap \Psi_{\text{d.d.r.}}$, the second representation is in an A-packet by (2) in 8.0.1.

8.1.2. *Case* $(0.1), \alpha = 0$ (3.4.6 of [Tad20]). Here all 5 irreducible subquotients are unitarizable. Two are square integrable, and another another two are their Aubert involution. Therefore we need to consider only

$$\pi_2 := L([0, 1]; \sigma).$$

Let $\psi := \psi_\sigma \oplus E^\rho_{2,2}$. Then $\pi_2 \in \Pi_\psi$ by Proposition 6.0.3 of [Mœg11] (construction "L-packet inside A-packet").

8.2. **Corank 3.**

8.2.1. *Case* $(\alpha - 1, \alpha, \alpha + 1), \alpha > 1$ (4.5 of [Tad20]). Here we have 4 irreducible unitarizable subquotients. One is square integrable, and another is its Aubert involution. Therefore, we need to consider the following representations

$$\pi_3 := L([\alpha - 1]; \delta([\alpha, \alpha + 1]; \sigma)), \quad \pi_4 := L([\alpha + 1], [\alpha], [\alpha - 1]; \sigma),$$

where $\pi_3^{\mathbf{t}} = \pi_4$. We have $\sigma \in \Pi_{\psi'_\sigma}$, where we get ψ'_σ from ψ_σ by replacing $(2\alpha - 3, 1)$ with $(1, 2\alpha - 3)$ in $\mathrm{Jord}_\rho(\psi)$, i.e. $\mathrm{Jord}_\rho(\psi'_\sigma) = \{\dots, (1, 2\alpha - 3), (2\alpha - 1, 1)\}$. First increase the last block two times for 2, and then previous block for 2, and denote new parameter by ψ. Then $\pi_3 \in \Pi_\psi$. Since this $\psi \in \Psi_{\text{ele.}} \cap \Psi_{\text{d.d.r.}}$, we get the claim also for π_4.

8.2.2. $(\alpha - 1, \alpha, \alpha), \alpha > 1$ (4.6 of [Tad20]). Here only one irreducible subquotient is unitarizable:

$$\pi_0 := L([\alpha - 1], [\alpha]; \delta([\alpha]; \sigma)).$$

C. Mœglin has shown that this representation is in an A-packet (Appendix A of [Tad20]; we have also reproved her result in this paper).

8.2.3. *Case* $(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}), \alpha = \frac{3}{2}$ (4.7.2 of [Tad20]). Here all 8 irreducible subquotients are unitarizable. Two of them are tempered, while another two are cotempered. Therefore it remains to consider representations

$$\pi_5 := L([\tfrac{3}{2}]; \delta([-\tfrac{1}{2}, \tfrac{1}{2}]) \rtimes \sigma), \quad \pi_6 := L([\tfrac{1}{2}], [\tfrac{1}{2}]; \delta([\tfrac{3}{2}]; \sigma)),$$
$$\pi_7 := L([-\tfrac{1}{2}, \tfrac{3}{2}]; \sigma), \quad \pi_8 := L([\tfrac{1}{2}]; \delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}]; \sigma)).$$

For π_5, consider ψ'_σ where we get ψ'_σ from ψ_σ by replacing $(2, 1)$ with $(1, 2)$ in $\mathrm{Jord}_\rho(\psi)$. Now increase $(1, 2)$ to $(1, 4)$ and denote new parameter by ψ'. We get $L([\frac{3}{2}]; \sigma)$ in the packet of ψ'. Now add $(2, 1), (2, 1)$ to $\mathrm{Jord}_\rho(\psi')$. Now applying [Tad09, Proposition 5.3] we get that π_5 is in this new packet.

For π_6, increase $(2, 1)$ to $(4, 1)$ in $\mathrm{Jord}_\rho(\sigma)$ and denote this packet b ψ'. Then $\delta([\frac{3}{2}]; \sigma)$ is in the new packet. Now add $(1, 2), (1, 2)$ to $\mathrm{Jord}_\rho(\psi')$, and we get π_8 in the packet of this new parameter.

[13]For $\alpha > \frac{3}{2}$ the above argument is complete. For $\alpha = \frac{3}{2}$ observe that we are in the "even" case, and the construction 3.9 allows increasement from 0 to 2 in this case.

Observe that $\pi_7 \in \Pi_\psi$, where $\psi := \psi_\sigma \oplus E^\rho_{3,2}$.

For π_8, recall that $(2,1) \in \mathrm{Jord}_\rho(\psi_\sigma)$. Then increasing $(2,1)$ to $(6,1)$ we would get $\delta([\frac{3}{2},\frac{5}{2}];\sigma)$ in the packet. Adding $(2,1)$ and then replacing it with $(4,1)$, we would get (in two steps) that $\delta_{\mathrm{s.p.}}([\frac{1}{2},\frac{3}{2}],[\frac{3}{2},\frac{5}{2}];\sigma)$. Adding $(1,2)$ to the previous packet, we would get $L([\frac{1}{2}];\delta_{\mathrm{s.p.}}([\frac{1}{2},\frac{3}{2}],[\frac{3}{2},\frac{5}{2}];\sigma))$ in the packet. This representation is (by our construction) in Π_ψ, where $\mathrm{Jord}_\rho(\psi) = ((6,1),(4,1),(1,2))$. Put on $\mathrm{Jord}_\rho(\psi)$ standard order. Denote by ψ' the A-parameter obtained from ψ such that $\mathrm{Jord}_\rho(\psi)$ is changed to $\mathrm{Jord}_\rho(\psi') = (4,1),(2,1),(1,2)$. Consider on $\mathrm{Jord}_\rho(\psi')$ standard order satisfying $(2,1) >_{\psi'} (1,2)$, and let $\varphi : \mathrm{Jord}_\rho(\psi) \to \mathrm{Jord}_\rho(\psi')$ be a standard bijection which preserves order. Then $\mathrm{Jord}_\rho(\psi)$ dominates $\mathrm{Jord}_\rho(\psi')$ with respect to $>_{\psi'}$. By [Mœg11, 3.1.2] or [Xu17a, section 8], $\mathrm{Jac}_{[\frac{5}{2}]} \circ \mathrm{Jac}_{[\frac{3}{2}]}(L([\frac{1}{2}];\delta_{\mathrm{s.p.}}([\frac{1}{2},\frac{3}{2}],[\frac{3}{2},\frac{5}{2}];\sigma)))$, we get an element of the packet of ψ' or 0. To compute the last representation, observe that

$$L([\tfrac{1}{2}];\delta_{\mathrm{s.p.}}([\tfrac{1}{2},\tfrac{3}{2}],[\tfrac{3}{2},\tfrac{5}{2}];\sigma)) \hookrightarrow [\tfrac{3}{2}] \times [-\tfrac{1}{2}] \rtimes \delta_{\mathrm{s.p.}}([\tfrac{1}{2}],[\tfrac{3}{2},\tfrac{5}{2}];\sigma),$$

and that the last representation has unique irreducible subrepresentation. This implies

$$L([\tfrac{1}{2}];\delta_{\mathrm{s.p.}}([\tfrac{1}{2},\tfrac{3}{2}],[\tfrac{3}{2},\tfrac{5}{2}];\sigma)) \hookrightarrow [\tfrac{3}{2}] \rtimes L([\tfrac{1}{2}];\delta_{\mathrm{s.p.}}([\tfrac{1}{2}],[\tfrac{3}{2},\tfrac{5}{2}];\sigma)),$$

which easily implies

$$\mathrm{Jac}_{[\frac{3}{2}]}(L([\tfrac{1}{2}];\delta_{\mathrm{s.p.}}([\tfrac{1}{2},\tfrac{3}{2}],[\tfrac{3}{2},\tfrac{5}{2}];\sigma))) = L([\tfrac{1}{2}];\delta_{\mathrm{s.p.}}([\tfrac{1}{2}],[\tfrac{3}{2},\tfrac{5}{2}];\sigma)).$$

Observe that

$$\delta_{\mathrm{s.p.}}([\tfrac{1}{2}],[\tfrac{3}{2},\tfrac{5}{2}];\sigma)) \hookrightarrow [\tfrac{1}{2}] \times [\tfrac{5}{2}] \rtimes \delta([\tfrac{3}{2}];\sigma) \cong [\tfrac{5}{2}] \times [\tfrac{1}{2}] \rtimes \delta([\tfrac{3}{2}];\sigma).$$

Since the last representation has unique irreducible subrepresentation, we get

$$\delta_{\mathrm{s.p.}}([\tfrac{1}{2}],[\tfrac{3}{2},\tfrac{5}{2}];\sigma)) \hookrightarrow [\tfrac{5}{2}] \rtimes \delta_{\mathrm{s.p.}}([\tfrac{1}{2}],[\tfrac{3}{2}];\sigma).$$

Now

$$L([\tfrac{1}{2}];\delta_{\mathrm{s.p.}}([\tfrac{1}{2}],[\tfrac{3}{2},\tfrac{5}{2}];\sigma)) \hookrightarrow [-\tfrac{1}{2}] \times [\tfrac{5}{2}] \rtimes \delta_{\mathrm{s.p.}}([\tfrac{1}{2}],[\tfrac{3}{2}];\sigma) \cong [\tfrac{5}{2}] \times [-\tfrac{1}{2}] \rtimes \delta_{\mathrm{s.p.}}([\tfrac{1}{2}],[\tfrac{3}{2}];\sigma).$$

Since the last representation has unique irreducible subrepresentation, we get

$$L([\tfrac{1}{2}];\delta_{\mathrm{s.p.}}([\tfrac{1}{2}],[\tfrac{3}{2},\tfrac{5}{2}];\sigma)) \hookrightarrow [\tfrac{5}{2}] \rtimes L([\tfrac{1}{2}];\delta_{\mathrm{s.p.}}([\tfrac{1}{2}],[\tfrac{3}{2}];\sigma)).$$

This implies $\mathrm{Jac}_{[\frac{5}{2}]}(L([\frac{1}{2}];\delta_{\mathrm{s.p.}}([\frac{1}{2}],[\frac{3}{2},\frac{5}{2}];\sigma))) = L([\frac{1}{2}];\delta_{\mathrm{s.p.}}([\frac{1}{2}],[\frac{3}{2}];\sigma))$, and completes the proof that π_8 is in an A-packet.

8.2.4. $(\alpha-2,\alpha-1,\alpha), \alpha > 2$ (4.8.1 of [Tad20]). Here all 8 irreducible subquotients are unitarizable. They are

$$\pi_1 = \delta_{\mathrm{s.p.}}([\alpha-2],[\alpha-1],[\alpha];\sigma), \quad \pi_2 = L([\alpha-2];\delta_{\mathrm{s.p.}}([\alpha-1],[\alpha];\sigma)),$$
$$\pi_3 := L([\alpha-1],[\alpha-2];\delta([\alpha];\sigma)), \quad \pi_4 := L([\alpha-2,\alpha-1];\delta([\alpha];\sigma)),$$
$$\pi_5 := L([\alpha],[\alpha-1],[\alpha-2];\sigma), \quad \pi_6 := L([\alpha],[\alpha-2,\alpha-1];\sigma),$$
$$\pi_7 := L([\alpha-1,\alpha],[\alpha-2];\sigma), \quad \pi_8 := L([\alpha-2,\alpha];\sigma).$$

We have $\pi_1^t = \pi_8,\ \pi_2^t = \pi_7,\ \pi_3^t = \pi_6,\ \pi_4^t = \pi_5$. Since π_1 is tempered, and π_8 cotempered, it remains to consider 6 representations.

For π_7 observe that $\delta_{\text{s.p.}}([\alpha-1],[\alpha];\sigma)$ is in A-packet corresponding to ψ_1, where we get ψ_1 from ψ_σ by replacing $(2\alpha-3,1),(2\alpha-1,1)$ with $(2\alpha-1,1),(2\alpha+1)$ in $\text{Jord}_\rho(\psi_\sigma)$. Note that $\text{Jord}_\rho(\psi_1)$ ends with $(2\alpha-5,1),(2\alpha-1,1),(2\alpha+1,1)$. Now $L([\alpha-1,\alpha];\sigma)$ (which is the Aubert dual of previous discrete series by (3) of Proposition 3.7 in [Tad20]) is in the A-packet of $\psi_1^{\mathbf{t}}$ and $\text{Jord}_\rho(\psi_1^{\mathbf{t}})$ ends with $(1,2\alpha-5),(1,2\alpha-1),(1,2\alpha+1)$. Increasing $(1,2\alpha-5)$ to $(1,2\alpha-3)$, we get π_7 in the new A-packet. Since the last A-packet is discrete and elementary, π_2 is also in an A-packet.

For π_5, recall that by 8.1.1, $L([\alpha-1];\delta([\alpha];\sigma))$ is in the packet of ψ, where $\text{Jord}_\rho(\psi)$ ends with $(2\alpha-5,1),(1,2\alpha-1),(2\alpha+1,1)$. This is an elementary discrete packet. Therefore $L([\alpha-1],[\alpha];\sigma)=L([\alpha-1];\delta([\alpha];\sigma))^{\mathbf{t}}$ is in an elementary discrete packet of $\psi^{\mathbf{t}}$, and $\psi^{\mathbf{t}}$ ends with $(1,2\alpha-5),(2\alpha-1,1),(1,2\alpha+1)$. Replace $(1,2\alpha-5)$ with $(1,2\alpha-3)$ in $\psi^{\mathbf{t}}$. Then in this new packet is unique irreducible subrepresentation of $[-(\alpha-2)]\rtimes L([\alpha-1],[\alpha];\sigma)$. It is easy to show that this unique irreducible subrepresentation is π_5. Therefore, π_5 is in an A-packet. Further π_4 is in an A-packets since π_5 is in an elementary discrete A-packets (and $\pi_4^{\mathbf{t}}=\pi_5$).

For π_3 consider ψ'_σ which we get from ψ_σ by replacing $(2\alpha-5,1),(2\alpha-3,1)$ with $(1,2\alpha-5),(1,2\alpha-3)$ in $\text{Jord}_\rho(\psi_\sigma)$. Then $\text{Jord}_\rho(\psi'_\sigma)$ ends with $(1,2\alpha-5),(1,2\alpha-3),(2\alpha-1,1)$. Now we proceed in usual way (increasing each of this blocks for 2), and we get π_3 in the packet. Further π_6 is in an A-packets since π_3 is in an elementary discrete A-packets (and $\pi_3^{\mathbf{t}}=\pi_6$).

8.2.5. *Case* $(0,1,2),\alpha=2$ (4.8.2 of [Tad20]). Here all 8 irreducible subquotients are unitarizable. Two of them are tempered, while another two are cotempered. Therefore it remains to consider representations

$$\pi_5=L([1];[0]\rtimes\delta([2];\sigma)),\quad \pi_6=L([2],[0,1];\sigma),$$
$$\pi_7=L([0,1];\delta([2];\sigma)),\quad \pi_8=L([2],[1];[0]\rtimes\sigma),$$

where $\pi_5^{\mathbf{t}}=\pi_6$ and $\pi_7^{\mathbf{t}}=\pi_8$.

For π_5 and π_7, recall that by 8.1.1, $L([1];\delta([2];\sigma))$ is in Π_ψ for some A-parameter ψ. Now each irreducible subquotient of $[0]\rtimes L([1];\delta([2];\sigma))$ (it is also a subrepresentation) is in the packet of $\psi\oplus E^\rho_{1,1}\oplus E^\rho_{1,1}$. One of them is π_5 (apply [Tad09, Proposition 5.3]). For another one, observe that $([0]\rtimes L([1];\delta([2];\sigma)))^{\mathbf{t}}=[0]\rtimes L([1];\delta([2];\sigma))^{\mathbf{t}}=[0]\rtimes L([2],[1];\sigma)$, and that here π_8 is a subquotient (again apply [Tad09, Proposition 5.3]). Then $\pi_7=\pi_8^{\mathbf{t}}$ is a subquotient of $[0]\rtimes L([1];\delta([2];\sigma))$. Therefore, π_7 is also in an A-packet, as well as π_5.

For π_6 and π_8, recall that by 8.1.1, $L([2],[1];\sigma)$ is in Π_ψ for some A-parameter ψ. Now each irreducible subquotient of $[0]\rtimes L([2],[1];\sigma)$ is in the packet of $\psi\oplus E^\rho_{1,1}\oplus E^\rho_{1,1}$. One of them is π_8 (by [Tad09, Proposition 5.3]). For another one, observe that $([0]\rtimes L([2],[1];\sigma))^{\mathbf{t}}=[0]\rtimes L([2],[1];\sigma)^{\mathbf{t}}=[0]\rtimes L([1];\delta([2];\sigma))$, and that here π_5 is subquotient (by [Tad09, Proposition 5.3]). Then $\pi_6=\pi_5^{\mathbf{t}}$ is a subquotient of $[0]\rtimes L([2],[1];\sigma)$. Therefore, π_6 is also in an A-packet, as well as π_8.

8.2.6. *Case* $(0,1,1),\alpha=1$ (5.2 of [Tad20]). Here all 7 irreducible subquotients are unitarizable. Two of them are tempered, while another two are cotempered. Therefore it remains to consider representations

$$\pi_1 = L([0,1],[1];\sigma),\ \pi_3 = L([0,1];\delta([1];\sigma)),$$
$$\pi_4^+ = L([1];\tau([0]_+;\delta([1];\sigma))),$$

where π_1 and π_3 are dual.

For π_1 (resp. π_3) consider ψ obtained from ψ_σ such that in $\mathrm{Jord}_\rho(\psi_\sigma)$, $(1,1)$ is replaced by the pair $(1,3),(2,2)$ (resp. $(3,1),(2,2)$). Now π_1 (resp. π_3) is in the L-packet inside Π_ψ (by Proposition 6.0.3) of [Mœg11]).

For π_4^+ consider $\psi := \psi_\sigma \oplus E^\rho_{3,1} \oplus E^\rho_{1,3}$. One easily shows that π_4 is in the the L-packet inside Π_ψ .

8.2.7. *Case* $(\frac{1}{2},\frac{1}{2},\frac{3}{2}),\alpha=\frac{1}{2}$ (5.4 of [Tad20]). Here we have 8 irreducible unitarizable subquotients (and two non-unitarizable). Two of them are square integrable, and another two cotempered. Therefore, we need to consider the following

$$\pi_3 = L([-\tfrac{1}{2},\tfrac{3}{2}];\sigma),\quad \pi_4 = L([\tfrac{1}{2},\tfrac{3}{2}];\delta([\tfrac{1}{2}];\sigma)),$$
$$\pi_7 = L([\tfrac{1}{2}];\delta([\tfrac{1}{2},\tfrac{3}{2}];\sigma)),\quad \pi_8 = L([\tfrac{3}{2}];\delta([-\tfrac{1}{2},\tfrac{1}{2}]_-;\sigma)),$$

where $\pi_3^{\mathbf{t}} = \pi_4$ and $\pi_7^{\mathbf{t}} = \pi_8$.

Theorem 6.2 implies that π_7 and π_8 are in A-packets. For π_3 (resp. π_4) consider $\psi := \psi_\sigma \oplus E^\rho_{3,2}$ (resp. $\psi := \psi_\sigma \oplus E^\rho_{2,3}$). One directly sees that π_3 (resp. π_4) is in the the L-packet inside A-packet Π_ψ .

8.2.8. *Case* $(\frac{1}{2},\frac{1}{2},\frac{1}{2}),\alpha=\frac{1}{2}$ (5.5 of [Tad20]). Here all 5 irreducible subquotients are unitarizable. One of them is tempered, and another one cotempered. Therefore, we need to consider the following representations

$$\pi_2 = [\tfrac{1}{2}] \rtimes \delta([-\tfrac{1}{2},\tfrac{1}{2}]_-;\sigma), \pi_3 = [\tfrac{1}{2}] \rtimes L([\tfrac{1}{2}];\delta([\tfrac{1}{2}];\sigma)),$$
$$\pi_5 = L([\tfrac{1}{2}];\delta([-\tfrac{1}{2},\tfrac{1}{2}]_+;\sigma)),$$

where π_2 and π_3 are dual.

Representations π_2 and π_5 are in the L-packet inside the A-packet of $\psi_\sigma \oplus E^\rho_{1,2} \oplus E^\rho_{2,1} \oplus E^\rho_{2,1}$. Representation π_3 is in the L-packet inside the A-packet of $\psi_\sigma \oplus E^\rho_{2,1} \oplus E^\rho_{1,2} \oplus E^\rho_{1,2}$.

8.2.9. *Case* $(0,1,1),\alpha=0$ (6.2 of [Tad20]). Here 6 irreducible subquotients are unitarizable. Two of them are tempered, and another two cotempered. Therefore, we need to consider the following representations

$$\pi_3^\pm = L([1];\delta([0,1]_\pm;\sigma)).$$

Here $(\pi_3^+)^{\mathbf{t}} = \pi_3^-$.

These two representations are contained in the L-packet inside the A-packet of $\psi_\sigma \oplus E^\rho_{1,3} \oplus E^\rho_{3,1}$.

8.2.10. *Case* $(0,0,1), \alpha=0$ (6.3 of [Tad20]). Here all 6 irreducible subquotients are unitarizable. Two of them are tempered, and another two cotempered. Therefore, we need to consider the following representations

$$\pi_2^{\pm}=L([0,1];\delta([0]_{\pm};\sigma)).$$

Here $(\pi_2^{+})^{\mathbf{t}}=\pi_2^{-}$.

Representations $\pi_2^{\pm}$ are in the L-packet inside the A-packet of $\psi_\sigma\oplus E^{\rho}_{2,2}\oplus E^{\rho}_{1,1}\oplus E^{\rho}_{1,1}$. □

We recall now of a Conjecture 8.15 from [Tad20], but give here a little bit more precise statement.

Conjecture 8.5. *Each isolated representation in the unitary dual is in an A-packet.*

Observe that Conjectures 8.2 and 8.3 imply the above conjecture.

9. Appendix: Some intermediate complementary series of A-class

C. Mœglin mentioned to us that it is possible that some intermediate complementary series representations can be of A-class. We present below an example of this type. Below ρ,σ and α are as in section 3.14.

Lemma 9.1. *Let* $\alpha\geq 1$, $x\geq 0$ *and* $\alpha-x\in\mathbb{Z}_{>0}$. *Then* $[x]\rtimes\sigma$ *is in an A-packet*[14].

Proof. If $x=0$, then we are in the tempered situation, and the claim obviously holds (the A-parameter is $\psi_\sigma\oplus E^{\rho}_{1,1}\oplus E^{\rho}_{1,1}$). Therefore, we suppose $x>0$, which implies $\alpha>1$.

First we show that $[\alpha-1]\rtimes\sigma$ is in an A-packet if $\alpha>1$. Denote by $(\psi'_\sigma,\epsilon'_\sigma)$ the parameter obtained from $(\psi_\sigma,\epsilon_\sigma)$ deforming $E^{\rho}_{2\alpha-3,1}$ to $E^{\rho}_{1,2\alpha-3}$ (then $\mathrm{Jord}_\rho(\psi'_\sigma)$ ends with $(1,2\alpha-3),(2\alpha-1,1)$). We have $\sigma=\pi(\psi'_\sigma,\epsilon'_\sigma)$.

Denote by (ψ_1,ϵ_1) the parameter obtained from $(\psi'_\sigma,\epsilon'_\sigma)$ deforming $E^{\rho}_{2\alpha-1,1}$ to $E^{\rho}_{2\alpha+1,1}$ (now $\mathrm{Jord}_\rho(\psi_1)$ ends with $(1,2\alpha-3),(2\alpha+1,1)$). Then $\pi(\psi_1,\epsilon_1)=\delta([\alpha];\sigma)$.

Let (ψ_2,ϵ_2) be obtained from (ψ_1,ϵ_1) deforming $E^{\rho}_{1,2\alpha-3}$ to $E^{\rho}_{1,2\alpha-1}$ (now $\mathrm{Jord}_\rho(\psi_1)$ ends with $(1,2\alpha-1),(2\alpha+1,1)$). Then $\pi(\psi_2,\epsilon_2)=L([\alpha-1];\delta([\alpha];\sigma))$. Denote by $>_{\psi_2}$ the standard order on $\mathrm{Jord}_\rho(\psi_2)$.

Let ψ_3 be the A-parameter obtained from ψ_2 replacing $(1,2\alpha+1)$ with $(1,2\alpha-1)$ (now $\mathrm{Jord}_\rho(\psi_3)$ ends with $(2\alpha-1,1),(1,2\alpha-1)$). Denote by $>_{\psi_3}$ on $\mathrm{Jord}_\rho(\psi_3)$ standard order which satisfies

$$(2\alpha-1,1)>_{\psi_3}(1,2\alpha-1)$$

($>_{\psi_3}$ is an admissible order but not natural; ψ_3 is multiplicity one parameter, but not discrete).

We denote by $\varphi:\mathrm{Jord}_\rho(\psi_2)\to\mathrm{Jord}_\rho(\psi_3)$ the standard bijection which preserves order. This implies that it carries

$$(2\alpha+1,1)\mapsto(2\alpha-1,1)$$

(on the remaining elements it is identity). Now $\mathrm{Jord}(\psi_2)$ dominates $\mathrm{Jord}(\psi_3)$ with respect to $>_{\psi_3}$. Here we need to consider the matrix $X^{\gg}_{(\rho,A,B,\zeta_{a,b})}$ (defined in section 5 of [Xu17a]),

[14] For $x=\alpha$, both irreducible subquotients are in A-packets.

which is in our case 1×1 matrix $X^{\gg}_{(\rho,\alpha-1,0,1)}=[\alpha]$. We get elements of Π_{ψ_3} from $\mathrm{Jord}(\psi_2)$ applying Jac_α to each element of Π_{ψ_2} (result is always either irreducible representation or 0). Observe that

$$
\begin{aligned}
(9.1)\quad L([\alpha-1];\delta([\alpha];\sigma)) &\hookrightarrow [-(\alpha-1)]\rtimes\delta([\alpha];\sigma)\\
&\hookrightarrow [-(\alpha-1)]\times[\alpha]\rtimes\sigma\cong[\alpha]\times[-(\alpha-1)]\rtimes\sigma\cong[\alpha]\times[\alpha-1]\rtimes\sigma.
\end{aligned}
$$

Now Frobenius reciprocity implies that $\mathrm{Jac}_\alpha(L([\alpha-1];\delta([\alpha];\sigma)))=[\alpha-1]\rtimes\sigma$. Therefore, $[\alpha-1]\rtimes\sigma$ is in the A-packet of ψ_3.

In a similar way we show next that $[\alpha-2]\rtimes\sigma$ is in an A-packet if $\alpha>2$. Denote now by $(\psi'_\sigma,\epsilon'_\sigma)$ the parameter obtained from $(\psi_\sigma,\epsilon_\sigma)$ deforming $E^\rho_{2\alpha-5,1}$ to $E^\rho_{1,2\alpha-5}$ (then $\mathrm{Jord}_\rho(\psi'_\sigma)$ ends with $(1,2\alpha-5),(2\alpha-3,1),(2\alpha-1,1)$). We have $\sigma=\pi(\psi'_\sigma,\epsilon'_\sigma)$.

Denote by (ψ_1,ϵ_1) the parameter obtained from $(\psi'_\sigma,\epsilon'_\sigma)$ deforming $E^\rho_{2\alpha-1,1}$ to $E^\rho_{2\alpha+1,1}$ and then $E^\rho_{2\alpha-3,1}$ to $E^\rho_{2\alpha11,1}$ (now $\mathrm{Jord}_\rho(\psi_1)$ ends with $(1,2\alpha-5),(2\alpha-1,1),(2\alpha+1,1)$). We get directly that $\pi(\psi_1,\epsilon_1)=\delta_{\text{s.p.}}([\alpha-1],[\alpha];\sigma)$.

Let (ψ_2,ϵ_2) be obtained from (ψ_1,ϵ_1) deforming $E^\rho_{1,2\alpha-5}$ to $E^\rho_{1,2\alpha-3}$ (now $\mathrm{Jord}_\rho(\psi_1)$ ends with $(1,2\alpha-3),(2\alpha-1,1),(2\alpha+1,1)$). Then $\pi(\psi_2,\epsilon_2)=L([\alpha-2];\delta_{\text{s.p.}}([\alpha-1],[\alpha];\sigma))$. Denote by $>_{\psi_2}$ the standard order on $\mathrm{Jord}_\rho(\psi_2)$.

Denote by ψ_3 A-parameter obtained from ψ_2 replacing $(2\alpha-1,1)$ with $(2\alpha-3,1)$ and then $(2\alpha+1,1)$ with $(2\alpha-1,1)$ (now $\mathrm{Jord}_\rho(\psi_3)$ ends with $(2\alpha-3,1),(1,2\alpha-3),(2\alpha-1,1)$). Denote by $>_{\psi_3}$ on $\mathrm{Jord}_\rho(\psi_3)$ standard order on $\mathrm{Jord}_\rho(\psi_3)$ which satisfies

$$(2\alpha-3,1)>_{\psi_3}(1,2\alpha-3).$$

Let $\varphi:\mathrm{Jord}_\rho(\psi_2)\to\mathrm{Jord}_\rho(\psi_3)$ be the standard bijection which preserves order. This implies that it carries

$$(2\alpha-1,1)\mapsto(2\alpha-3,1),\qquad(2\alpha+1,1)\mapsto(2\alpha-1,1),$$

(on the remaining elements it is identity). Now $\mathrm{Jord}(\psi_2)$ dominates $\mathrm{Jord}(\psi_3)$ with respect to $>_{\psi_3}$. Here we need to consider the matrices $X^{\gg}_{(\rho,A,B,\zeta_{a,b})}$, which in our case are 1×1 matrices $X^{\gg}_{(\rho,\alpha-2,0,1)}=[\alpha-1]$ and $X^{\gg}_{(\rho,\alpha-1,0,1)}=[\alpha]$. We need to apply them in descending order on $L([\alpha-2];\delta_{\text{s.p.}}([\alpha-1],[\alpha];\sigma))$, i.e. we need to apply $\mathrm{Jac}_\alpha\circ\mathrm{Jac}_{\alpha-1}$ to the last representation (and we will get either 0 or an element of Π_{ψ_3}).

Now we will compute the action of the above operator on $L([\alpha-2];\delta_{\text{s.p.}}([\alpha-1],[\alpha];\sigma)$. Observe that

$$
\begin{aligned}
(9.2)\quad L([\alpha-2];\delta_{\text{s.p.}}([\alpha-1],[\alpha];\sigma) &\hookrightarrow [-(\alpha-2)]\rtimes\delta_{\text{s.p.}}([\alpha-1],[\alpha];\sigma)\\
&\hookrightarrow [-(\alpha-2)]\times[\alpha-1]\rtimes\delta([\alpha];\sigma)\cong[\alpha-1]\times[-(\alpha-2)]\rtimes\delta([\alpha];\sigma).
\end{aligned}
$$

Note that $[-(\alpha-2)]\rtimes\delta([\alpha];\sigma)$ is irreducible. Now Frobenius reciprocity implies that

$$\mathrm{Jac}_{\alpha-1}(L([\alpha-2];\delta_{\text{s.p.}}([\alpha-1],[\alpha];\sigma)))=[-(\alpha-2)]\rtimes\delta([\alpha];\sigma).$$

Further

$$[-(\alpha-2)]\rtimes\delta([\alpha];\sigma)\hookrightarrow[-(\alpha-2)]\times[\alpha]\times\sigma\cong[\alpha]\times[-(\alpha-2)]\times\sigma\cong[\alpha]\times[\alpha-2]\times\sigma,$$

and one directly concludes that $\text{Jac}_\alpha([-(\alpha-2)] \rtimes \delta([\alpha];\sigma)) = [\alpha-2] \times \sigma$. Therefore, $[\alpha-2] \rtimes \sigma$ is in the A-packet of ψ_3.

Continuing this procedure, we complete the proof of the lemma. □

Definition 9.2. *Suppose that an irreducible representation π of a classical group S_n is in an A-packet, and that there do not exist Speh representations $\tau_1, \dots, \tau_k$ and an irreducible representation π_0 of a classical group S_m with $m < n$, contained in some A-packet, such that*

$$\pi \hookrightarrow \tau_1 \times \cdots \times \tau_k \rtimes \pi_0.$$

Then π_0 will be called primitive representation of A-type.

L. Clozel introduced in [Clo07] the notion of automorphic dual. Motivated with [Tad10], we ask the following

Question. Is each primitive representation of A-type isolated in the automorphic dual?

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