ON UNITARIZABILITY AND ARTHUR PACKETS

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ABSTRACT. In this paper we begin to explore the relation between the question of unitarizability of classical p-adic groups, and Arthur packets, starting from [Tad20].

CONTENTS

1. INTRODUCTION

In [Tad18] we proposed a strategy to approach unitarizability of classical groups over a p-adic field F of characteristic 0. In that strategy, the only relevant information is the cuspidal reducibility exponent, which is an element of

$$\tfrac{1}{2}\mathbb{Z}_{\geq 0}$$

(therefore, these are the parameters with which we work). We applied this strategy in [Tad20] to classify unitarizability in coranks ≤ 3. The key to control unitarizability in [Tad20] is to understand it in the case of the so-called critical points (see Definition 8.1). These are the places where the most important irreducible unitarizable representations show up. Non-unitarizable representations also give some key information for proving exhaustion. The majority of irreducible subquotients are non-unitarizable. Still, approximately 100 types of irreducible subquotients are unitarizable. Unitarizability of these

representations was proved using standard methods of representation theory, except in the case of the representations given in the Langlands classification by

$$(1.1) \qquad\qquad L(\nu^{\alpha-1}\rho, \nu^{\alpha}\rho; \delta([\nu^{\alpha}\rho]; \sigma)),$$

where ρ and σ are irreducible cuspidal representations of a general linear and a classical group, respectively, and α is corresponding (exceptional) reducibility point which satisfies $\alpha \geq \frac{3}{2}$ (see 2.6 and 2.9 for a description of the notation). C. Mœglin proved its unitarizability using her explicit characterisation of Arthur packets (see her Appendix A in [Tad20]). This is the single place in [Tad20] where Arthur packets interact (explicitly) with questions of unitarizability. The lowest rank cases when representations (1.1) show up are $\mathrm{Sp}(10, F)$ and $\mathrm{SO}(11, F)$.

We expect that the role of Arthur packets is much deeper in the unitarizability problem. This paper is a step in trying to understand (and an attempt to formulate more precisely) this interplay. In this paper we consider symplectic and split special odd-orthogonal groups over F (we expect that the results of this paper also hold for other classical groups).

The starting point of this paper in the direction of Arthur packets are the Mœglin representations (1.1) which we considered in [Tad20]. In this paper we extend the Mœglin construction to a two-parameter family

$$\pi_{m,n} := L([\nu^{\alpha-1}\rho], [\nu^{\alpha}\rho], \ldots, [\nu^{\alpha+m}\rho]; \delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho]; \sigma)), \quad m, n \geq 0$$

(see Theorem 4.2). We have seen in [Tad20] that the representations $\pi_{0,0}$ (i.e. (1.1)) are isolated in the unitary duals. We expect that all the representations $\pi_{n,m}$ are isolated as well. Further, these representations satisfy the following very simple formula for the Aubert duality

$$(1.2) \qquad\qquad \pi_{m,n}^{\mathbf{t}} = \pi_{n,m}.$$

Recall that for the Speh representations we have analogous formulas for duality

$$u_{\rho}(m, n)^{\mathbf{t}} = u_{\rho}(n, m)$$

(see 2.3 for explanation of the notation).

Further, in Theorems 5.1 and 6.2 we describe analogous two-parameter families in the cases of non-exceptional reducibilities 0 and $\frac{1}{2}$ (excluding finitely many ρ's, all reducibilities are of this type when we fix σ), and also handle the case of exceptional reducibility at 1 (Theorem 7.3). We also compute the Aubert duals of these representations, and get formulas similar to formula (1.2) (see Propositions 5.2, 6.1 and 7.2). We also expect for these representations to be isolated, excluding few of them (with very low indexes).

The second starting point of this paper in the direction of the Arthur packets is the list of unitarizable irreducible subquotients at critical points in [Tad20]. As we already mentioned, this is a list of around 100 types of them. The list (which was essentially obtained through case-by-case considerations; it took about half of [Tad20] to get it) does not reveal a clear pattern; in fact, at first glance, it seems somewhat random.

Recall that we know from our present (very limited) understanding of unitarizability: In the case of general linear groups, as well as generic and spherical representations of

classical groups (see [Tad86], [LMT04] and [MT11]), automorphic representations play a key role in describing unitarizability. It may be natural to try to see if this is also the case for classical groups.

Arthur packets provide us with a significant number of unitarizable representations (their unitarizability follows from the fundamental work [Art13] of J. Arthur). For a number of representations of Arthur type we do not know to prove their unitarizability by other methods (at least not at the present time). The most interesting representations of Arthur type seem to be those which are irreducible subquotients at critical points.

Unitary duals carry natural topology ([Fel62]), and the most interesting representations in the unitary duals are the isolated representations. The following conjecture is an attempt to relate Arthur type representations, critical points and isolated representations. It may easily happen that the following conjecture is not true, but still we expect that this line of thinking is useful. This conjecture may be very hard to prove (if it is true).

Conjecture 1.1.

(1) *Let π be an irreducible subquotient at a critical point. Then*
   the following two claims are equivalent:
   (a) π is unitarizable.
   (b) π is a member of some Arthur packet.
(2) *If π is an isolated representation in the unitary dual, then π is a representation of critical type.*
(3) *Each isolated representation π in the unitary dual is in an Arthur packet.*

Note that (1) and (2) would imply (3) (claim (2) of the above conjecture was stated as Conjecture 8.16 in [Tad20]).

Now we briefly comment claim (1) of the above conjecture. As we already noted, the statement that (1b) implies (1a) is known (so claim (1) is that (1a) implies (1b)). This claim would give a relatively simple pattern of understanding unitarizability in the most delicate cases for classical groups. It relates a highly mysterious and very hard question of unitarizability/non-unitarizability at the critical points to (at least a little bit) less mysterious and more combinatorial question of belonging to Arthur packets (a problem which seems easier to handle).

If the above claim is true, it could also provide upper bounds for unitarizability in general (and therefore, be useful for exhaustion questions). Namely, the exhaustion is obtained in [Tad20] (as well as in the other papers on unitarizability in the corank two) by reducing the questions of the non-unitarizability to the known non-unitarizability at the critical points.

We have the following (very limited) support to the above conjecture:

Theorem 1.2. *Conjecture 1.1 holds if π is an irreducible representation which is a subquotient at a critical point in corank ≤ 3, or an unramified representation, or a generic representation.*

The fact that the above conjecture holds if π is an unramified (resp. a generic) representation follows easily from [MT11] (resp [LMT04]). If π is an irreducible subquotient at

a critical point in corank ≤ 3, then claims (2) and (3) follow directly from [Tad20], while claim (1) is Theorem 8.2 of this paper.

Very important parts of unitary duals are automorphic duals (introduced in [Clo07] by L. Clozel; we recall this notion in section 8 of this paper). In the cases where unitarizability is known, automorohic duals usually contain the most important representations of the unitary duals (like isolated representations). If (3) of Conjecture 1.1 holds, then each isolated representation in the unitary dual would be isolated in the automorphic dual. Such a representation must be primitive, i.e. it must not be a subrepresentation of a representation parabolically induced by an Arthur type representation of a proper Levi subgroup (see Definition 8.4).

The question of the isolated representations in the unitary duals is one of the most delicate problems of the unitarizability (see [Tad87] for the case of unitary duals of general linear groups and [MT11] for the case of unramified unitary duals of classical groups). The study of automorphic duals in [Tad10] (under assumption of the generalized Ramanujan conjecture and the "Arthur + ϵ" conjecture of Clozel from [Clo07]) suggests that the lists of isolated representations in the automorphic duals could be considerably simpler than the same lists in the unitary duals. While we do not see, at the moment, a way to conjecture much regarding a list of isolated representations in the unitary duals, we may try to guess the following qualitative characterization of the isolated representations in the automorphic duals (which could yield a quantitative description):

Question 1.3. Is each primitive representation of Arthur type isolated in the automorphic dual?

The key for handling Arthur packets in our paper is the Mœglin explicit construction of Arthur packets (together with the work of B. Xu related to this). Let us note that the techniques of Mœglin seem to fit well with the approach to the unitarizability based only on reducibility points. In construction of Arthur packets, knowledge of the Aubert involution of representations is very useful. Some crucial ideas for computation of the involution belong to C. Jantzen. These are principal tools that we use in this paper.

Recently H. Atobe and A. Mínguez in [AM20] and H. Atobe in [Ato21] made a crucial breakthrough finding algorithms for the Aubert involution and construction of Arthur packets.

We are very thankful to C. Mœglin for a series of discussions and for sharing her results with us. Discussions with E. Lapid helped us better understand some ideas on which this paper is based. P. Bakić has read the paper and gave us a number of useful suggestions, which helped us a lot to improve the style of the paper. We are very thankful to the referee for a very careful reading and a number of important suggestions and corrections. Thanks to them, this paper is much easier to read.

The structure of the paper is the following. Section 2 introduces the notation of the representation theory of classical p-adic groups, while section 3 collects the notation and some facts about the Mœglin construction of Arthur packets. Sections 4, 5, 6 and 7 bring the constructions of families of Arthur representations corresponding to four types of reducibility points. In section 4 we consider the case of reducibility > 1, where details of

all the proofs are presented. In sections 5, 6 and 7 the cases of reducibility points 0, $\frac{1}{2}$ and 1 are considered respectively. The proofs of the claims in sections 5, 6 and 7 are obtained using similar ideas and techniques as the proofs of the corresponding claims in section 4. Because of this, we completely omit proofs in sections 5, 6 and 7. In the last section we show that each irreducible unitarizable subquotient of a critical point in corank ≤ 3 is of Arthur class. In the Appendix we show that some of the simplest complementary series can be of Arthur class when reducibility exponent is > 1.

2. NOTATION

We first recall briefly the well-known notation for p-adic general linear groups established by J. Bernstein and A. V. Zelevinsky ([Zel80]; see also [Rod82]), and its natural extension to classical p-adic groups.

A p-adic field F of characteristic zero will be fixed. All representations considered in this paper will be smooth. The Grothendieck group of the category $\mathrm{Alg}_{\mathrm{f.l.}}(G)$ of all finite length representations of a connected reductive p-adic group G is denoted by

$$\mathfrak{R}(G).$$

We have a natural map s.s. : $\mathrm{Alg}_{\mathrm{f.l.}}(G) \to \mathfrak{R}(G)$ called semi-simplification. There is a natural partial order on $\mathfrak{R}(G)$. For two finite length representations π_1 and π_2 of G, the fact $\mathrm{s.s.}(\pi_1) \leq \mathrm{s.s.}(\pi_2)$ we will write simply as $\pi_1 \leq \pi_2$. The contragredient representation of π will be denoted by $\tilde{\pi}$. We can lift the mapping $\pi \to \tilde{\pi}$ to an additive group homomorphism $\sim: \mathfrak{R}(G) \to \mathfrak{R}(G)$.

If Π (resp. π) is a representation (resp. an irreducible representation) of G , then

$$\pi \underset{\mathrm{u.i.sub.}}{\hookrightarrow} \Pi$$

will mean that π is a unique irreducible subrepresentation of Π.

2.1. **Hopf algebra for general linear groups.** The modulus character on F is denoted by $|\ |_F$, and the character $|\det|_F$ of $\mathrm{GL}(n, F)$ by ν. Let $n = n_1 + n_2, n_i \geq 0$. Denote by $P_{(n_1,n_2)} = M_{(n_1,n_2)} N_{(n_1,n_2)}$ the parabolic subgroup of $\mathrm{GL}(n, F)$ which is standard with respect to the minimal parabolic subgroup of upper triangular matrices, such that $M_{(n_1,n_2)}$ is naturally isomorphic to $\mathrm{GL}(n_1, F) \times \mathrm{GL}(n_2, F)$. For representations π_i, $i = 1, 2$, of $\mathrm{GL}(n_i, F)$, denote

$$\pi_1 \times \pi_2 := \mathrm{Ind}_{P_{(n_1,n_2)}}^{\mathrm{GL}(n,F)}(\pi_1 \otimes \pi_2).$$

Let $R := \underset{n \geq 0}{\oplus} \mathfrak{R}(\mathrm{GL}(n, F))$. Then $\times$ lifts naturally to a multiplication on R, and in this way we get commutative graded $\mathbb{Z}$-algebra structure on R. We can factorise $\times : R \times R \to R$ through $m : R \otimes R \to R$.

The normalised Jacquet module with respect to $P_{(n_1,n_2)}$ of a representation π of $\mathrm{GL}(n, F)$ is denoted by $r_{(n_1,n_2)}(\pi)$. If π is of finite length, then we can set

$$m^*(\pi) := \sum_{k=0}^{n} \mathrm{s.s.}\big(r_{(k,n-k)}(\pi)\big) \in R \otimes R.$$

One extends additively m^* on whole R, and gets graded Hopf algebra structure on R.

2.2. Segments and corresponding irreducible subrepresentations. Denote by $\mathcal{C}$ (resp. $\mathcal{D}$) the set of all equivalence classes of irreducible cuspidal (resp. essentially square integrable) representations of all $\mathrm{GL}(n, F)$, $n \geq 1$. For $\delta \in \mathcal{D}$, there exists unique $e(\delta) \in \mathbb{R}$ and unitarizable $\delta^u \in \mathcal{D}$ such that

$$\delta = \nu^{e(\delta)} \delta^u.$$

For $\rho \in \mathcal{C}$ and $x, y \in \mathbb{R}$ such that $y - x \in \mathbb{Z}_{\geq 0}$, the set $[\nu^x \rho, \nu^y \rho] := \{\nu^x \rho, \nu^{x+1} \rho, \dots, \nu^y \rho\}$ is called a segment of cuspidal representations of general linear groups (one-point segment $[\nu^x \rho, \nu^x \rho]$ will be denoted simply by $[\nu^x \rho]$).We denote it also by

$$[x, y]^{(\rho)},$$

or simply by $[x, y]$ when we will work with a fixed ρ (usually this will be the case later). The set of all segments of cuspidal representations is be denoted by $\mathcal{S}(\mathcal{C})$. The representation $\nu^x \rho \times \nu^{x+1} \rho \times \cdots \times \nu^y \rho$ (resp. $\nu^y \rho \times \nu^{y-1} \rho \times \cdots \times \nu^x \rho$) contains a unique irreducible subrepresentation which will be denoted by

$$\langle \nu^x \rho, \nu^{x+1} \rho, \dots, \nu^y \rho \rangle \text{ (resp. } \langle \nu^y \rho, \nu^{y-1} \rho, \dots, \nu^x \rho \rangle).$$

When we deal with a fixed ρ, these representations will be denoted simply by $\langle x, x + 1, \dots, y \rangle$, (resp. $\langle y, y - 1, \dots, x \rangle$). For a segment $[x, y]^{(\rho)} \in \mathcal{S}(\mathcal{C})$ denote $\delta([x, y]^{(\rho)}) := \langle \nu^y \rho, \nu^{y-1} \rho, \dots, \nu^x \rho \rangle$. Then $\delta([x, y]^{(\rho)}) \in \mathcal{D}$. For $n \geq 1$ set $\delta(\rho, n) := \delta([-\frac{n-1}{2}, \frac{n-1}{2}]^{(\rho)})$.

   Let π be an irreducible representation of a general linear group. Then there exist $\rho_1, \dots, \rho_k \in \mathcal{C}$ such that $\pi \hookrightarrow \rho_1 \times \cdots \times \rho_k$. The multiset $(\rho_1, \dots, \rho_k)$ is called the (cuspidal) support of π, and is denoted by $\mathrm{supp}(\pi)$.

2.3. Langlands classification for general linear groups. For a set X, denote by $M(X)$ the set of all finite multisets in X. For $d = (\delta_1, \dots, \delta_k) \in M(\mathcal{D})$ chose a permutation p of $\{1, \dots, k\}$ such that $e(\delta_{p(1)}) \geq \cdots \geq e(\delta_{p(k)})$. Then the representation $\lambda(d) := \delta_{p(1)} \times \cdots \times \delta_{p(k)}$ has a unique irreducible quotient, denoted by $L(d)$. This defines a bijection from $M(\mathcal{D})$ onto the set of equivalence classes of irreducible representations of all groups $\mathrm{GL}(n, F)$, $n \geq 0$ (Langlands classification). Another way to express this classification is by $M(\mathcal{S}(\mathcal{C}))$. To $a = (\Delta_1, \dots, \Delta_k) \in M(\mathcal{S}(\mathcal{C}))$ attach

$$L(a) := L(\delta(\Delta_1), \dots, \delta(\Delta_k)).$$

This is the version of the Langlands classification which we will use in the paper. For $n, m \geq 1$ and $\rho \in \mathcal{C}$ we denote by

$$u_\rho(n, m) := L(\nu^{\frac{m-1}{2}} \delta(\rho, n), \nu^{\frac{m-1}{2}-1} \delta(\rho, n), \dots, \nu^{-\frac{m-1}{2}} \delta(\rho, n)),$$

and call it a Speh representation.

2.4. **Module and comodule structures for classical groups.** In this paper we consider classical groups $\mathrm{Sp}(2n, F)$ and split $\mathrm{SO}(2n+1, F)$, $n \geq 0$. We will use their matrix realisation from [Tad95]. Such a group of rank n will be denoted by S_n (we will always work with a fixed series of groups). We fix in S_n a minimal parabolic subgroup consisting of all upper triangular matrices in the group. Now for each $0 \leq k \leq n$, there is a standard parabolic subgroup $P_{(k)} = M_{(k)} N_{(k)}$ such that $M_{(k)}$ is naturally isomorphic to the direct product $\mathrm{GL}(k, F) \times S_{n-k}$. For representations π and σ of $\mathrm{GL}(k, F)$ and S_{n-k} respectively, one defines $\pi \rtimes \sigma := \mathrm{Ind}_{P_{(k)}}^{S_n}(\pi \otimes \sigma)$. Denote $R(S) := \underset{n \geq 0}{\oplus} \mathfrak{R}(S_n)$. Then $\rtimes$ lifts in a natural way to $\rtimes : R \times R(S) \to R(S)$, and in this way $R(S)$ becomes an R-module. The normalised Jacquet module with respect to $P_{(k)}$ of a representation π of S_n is denoted by $s_{(k)}(\pi)$. Let π be of finite length. Then we set

$$\mu^*(\pi) := \sum_{k=0}^{n} \mathrm{s.s.}(s_{(k)}(\pi)) \in R \otimes R(S),$$

and extend μ^* additively to $\mu^* : R(S) \to R \otimes R(S)$. In this way, $R(S)$ becomes an R-comodule.

2.5. **Twisted Hopf algebra.** Denote by $\kappa : R \otimes R \to R \otimes R$ the transposition map $\sum_i x_i \otimes y_i \mapsto \sum_i y_i \otimes x_i$, and by

(2.1) $$M^* := (m \otimes \mathrm{id}_R) \circ (\sim \otimes m^*) \circ \kappa \circ m^* : R \to R \otimes R.$$

Then for finite length representations π and σ of $\mathrm{GL}(n, F)$ and S_k respectively, by [Tad95] we have

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\pi).$$

Denote by $M_{GL}^*(\pi) \otimes 1$ the component of $M^*(\pi)$ which is in $\mathfrak{R}(\mathrm{GL}(n, F)) \otimes \mathfrak{R}(\mathrm{GL}(0, F))$. We calculate $M_{GL}^*(\pi)$ by the following simple formula: if $m^*(\pi) = \sum_i x_i \otimes y_i$, then $M_{GL}^*(\pi) = \sum_i x_i \times \tilde{y}_i$. The component of $M^*(\pi)$ which is in $\mathfrak{R}(\mathrm{GL}(0, F)) \otimes \mathfrak{R}(\mathrm{GL}(n, F))$ is $1 \otimes \pi$.

Let additionally σ be an irreducible cuspidal representation of a classical group, and τ a subquotient of $\pi \rtimes \sigma$. Then we denote $s_{\mathrm{GL}}(\tau) := s_{(n)}(\tau)$. Now for a finite length representation π' of $\mathrm{GL}(n, F)$ we have $\mathrm{s.s.}(s_{\mathrm{GL}}(\pi' \rtimes \tau)) = M_{GL}^*(\pi') \times \mathrm{s.s.}(s_{GL}(\tau))$.

2.6. **Langlands classification for classical groups.** Denote by Irr^{cl} the set of equivalence classes of all irreducible representations of groups $S_n, n \geq 0$, and by $\mathcal{T}^{cl}$ the subset of the tempered representations in Irr^{cl}. Let $\mathcal{D}_+ = \{\delta \in \mathcal{D} : e(\delta) > 0\}$. Take $t = ((\delta_1, \delta_2, \ldots, \delta_k), \tau) \in M(\mathcal{D}_+) \times \mathcal{T}^{cl}$. Chose a permutation p of $\{1, \ldots, k\}$ such that $e(\delta_{p(1)}) \geq e(\delta_{p(2)}) \geq \cdots \geq e(\delta_{p(k)})$. Then, the representation $\lambda(t) := \delta_{p(1)} \times \delta_{p(2)} \times \cdots \times \delta_{p(k)} \rtimes \tau$ has a unique irreducible irreducible quotient, denoted by $L(t)$. The mapping $t \mapsto L(t)$ defines a bijection between $M(\mathcal{D}_+) \times \mathcal{T}^{cl}$ and Irr^{cl}, and it is the Langlands classification for classical groups (the multiplicity of $L(t)$ in $\lambda(t)$ is one). If $t = (d; \tau)$, then $L(d; \tau)^{\tilde{}} \cong L(d; \tilde{\tau})$ and $L(d; \tau)^- \cong L(\bar{d}; \bar{\tau})$, where $\bar{\pi}$ denotes the complex conjugate representation of π.

Introducing $\mathcal{S}(\mathcal{C})_+ = \{\Delta \in \mathcal{S}(\mathcal{C}); e(\delta(\Delta)) > 0\}$, we can define in a natural way the Langlands classification $(a, \tau) \mapsto L(a; \tau)$ using parameters in $M(\mathcal{S}(\mathcal{C})_+) \times \mathcal{T}^{cl}$. We will use this classification in this paper.

2.7. **Duality.** There is a natural involution D_G on the Grothendieck group of the representations of any connected reductive p-adic group G ([Aub95] and [SS97], see also [BBK18]). It takes any irreducible representation to an irreducible representation up to a sign. For any irreducible representation π, let π^t be the irreducible representation such that $D_G(\pi) = \pm\pi^t$. We call π^t the Aubert involution of π, or DL dual of π. This involution is compatible with parabolic induction in the sense that $(\pi \rtimes \tau)^t = \pi^t \rtimes \tau^t$ (on the level of Grothendieck groups). Furthermore, regarding Jacquet modules, the mapping

$$\pi_1 \otimes \ldots \pi_l \otimes \mu \mapsto \tilde{\pi}_1^t \otimes \ldots \tilde{\pi}_l^t \otimes \mu^t,$$

is a bijection from the semi-simplification of $s_\beta(\pi)$ onto the semi simplification of $s_\beta(\pi^t)$ (β is the partition which parameterises the corresponding parabolic subgroup).

We will use the following result: for $\Delta \in \mathcal{S}(\mathcal{C})$ and cuspidal $\sigma \in \mathrm{Irr}^{cl}$ we have

$$\delta(\Delta) \rtimes \sigma \text{ is reducible} \iff \theta \rtimes \sigma \text{ is reducible for some } \theta \in \Delta.$$

This result follows from Theorem 13.2. of [Tad98a]. To get the above result from the this theorem, one needs to know that the cuspidal reducibility exponents are in $\frac{1}{2}\mathbb{Z}$, which is implied by the basic assumption from [MT02]. This assumption follows from (ii) in Remarks 4.5.2 of [MW06] and Theorem 1.5.1 in [Art13].

An irreducible representation will be called cotempered, if it is the Aubert involution of a tempered representation.

2.8. **Some formulas for M^*.** Let $\rho \in \mathcal{C}$ be selfcontragredient and $[x, y]^{(\rho)} \in \mathcal{S}(\mathcal{C})$. Then, one easily gets

$$(2.2) \qquad M^*\big(\delta([x, y]^{(\rho)})\big) = \sum_{i=x-1}^{y} \sum_{j=i}^{y} \delta([-i, -x]^{(\rho)}) \times \delta([j+1, y]^{(\rho)}) \otimes \delta([i+1, j]^{(\rho)}).$$

In the above formula and the formulas below, we take terms of the form $[t, t-1]^{(\rho)}$ to be the identity of R, i.e. to be $L(\emptyset)$. In particular

$$(2.3) \qquad M_{\mathrm{GL}}^*\big(\delta([x, y]^{(\rho)})\big) = \sum_{i=x-1}^{y} \delta([-i, -x]^{(\rho)}) \times \delta([i+1, y]^{(\rho)}).$$

We denote the multiset $([x]^{(\rho)}, [x+1]^{(\rho)}, \ldots, [y]^{(\rho)}) = ([x]^{(\rho)}) + ([x+1]^{(\rho)}) + \cdots + ([y]^{(\rho)})$ by

$$([x, y]^{(\rho)})^{\mathbf{t}}.$$

Then $\langle \nu^x \rho, \nu^{x+1}\rho, \ldots, \nu^y\rho \rangle = L(([x, y]^{(\rho)})^{\mathbf{t}})$. Now

$(2.4) \quad M^*(L(([x, y]^{(\rho)})^{\mathbf{t}})) =$

$$\sum_{x-1 \leq i \leq y} \sum_{x-1 \leq j \leq i} L(([-y, -i-1]^{(\rho)})^{\mathbf{t}}) \times L(([x, j]^{(\rho)})^{\mathbf{t}}) \otimes L(([j+1, i]^{(\rho)})^{\mathbf{t}}),$$

$$(2.5) \qquad M_{GL}^*(L(([x, y]^{(\rho)})^{\mathbf{t}})) = \sum_{i=x-1}^{y} L(([-y, -i-1]^{(\rho)})^{\mathbf{t}}) \times L(([x, i]^{(\rho)})^{\mathbf{t}}).$$

2.9. **Some very simple irreducible square integrable and tempered representations of classical groups.** Let ρ and σ be irreducible cuspidal representations of a general linear and a classical group respectively, and suppose that ρ is selfcontragredient (i.e. $\rho \cong \tilde{\rho}$). Then there exists a unique $\alpha_{\rho,\sigma} \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ such that

$$\nu^{\alpha_{\rho,\sigma}} \rho \rtimes \sigma$$

reduces. We denote $\alpha_{\rho,\sigma}$ simply by α once we fix ρ and σ.

Suppose $\alpha > 0$ and $n \geq 0$. Then the representation $\delta([\nu^\alpha \rho, \nu^{\alpha+n}\rho]) \rtimes \sigma$ contains a unique irreducible representation, which is denoted by

$$\delta([\nu^\alpha \rho, \nu^{\alpha+n}\rho]; \sigma).$$

This representation is square integrable, and it is called a generalised Steinberg representation. Further, $\delta([\nu^\alpha \rho, \nu^{\alpha+n}\rho]; \sigma)$ is the unique irreducible subrepresentation of $\nu^{\alpha+n}\rho \rtimes \delta([\nu^\alpha \rho, \nu^{\alpha+n-1}\rho]; \sigma)$. By [Tad98b, Theorem 6.3, (viii)] we have

$$(2.6) \qquad \mu^*(\delta([\nu^\alpha \rho, \nu^{\alpha+n}\rho], \sigma)) = \sum_{k=-1}^{n} \delta([\nu^{\alpha+k+1}\rho, \nu^{\alpha+n}\rho]) \otimes \delta([\nu^\alpha \rho, \nu^{\alpha+k}\rho], \sigma),$$

where we take $\delta(\emptyset, \sigma) = \sigma$.

Starting from generalised Steinberg representations, one can construct further (strongly positive) square integrable representations ([Mœg02] and [MT02] contain a general construction of such representations; see also section 34 of [Tad12]). We will describe here only the first step of the construction. Let $\alpha \geq \frac{3}{2}$. Take $m \in \mathbb{Z}_{\geq 0}$ such that $m \leq n$. Then the representation $\delta([\nu^{\alpha-1}\rho, \nu^{\alpha-1+m}\rho]) \rtimes \delta([\nu^\alpha \rho, \nu^{\alpha+n}\rho]; \sigma)$ has a unique irreducible subrepresentation, denoted by

$$\delta_{\mathrm{s.p.}}([\nu^{\alpha-1}\rho, \nu^{\alpha-1+m}\rho], [\nu^\alpha \rho, \nu^{\alpha+n}\rho]; \sigma).$$

This representation is square integrable.

Suppose $\alpha > 0$. Take $x, y \in \mathbb{R}$ such that $x \leq y$ and $x - \alpha,\ y - \alpha \in \mathbb{Z}_{\geq 0}$. Then the representation $\delta([\nu^{-x}\rho, \nu^y\rho]) \rtimes \sigma$ contains precisely two irreducible subrepresentations. If $\alpha \in \mathbb{Z}$ (resp. $\frac{1}{2} + \mathbb{Z}$), then precisely one of these subrepresentations contains in its Jacquet module the term $\delta([\nu\rho, \nu^x\rho]) \times \delta([\rho, \nu^y\rho]) \otimes \sigma$ (resp. $\delta([\nu^{\frac{1}{2}}\rho, \nu^x\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^y\rho]) \otimes \sigma$). We denote this irreducible subrepresentation by

$$\delta([\nu^{-x}\rho, \nu^y\rho]_+; \sigma)$$

and the other irreducible subrepresentation by $\delta([\nu^{-x}\rho, \nu^y\rho]_-; \sigma)$. Both subrepresentations are square integrable if $x < y$, and tempered (but not square integrable) otherwise (see [Tad99] for more details).

Assume now $\alpha = 0$ and $n \geq 0$. Take irreducible tempered representations $\delta([\rho]_\pm; \sigma)$ such that

$$(2.7) \qquad\qquad\qquad \rho \rtimes \sigma := \delta([\rho]_+; \sigma) \oplus \delta([\rho]_-; \sigma).$$

If σ is generic, then we take $\delta([\rho]_+; \sigma)$ to be a generic summand. Then the representation $\delta([\nu\rho, \nu^{\alpha+n}\rho]) \rtimes \delta(\rho_\pm; \sigma)$ contains a unique irreducible subrepresentation, which is denoted

by $\delta([\rho, \nu^n \rho]_\pm; \sigma)$. These representations are square integrable for $n \geq 1$. Further for $n \geq 1$, $\delta([\rho, \nu^{\alpha+n} \rho]_\pm; \sigma)$ is the unique irreducible subrepresentation of $\nu^{\alpha+n} \rho \rtimes \delta([\rho, \nu^{\alpha+n-1} \rho]_\pm; \sigma)$.

Let now $\alpha = 1$ and $n \geq 1$. Then $\rho \rtimes \delta([\nu\rho, \nu^n \rho]; \sigma)$ decomposes into a direct sum of two irreducible (tempered) representations, which we denote by

$$\tau([\rho]_\pm; \delta([\nu\rho, \nu^n \rho]; \sigma)).$$

The representation $\tau([\rho]_+; \delta([\nu\rho, \nu^n \rho]; \sigma))$ is characterised by the fact that $\delta([\rho, \nu^n \rho]) \otimes \sigma$ is in its Jacquet module.

2.10. Jantzen lemma.

Definition 2.1. *Let π be an irreducible representation of some S_n and $\rho \in \mathcal{C}$.*

(1) *We let $\mu^*_\rho(\pi)$ be the sum (in the corresponding Grothendieck group) of all irreducible terms in $\mu^*(\pi)$ of the form $\rho \otimes \tau$.*

(2) *We let*

$$\mathrm{Jac}_\rho(\pi)$$

be the sum in $R(S)$ of all τ when irreducible $\rho \otimes \tau$ runs over $\mu^*_\rho(\pi)$.

Observe that $\mu^*_\rho(\pi) = \rho \otimes \mathrm{Jac}_\rho(\pi)$. By Lemma 5.6 of [Xu17b] we have $\mathrm{Jac}_{\rho_1} \circ \mathrm{Jac}_{\rho_2} = \mathrm{Jac}_{\rho_2} \circ \mathrm{Jac}_{\rho_1}$ if $\rho_1 \notin \{\nu\rho_2, \nu^{-1}\rho_2\}$. Below we recall of Lemma 3.1.3 from [Jan14] (in a slightly less general form).

Definition 2.2. *Let π be an irreducible representation of some S_n and $\rho \in \mathcal{C}$. Denote by $f_\pi(\rho)$ the largest value of f such that some Jacquet module of π contains an irreducible subquotient of the form $\rho \otimes \cdots \otimes \rho \otimes \tau$, where ρ shows up f times. We let*

$$\mu^*_{\{\rho\}}(\pi)$$

be the sum of all irreducible terms in $\mu^*(\pi)$ of the form $\rho \times \cdots \times \rho \otimes \tau$, where ρ shows up $f_\pi(\rho)$ times in the last formula and τ is irreducible.

Lemma 2.3. *Let π be an irreducible representation of some S_n and $\rho \in \mathcal{C}$. Then there is a unique irreducible representation θ and unique $f \geq 0$ such that the following are all satisfied:*

(1) *$\pi \hookrightarrow \lambda \rtimes \theta$, where $\lambda := \rho \times \cdots \times \rho$ and ρ shows up f times in the last formula.*

(2) *$\mu^*_\rho(\theta) = 0$.*

Furthermore, $f = f_\pi(\rho)$ and all irreducible subquotients of $\mu^*_{\{\rho\}}(\pi)$ are isomorphic to $\lambda \otimes \theta$.

If $\rho \not\cong \tilde{\rho}$, then $\mu^*_{\{\rho\}}(\pi)$ is irreducible (i.e. the multiplicity of $\lambda \otimes \theta$ in $\mu^*_{\{\rho\}}(\pi)$ is one) and $\pi \hookrightarrow \lambda \rtimes \theta$ is its unique irreducible subrepresentation. In particular, if π' is an irreducible representation with $\mu^*_{\{\rho\}}(\pi) = \mu^*_{\{\rho\}}(\pi')$, then $\pi \cong \pi'$.

Remark 2.4. (1) *If $\mu^*_\rho(\pi) = \rho \otimes \theta$, then $\mu^*_{\tilde{\rho}}(\pi^{\mathbf{t}}) = \tilde{\rho} \otimes \theta^{\mathbf{t}}$.*

(2) *$\mathrm{Jac}_\rho(\pi)^{\mathbf{t}} = \mathrm{Jac}_{\tilde{\rho}}(\pi^{\mathbf{t}})$.*

(3) *If $\rho \not\cong \tilde{\rho}$ and $\mu^*_{\{\rho\}}(\pi) = \lambda \otimes \theta$, then $\mu^*_{\{\tilde{\rho}\}}(\pi^{\mathbf{t}}) = \tilde{\lambda} \otimes \theta^{\mathbf{t}}$ and $\pi^{\mathbf{t}} \hookrightarrow \tilde{\lambda} \rtimes \theta^{\mathbf{t}}$ as the unique irreducible subrepresentation.*

(4) Let $\rho \in \mathcal{C}$, $x \in \mathbb{R}$, $x \neq 0$, and let π, π' be irreducible representations of S_n and S_m. Suppose $\pi \hookrightarrow \nu^x \rho \rtimes \pi'$, $\mathrm{Jac}_{\nu^{-x}\rho}(\pi) \neq 0$ and $\mathrm{Jac}_{\nu^{-x}\rho}(\pi') = 0$. Then $\nu^x \rho \rtimes \pi'$ is irreducible ([Mœg06, Remark in 2.3]).

3. PARAMETERS OF A-PACKETS

In this section we recall the well-known terminology related to A-packets following mainly C. Mœglin.

3.1. **A-parameters.** For an irreducible cuspidal representation ρ of $GL(n_\rho, F)$ (this defines n_ρ), ρ will also denote the corresponding irreducible representation of the Weil group W_F under the local Langlands correspondence for general linear groups. The irreducible algebraic representation of $SL(2, \mathbb{C})$ of dimension a over $\mathbb{C}$ is denoted by E_a. A triple (ρ, a, b), $\rho \in \mathcal{C}$, $a, b \in \mathbb{Z}_{>0}$ is called a Jordan block. To shorten notation in the paper, we denote

$$E^\rho_{a,b} := \rho \otimes E_a \otimes E_b.$$

For a connected reductive group G over F, the connected component of the dual group ^LG is denoted by $^LG^0$, and called the complex dual group. Then $^L\mathrm{Sp}(2n, F)^0 = \mathrm{SO}(2n+1, \mathbb{C})$ and $^L\mathrm{SO}(2n+1, F)^0 = \mathrm{Sp}(2n, \mathbb{C})$. Set $n^* = 2n+1$ (resp. $n^* = 2n$) if $G = \mathrm{Sp}(2n, F)$ (resp. $G = \mathrm{SO}(2n+1, F)$).

Definition 3.1. *An A-parameter for the group S_n is a continuous homomorphism $\psi : W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \to {}^LS^0_n$, which is bounded on W_F and is (complex) algebraic on $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$.*

We can decompose ψ as above into a the sum of irreducible representations

$$(3.1) \qquad \psi = \bigoplus_{(\rho,a,b) \in \mathrm{Jord}(\psi)} E^\rho_{a,b}$$

where $Jord(\psi)$ is a finite multiset, which is called the Jordan block of ψ. Then we have $\sum_{(\rho,a,b) \in \mathrm{Jord}(\psi)} n_\rho ab = n^*$. Clearly, $\mathrm{Jord}(\psi)$ determines ψ (up to an equivalence). We can work with $\mathrm{Jord}(\psi)$ as the A-parameter instead of ψ. For a finite multiset of (ρ, a, b)'s, some additional conditions may be needed so that this multiset is the set of Jordan blocks of an A-parameter (we will not discuss these conditions here). Denote

$$\mathrm{Jord}_\rho(\psi) = ((a,b); (\rho, a, b) \in \mathrm{Jord}(\psi)).$$

The set of all equivalence classes of A-parameters of S_n will be denoted by $\Psi(S_n)$, and $\Psi = \cup_{n \geq 0} \Psi(S_n)$ (we will indicate the series of groups we are woring with when this is necessary).

One says that $(\rho, a, b) \in Jord(\psi)$ has good parity (with respect to S_n) if there exist $z \in \mathbb{Z}$ such that $\nu^{\frac{a+b}{2}+z} \rho \rtimes 1_{S_0}$ reduces. We say that $\mathrm{Jord}(\psi)$ has good parity if each of its elements has good parity. The subset of A-parameters of good parity will be denoted by

$$\Psi_{\mathrm{g.p.}}.$$

In this paper we will only work with A-parameters of good parity. This implies that the cuspidal representation ρ will always be selfcontragredient.

To an A-parameter ψ one can associate an irreducible unitarizable representation

$$\pi_\psi := \underset{(\rho,a,b)\in Jord(\psi)}{\times} u_\rho(a,b)$$

of a general linear group over F (we can work with π_ψ as the A-parameter instead of ψ).

3.2. Another parameterisation of Jordan blocks.
Let $(\rho,a,b) \in Jord(\psi)$, $\psi \in \Psi_{\text{g.p.}}$. Put

$$A = \frac{a+b}{2} - 1, \quad B = \frac{|a-b|}{2}$$

and $\zeta_{a,b} = sign(a-b)$ if $a \neq b$, and $\zeta_{a,b} = 1$ arbitrary element of $\{\pm 1\}$ otherwise. Obviously either $A, B \in \mathbb{Z}_{\geq 0}$ or $A, B \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$. Observe that

$$a = A + 1 + \zeta_{a,b}B, \qquad b = A + 1 - \zeta_{a,b}B.$$

The Jordan block (ρ,a,b) will be also denoted by

$$(\rho, A, B, \zeta_{a,b}).$$

3.3. Modifying A-parameters.
Fix a series of classical groups and $\psi \in \Psi_{\text{g.p.}}$. Let (ρ,a,b) and (ρ,a',b') be two Jordan blocks which satisfy

(3.2) $$\tilde{\rho} \cong \rho, \qquad a \equiv a' (\text{mod } 2\mathbb{Z}), \qquad b \equiv b' (\text{mod } 2\mathbb{Z}).$$

Then:

(1) If $(\rho,a,b) \in \text{Jord}_\rho(\psi)$, and if we define a new parameter ψ' by replacing (ρ,a,b) by (ρ,a',b') in $\text{Jord}(\psi)$, then $\psi' \in \Psi_{\text{g.p.}}$.
(2) If (ρ,a,b) has good parity, then $\psi \oplus E^\rho_{a,b} \oplus E^\rho_{a',b'}$ has good parity as well.
(3) If (ρ,a,b) has good parity and $ab \in 2\mathbb{Z}$, then $\psi \oplus E^\rho_{a,b}$ has good parity as well.

3.4. Elementary A-parameters.
A Jordan block (ρ,a,b) will be called elementary if $1 \in \{a,b\}$. An A-parameter ψ will be called elementary if it has good parity and if each $(\rho,a,b) \in Jord(\psi)$ is elementary. The last condition means that for each $(\rho,A,B,\zeta) \in Jord(\psi)$, we have $A = B$. The subset of elementary A-parameters in Ψ (and $\Psi_{\text{g.p.}}$) is denoted by

$$\Psi_{\text{ele.}}.$$

Let ψ be elementary. Then using the parameterisation introduced in 3.2, each element in the Jordan block can be written as $(\rho, \frac{c-1}{2}, \frac{c-1}{2}, \delta_c)$, where $c \in \mathbb{Z}_{>0}$, $\delta_c \in \{-1,1\}$, and we denote this Jordan block simply by

$$(\rho, c, \delta_c)$$

In the case of elementary A-parameters, we take $\delta_1 = 1$. Observe that if $\delta_c = 1$ (resp. $\delta_c = -1$), the corresponding Speh representation is square integrable (resp. the Aubert dual of a square integrable representation).

Definition 3.2. *We say that an A-parameter ψ is tempered (resp. cotempered) if $b = 1$ (resp. $a = 1$) for each $(\rho,a,b) \in \text{Jord}(\psi)$.*

3.5. **Discrete A-parameters.** Denote by $\Phi(S_n)$ the set of equivalence classes of admissible homomorphisms $W_F \times SL(2,\mathbb{C}) \to {}^L S_n^0$, and let $\Phi = \cup_{i \geq 0} \Phi(S_n)$. Let Φ_2 be the subset corresponding to the square integrable L-packets. For $\phi \in \Phi$, one defines $\mathrm{Jord}(\phi)$ and $\mathrm{Jord}_\rho(\phi)$ analogously as in the case of A-packets.

Denote by $\Delta : SL(2,\mathbb{C}) \to SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ the diagonal map. Let ψ be an A-parameter. Then the composition $\psi \circ \Delta$ is given by $(w, g) \mapsto \psi(w, g, g)$, and this element of Φ is denoted by ψ_d. Then

$$(3.3) \qquad \psi_d = \bigoplus_{(\rho,a,b) \in \mathrm{Jord}(\psi)} \bigoplus_{j \in [B,A]} \rho \otimes E_{2j+1},$$

where B and A are defined in 3.2 (i.e. $B = \frac{|a-b|}{2}$ and $A = \frac{a+b}{2} - 1$). One says that an A-parameter ψ is discrete (or that has discrete diagonal restriction) if $\psi_d \in \Phi_2$. It is equivalent to the fact that ψ has good parity and that ψ_d is a multiplicity one representation (in particular, then ψ is a multiplicity one representation). The subset of all $\psi \in \Psi$ which are discrete is denoted by

$$\Psi_{\mathrm{d.d.r.}}.$$

By 3.3, an A-parameter $\psi \in \Psi$ is in $\Psi_{\mathrm{d.d.r.}}$ if and only if ψ has good parity and for each fixed ρ, all the segments $[B, A]$ when (ρ, A, B, ζ) runs over $\mathrm{Jord}(\psi)$, are disjoint.

3.6. **Characters of the component group - good parity case.** Let $\psi \in \Psi_{\mathrm{g.p.}}$. Then the characters of the component group can be identified with the functions ϵ on the multiset $\mathrm{Jord}(\psi)$ into $\{\pm 1\}$ which satisfy $\prod_{(\rho,a,b) \in \mathrm{Jord}(\psi)} \epsilon(\rho, a, b) = 1$ and

$$\epsilon(\rho, a, b) = \epsilon(\rho', a', b') \text{ whenever } (\rho, a, b) = (\rho', a', b').$$

Sometimes we will look at characters of the component group of ψ as functions on irreducible constituents of ψ in a natural way.

Definition 3.3. *Let $\psi \in \Psi_{\mathrm{g.p.}}$, let ϵ be a character of the component group of ψ, $(\rho, a, b) \in \mathrm{Jord}_\rho(\psi)$ and let (ρ, a', b') be a Jordan block such that $(\rho, a', b') \notin \mathrm{Jord}_\rho(\psi)$, $a \equiv a' (mod\ 2\mathbb{Z})$ and $b \equiv b' (mod\ 2\mathbb{Z})$. Denote by ψ' the A-parameter obtained from ψ by replacing (ρ, a, b) by (ρ, a', b') in $\mathrm{Jord}_\rho(\psi)$ (then $\psi' \in \Psi_{\mathrm{g.p.}}$). Denote by ϵ' the character of the component group of ψ' such that $\epsilon'(\rho, a', b') = \epsilon(\rho, a, b)$, and that ϵ' and ϵ coincide on the remaining elements. We say that (ψ', ϵ') is obtained from (ψ, ϵ) deforming (ρ, a, b) to (ρ, a', b') (or deforming $E_{a,b}^\rho$ to $E_{a',b'}^\rho$).*

3.7. **A-packets.** To each A-parameter ψ of S_n, J. Arthur has attached in [Art13, Theorem 2.2.1] a finite multiset Π_ψ of irreducible unitarizable representations of S_n, called the A-packet of ψ, such that endoscopic distribution properties are satisfied. We will not recall these properties, but we will follow C. Mœglin explicit representation-theoretic construction of A-packets ([Mœg06], [Mœg09] and [Mœg11]). Unlike L-packets, A-packets do not need to be disjoint for different conjugacy classes of ψ (see Corollary 4.2 of [MW06] for more information in that direction). Another difference from L-packets is that A-packets always consist only of unitarizable representations.

More precisely, Arthur has attached to each character ϵ of the component group of ψ a multiset $\pi(\psi, \epsilon)$ of irreducible representations. Their sum is Π_ψ. Mœglin has proved that

$\pi(\psi, \epsilon)$ is multiplicity one ([Mœg11]), and that for a fixed ψ, the $\pi(\psi, \epsilon)$'s are disjoint for different ϵ's. Therefore, she has proved that Π_ψ's also have multiplicity one. Mœglin and Arthur definitions of $\pi(\psi, \epsilon)$ are not the same, but they are simply related (see [Xu17a]). In this paper we will follow the Mœglin definition of $\pi(\psi, \epsilon)$.

Mœglin has also proved that in the case of elementary discrete A-parameters, $\pi(\psi, \epsilon)$ are always irreducible representations ([Mœg06]). Note that in the case of elementary discrete A-parameter ψ, the number of characters of the component group of ψ is the same as the number of characters of the component group of ψ_d, and we can identify them in an obvious way.

Remark 3.4. *For $\psi_0 \in \Psi$ and $\psi_1 = \rho \otimes E_a \otimes E_b$ denote $\psi = \psi_1 \oplus \psi_o \oplus \tilde{\psi}_1$. Then there exists a canonical injection $\Pi_{\psi_0} \hookrightarrow \Pi_\psi$ and all irreducible constituents of $u_\rho(a,b) \rtimes \pi_0$, $\pi_0 \in \Pi_{\psi_0}$, are contained in the image of this injection ([AM20, section 5]; there is a more precise statement regarding $u_\rho(a,b) \rtimes \pi_0$).*

3.8. **Notation $b_{\rho,\psi,\epsilon}$ and $a_{\rho,\psi,\epsilon}$.** Fix $\psi \in \Psi_{\text{ele.}} \cap \Psi_{\text{g.p.}}$, a character ϵ of the component group of ψ and selfcontragredient $\rho \in \mathcal{C}$. Let X be a subset of $\text{Jord}(\psi)$ of the form $X = \{(\rho, c_1, \delta_{c_1}), \ldots, (\rho, c_k, \delta_{c_k})\}$, and chose an enumeration such that $c_1 < c_2 < \cdots < c_k$. We say that ϵ is cuspidal on X if

(1) $c_1 \in \{1, 2\}$.
(2) $c_{i+1} - c_i = 2$ for $1 \leq i \leq k-1$.
(3) $\epsilon(\rho, c_{i+1}, \delta_{c_{i+1}}) = -\epsilon(\rho, c_i, \delta_{c_i})$ for $1 \leq i \leq k-1$.
(4) $\epsilon(\rho, 2, \delta_2) = -1$ if $c_1 = 2$.

Denote by

$$b_{\rho,\psi,\epsilon}$$

the maximal positive integer (if it exists) such that ϵ is cuspidal on $\{(\rho, c, \delta_c) \in \text{Jord}(\psi); c \leq b_{\rho,\psi,\epsilon}\}$. If there is no integer as above, we take $b_{\rho,\psi,\epsilon} = -1$ if elements of $\text{Jord}_\rho(\psi_d)$ are odd, and $b_{\rho,\psi,\epsilon} = 0$ if elements of $\text{Jord}_\rho(\psi_d)$ are even. Further, let

$$a_{\rho,\psi,\epsilon}$$

be the minimum of the set $\{c; (\rho, c, \delta_c) \in \text{Jord}(\psi), c > b_{\rho,\psi,\epsilon}\}$ if the above set is nonempty. Otherwise, put $a_{\rho,\psi,\epsilon} = \infty$. Note that $a_{\rho,\psi,\epsilon} \geq 3$ if $\text{Jord}_\rho(\psi_d) \subseteq 1 + 2\mathbb{Z}$. Since $a_{\rho,\psi,\epsilon} \geq b_{\rho,\psi,\epsilon} + 2$, we have the following definition:

Definition 3.5. *If $a_{\rho,\psi,\epsilon} = b_{\rho,\psi,\epsilon} + 2$, then we say that we are in the boundary case.*

We use Mœglin construction of A-packets in the paper, but we do not recall it here. We recall only the following simple step which we use most often:

3.9. **Simple reduction step: case of $a_{\rho,\psi,\epsilon} > b_{\rho,\psi,\epsilon} + 2$ or $b_{\rho,\psi,\epsilon} = 0$** ([Mœg06, section 2.4, 1] or [Xu17a, Definition 6.3, (2)]). We consider two possibilities.

If $a_{\rho,\psi,\epsilon} > 2$, then the pair (ψ', ϵ') is is obtained from (ψ, ϵ) deforming $(\rho, a_{\rho,\psi,\epsilon}, \delta_{a_{\rho,\psi,\epsilon}})$ to $(\rho, a_{\rho,\psi,\epsilon} - 2, \delta_{a_{\rho,\psi,\epsilon}})$. If $a_{\rho,\psi,\epsilon} = 2$, then the pair (ψ', ϵ') is defined by deleting $(\rho, a_{\rho,\psi,\epsilon}, \delta_{a_{\rho,\psi,\epsilon}})$ from $\text{Jord}_\rho(\psi)$, and taking ϵ' to be the restriction of ϵ.

Definition 3.6. *If $a_{\rho,\psi,\epsilon} > b_{\rho,\psi,\epsilon} + 2$ or $b_{\rho,\psi,\epsilon} = 0$, one defines*

$$\pi(\psi,\epsilon) \underset{u.i.sub.}{\hookrightarrow} \nu^{\delta_{a_{\rho,\psi,\epsilon}} \frac{a_{\rho,\psi,\epsilon}-1}{2}} \rho \rtimes \pi(\psi',\epsilon')$$

to be the unique irreducible subrepresentation of the right-hand side.

3.10. **Irreducible square integrable representations.** These representations of S_n decompose into a disjoint union $\bigsqcup \Pi_\Psi$ when ψ runs over all tempered discrete A-parameters of S_n. For any character ψ of the component group, $\pi(\psi,\epsilon)$ is an irreducible representation. In this situation one usually works with the Weil-Deligne group (and drops the b's which are always one in this case). Then we are in the case of local Langlands correspondence for square integrable representations.

The classification (modulo cuspidal data) of irreducible square integral representations of the groups S_n is completed in [MT02]. To an irreducible square integral representation one attaches an admissible triple $(\mathrm{Jord}(\pi), \epsilon_\pi, \pi_{cusp})$ consisting of Jordan blocks, a partially defined function and a partial cuspidal support of π. Such triples classify irreducible square integrable representations (see [MT02] for details). Then $\mathrm{Jord}(\pi) = \mathrm{Jord}(\psi)$ if and only if $\pi \in \Pi_\psi$ ([Mœg11, Theorem 1.3.1] or Theorem 10.1 of [Xu17b]). Further, ϵ_π is the restriction of the character of the component group of ψ which is attached to π by Arthur (Theorem 10.1 of [Xu17b], see also Proposition 8.1 there).

3.11. **A consequence of the involution.** Let (ψ,ϵ) be a pair consisting of $\psi \in \Psi$ and a character ϵ of the component group of ψ. One defines a pair

$$(\psi^{\mathbf{t}}, \epsilon^{\mathbf{t}})$$

where $\psi^{\mathbf{t}} \in \Psi$ and $\epsilon^{\mathbf{t}}$ is a character of the component group of $\psi^{\mathbf{t}}$, by the requirement that $\mathrm{Jord}(\psi^{\mathbf{t}})$ consists of all (ρ,b,a), (ρ,a,b) in $\mathrm{Jord}(\psi)$, and $\epsilon^{\mathbf{t}}$ is defined using natural bijection between $\mathrm{Jord}(\psi^{\mathbf{t}})$ and $\mathrm{Jord}(\psi)$.

Let $\psi \in \Psi_{\mathrm{ele.}} \cap \Psi_{\mathrm{d.d.r.}}$. Obviously $\psi_d = (\psi^{\mathbf{t}})_d$. Using this, we identify characters of component groups of ψ and ψ_d. Therefore, if $\psi', \psi'' \in \Psi_{\mathrm{ele.}} \cap \Psi_{\mathrm{d.d.r.}}$ such that $(\psi')_d = (\psi'')_d$, their characters of component groups can be identified in a natural way.

C. Mœglin defined in [Mœg06] involutions on irreducible representations, which generalise the Aubert involution, and showed that each element $\pi(\psi,\epsilon)$ of an elementary discrete A-packet can be obtained from square integrable representation corresponding to ϵ in the L-packet of ψ_d by applying the involution (see [Mœg06, Theorem 5]). A consequence of it for classical Aubert involution is that

(3.4) $$\pi(\psi,\epsilon)^{\mathbf{t}} = \pi(\psi^{\mathbf{t}}, \epsilon) \quad \text{for} \quad \psi \in \Psi_{\mathrm{ele.}} \cap \Psi_{\mathrm{d.d.r.}}$$

([Mœg06, Theorem 5], Theorem 6.10 of [Xu17a]).

3.12. **Cuspidal representations in elementary discrete A-packets.** Let $\psi \in \Psi_{\mathrm{ele.}} \cap \Psi_{\mathrm{d.d.r.}}$. From our observations in 3.7 (or in 3.11) it follows that the cardinality of the L-packet of ψ_d is equal to the cardinality of the A-packet of ψ.

Partial Aubert involutions (defined in section 4 of [Mœg06]) carry irreducible non-cuspidal representations to non-cuspidal ones, and they cannot carry non-cuspidal to cuspidal ones. This implies that for an irreducible cuspidal representation σ of a classical group, σ belongs to the L-packet of ψ_d if and only if it belongs to the A-packet of $\psi \in \Pi_\psi$. Moreover, they determine the same character of the component groups (after we identify them).

3.13. **Orders on Jordan blocks.** Let $\psi \in \Psi$. Any total order $>_\psi$ on $\mathrm{Jord}_\rho(\psi)$ which satisfies

$$(\mathcal{P}) \qquad a + b > a' + b', \quad |a - b| > |a' - b'|, \quad \zeta_{a,b} = \zeta_{a',b'} \quad \Longrightarrow \quad (\rho, a, b) >_\psi (\rho, a', b')$$

for any $(\rho, a, b), (\rho, a', b') \in \mathrm{Jord}_\rho(\psi)$ will be called an **admissible order**.

We will always fix some total order $>'$ on the set $\{\rho; \mathrm{Jord}_\rho(\psi) \neq \emptyset\}$, and assume for each $(\rho, a, b), (\rho', a', b') \in \mathrm{Jord}_\rho(\psi)$ that if $\rho >' \rho'$, then $(\rho, a, b) >_\psi (\rho', a', b')$ (we do not need to assume this, but it simplifies descriptions of admissible orders). Therefore, for describing an admissible order on $\mathrm{Jord}_\rho(\psi)$, it is enough to describe it on each $\mathrm{Jord}_\rho(\psi)$.

Suppose $\psi \in \Psi_{\mathrm{d.d.r.}}$. Then in $(\mathcal{P})$ we have $a + b > a' + b' \iff |a - b| > |a' - b'|$ (and the condition $|a - b| > |a' - b'|$ is redundant in $(\mathcal{P})$ in this case). Actually, in this case we can find an admissible order $>_\psi$ satisfying

$$(\rho, a, b) >_\psi (\rho, a', b') \iff a + b > a' + b'.$$

Such an order will be called **natural**.

One says that $\psi_\gg \in \Psi_{\mathrm{d.d.r.}}$ with a natural order $>_{\psi_\gg}$ **dominates** $\psi \in \Psi_{\mathrm{g.p.}}$ **with respect to an admissible order** $>_\psi$ on $\mathrm{Jord}_\rho(\psi)$ if $\{\rho; \mathrm{Jord}_\rho(\psi_\gg) \neq \emptyset\} = \{\rho; \mathrm{Jord}_\rho(\psi) \neq \emptyset\}$, and if for each ρ from the last set we have an order preserving bijection $(a_\gg, b_\gg) \mapsto (a, b)$ from $\mathrm{Jord}_\rho(\psi_\gg)$ onto $\mathrm{Jord}_\rho(\psi)$ which satisfies

$$A_\gg - A = B_\gg - B \geq 0 \quad \text{and} \quad \zeta_{a,b} = \zeta_{a_\gg, b_\gg}.$$

Define the function $\boldsymbol{T} : \mathrm{Jord}(\psi) \to \mathbb{Z}_{\geq 0}$ by $\boldsymbol{T}(\rho, a, b) = A_\gg - A = B_\gg - B$. Observe that

$$(\rho, a_\gg, b_\gg) = \big(\rho, \quad a + (1 + \zeta_{(\rho,a,b)}) \boldsymbol{T}(\rho, a, b), \quad b + (1 - \zeta_{(\rho,a,b)}) \boldsymbol{T}(\rho, a, b) \big)$$

(the bigger of the numbers a and b is increased by $2\boldsymbol{T}(\rho, a, b)$, and the smaller one is unchanged; in the case $a = b$, we increase the first a or the second a by $2\boldsymbol{T}(\rho, a, a)$ and leave the other one unchanged, depending on whether we took $\zeta_{a,a}$ to be 1 or -1).

3.13.1. *Orders on elementary packets.* Let $\psi \in \Psi_{\mathrm{ele.}}$. Any total order $>$ satisfying the condition $a + b > a' + b' \implies (a, b) > (a', b')$ for any ρ and any $(a, b), (a', b') \in \mathrm{Jord}_\rho(\psi)$ will be called standard. Any standard order is obviously admissible. Let $\psi \in \Psi_{\mathrm{ele.}} \cap \Psi_{\mathrm{d.d.r.}}$. Since we have fixed a total order on $\{\rho; \mathrm{Jord}_\rho(\psi) \neq \emptyset\}$, there is only one natural order on $\mathrm{Jord}_\rho(\psi)$ (the standard one).

Let $\psi, \psi' \in \Psi_{\mathrm{ele.}}$ and assume $\psi' \in \Psi_{\mathrm{d.d.r.}}$. Suppose that we have a bijection $\varphi : \mathrm{Jord}(\psi') \to \mathrm{Jord}(\psi)$ which for any ρ induces a bijection $(\rho, a', b') \mapsto (\rho, a, b)$ from $\mathrm{Jord}_\rho(\psi')$ onto $\mathrm{Jord}_\rho(\psi)$ which satisfies

$$\max(k', l') > \max(k, l) \implies \max \varphi(k', l') \geq \max \varphi(k, l).$$

Then any such a bijection will be called **standard**.

3.14. **Cuspidal representations and reducibility exponent.** Fix an irreducible cuspidal selfcontragredient representation ρ of a general linear group and an irreducible cuspidal representation σ of a classical group. The representation $\nu^{\alpha_{\rho,\sigma}}\rho \rtimes \sigma$ reduces for a unique $\alpha_{\rho,\sigma} \geq 0$ (this defines $\alpha_{\rho,\sigma}$). Further, $\alpha_{\rho,\sigma} \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, and we denote $\alpha_{\rho,\sigma}$ simply by

$$\alpha.$$

Denote by $(\phi_\sigma, \epsilon_\sigma)$ a pair of an admissible homomorphism of the Weil-Deligne group and a character of the component group of ϕ_σ, corresponding to σ under the local Langlands correspondence. Let

$$\psi_\sigma := \phi_\sigma \otimes E_1,$$

and lift ϵ_σ to a character of the component group of ψ in the natural way, and denote it again by ϵ_σ. Then $a_{\rho',\psi_\sigma,\epsilon_\sigma} = \infty$ for all selfcontragredient $\rho' \in \mathcal{C}$. Further:

(1) Suppose $\alpha \geq 1$. This is equivalent to $\mathrm{Jord}_\rho(\phi) \neq \emptyset$. Then $\alpha = \frac{\max(\mathrm{Jord}_\rho((\psi_\sigma)_d))+1}{2}$.

(2) Suppose $\alpha < 1$. Then $\alpha = 0$ (resp. $\alpha = \frac{1}{2}$) if and only if $\nu^{\frac{1}{2}}\rho \rtimes 1_{S_0}$ is irreducible (resp. reducible), where 1_G denotes the trivial (one-dimensional) representation of a group G.

4. Case of reducibility > 1

In this section we assume that ρ, σ and α are as in 3.14, and we assume that $\alpha > 1$.

4.1. **Involution.** The proof of the following proposition and other claims in this paper that compute the Aubert involution, is based on the basic idea of [Jan18] (another possibility is to apply [AM20]).

Proposition 4.1. *Let* $\alpha \geq \frac{3}{2}$ *and* $m, n \in \mathbb{Z}_{\geq 0}$. *Denote*

$$\pi_{m,n} := L([\alpha - 1, \alpha + m]^{\mathbf{t}}; \delta([\alpha, \alpha + n]; \sigma)).$$

Then

(4.1) $$\pi^{\mathbf{t}}_{n,m} = \pi_{m,n}.$$

Proof. We prove formula (4.1) by induction with respect to

(4.2) $$\min(m, n).$$

We start with the basis of the induction, the case $\min(m, n) = 0$. To prove the formula in this case it is enough to prove that

$$\pi^{\mathbf{t}}_{0,n} = \pi_{n,0}$$

for $n \geq 0$. We prove it by a new induction. For $n = 0$, $\pi^{\mathbf{t}}_{0,0} = \pi_{0,0}$ by [Tad20, Proposition 4.6, (2)]. Suppose that the claim holds for some $n \geq 0$. Observe that

(4.3) $\pi_{0,n+1} \hookrightarrow [-\alpha] \times [-(\alpha - 1)] \rtimes \delta([\alpha, \alpha + n + 1]; \sigma)$
$$\hookrightarrow [-\alpha] \times [-(\alpha - 1)] \times [\alpha + n + 1] \rtimes \delta([\alpha, \alpha + n]; \sigma)$$
$$\cong [\alpha + n + 1] \times [-\alpha] \times [-(\alpha - 1)] \rtimes \delta([\alpha, \alpha + n]; \sigma).$$

From $\pi_{0,n} \hookrightarrow [-\alpha] \times [-(\alpha - 1)] \rtimes \delta([\alpha, \alpha + n]; \sigma)$ we get

$$(4.4) \qquad [\alpha + n + 1] \rtimes \pi_{0,n} \hookrightarrow [\alpha + n + 1] \times [-\alpha] \times [-(\alpha - 1)] \rtimes \delta([\alpha, \alpha + n]; \sigma).$$

One checks directly that the right hand side has a unique irreducible subrepresentation. This together with (4.3) and (4.4) implies that $\pi_{0,n+1} \hookrightarrow [\alpha + n + 1] \rtimes \pi_{0,n}$. Using this, one easily gets $\mu^*_{\{[\alpha+n+1]\}}(\pi_{0,n+1}) = [\alpha + n + 1] \otimes \pi_{0,n}$ (see Definition 2.2 for notation). Now

$$(4.5) \qquad \pi^{\mathbf{t}}_{0,n+1} \hookrightarrow [-(\alpha + n + 1)] \rtimes \pi^{\mathbf{t}}_{0,n} = [-(\alpha + n + 1)] \rtimes \pi_{n,0},$$

and $\pi^{\mathbf{t}}_{0,n+1}$ is the unique irreducible subrepresentation of the right-hand side (use (3) of Remark 2.4, and the inductive assumption). Since $\pi_{n+1,0}$ is a unique irreducible subrepresentation of the right-hand side of (4.5), we get that $\pi^{\mathbf{t}}_{0,n+1} = \pi_{n+1,0}$.

Now we prove the inductive step (for induction with respect to (4.2)). Fix $k \geq 0$, and suppose that formula (4.1) holds for all pairs (m, n) such that $\min(m, n) = k$. Now take a pair (m', n') such that $\min(m', n') = k + 1$. It is enough to prove formula (4.1) in the case of $n' \leq m'$. Denote $m' = m$ and $n' = n + 1$. Then we need to prove formula (4.1) for the pair $(m, n + 1)$, where $n < m$ and the inductive assumption implies that formula (4.1) holds for the pair (m, n).

Observe that

$$(4.6) \quad \pi_{m,n+1} \hookrightarrow L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}) \rtimes \delta([\alpha, \alpha + n + 1]; \sigma)$$
$$\hookrightarrow L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}) \times [\alpha + n + 1] \rtimes \delta([\alpha, \alpha + n]; \sigma)$$
$$\cong [\alpha + n + 1] \times L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}) \rtimes \delta([\alpha, \alpha + n]; \sigma).$$

Further, $\pi_{m,n} \hookrightarrow L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}) \rtimes \delta([\alpha, \alpha + n]; \sigma)$ implies

$$(4.7) \quad [\alpha + n + 1] \rtimes \pi_{m,n} \hookrightarrow [\alpha + n + 1] \times L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}) \rtimes \delta([\alpha, \alpha + n]; \sigma).$$

Denote the representation on the right-hand side by Π. We will show that Π has a unique irreducible subrepresentation by showing that

$$\gamma := [\alpha + n + 1] \times L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}) \otimes \delta([\alpha, \alpha + n]; \sigma)$$

has multiplicity one in $\mu^*(\Pi)$. This will imply $\pi_{m,n+1} \hookrightarrow [\alpha + n + 1] \rtimes \pi_{m,n}$.

Recall

$$(4.8) \quad \mu^*(\Pi) = M^*([\alpha + n + 1]) \times M^*(L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}})) \rtimes \mu^*(\delta([\alpha, \alpha + n]; \sigma)).$$

Suppose that $\tau_1 \otimes \tau_2$ is an irreducible subquotient of $\mu^*(\Pi)$ such that

$$\mathrm{supp}(\tau_1) = \mathrm{supp}([\alpha + n + 1] \times L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}))$$

(obviously, γ satisfies the assumption of $\tau_1 \otimes \tau_2$). Now we will analyze when we can get $\tau_1 \otimes \tau_2$ from the right hand side of (4.8). Suppose that $\tau_1 \otimes \tau_2 \leq \gamma_1 \times \gamma_2 \rtimes \gamma_3$, where γ_1, γ_2 and γ_3 are terms from the sums of

$$M^*([\alpha + n + 1]), M^*(L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}})) \quad \text{and} \quad \mu^*(\delta([\alpha, \alpha + n]; \sigma))$$

respectively. Considering the support of τ_1, formula (2.6) implies that γ_3 must be $1 \otimes \delta([\alpha, \alpha + n]; \sigma)$.

Further, if $\gamma_1 = [\alpha + n + 1] \otimes 1$, then considering the support of τ_1, formula (2.4) (actually (2.5) is enough) implies that γ_2 must be $L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}) \otimes 1$, and then $\tau_1 \otimes \tau_2 = \gamma_1 \times \gamma_2 \rtimes \gamma_3 = [\alpha + n + 1] \times L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}) \otimes \delta([\alpha, \alpha + n]; \sigma) = \gamma$.

Suppose $\gamma_1 \neq [\alpha + n + 1] \otimes 1$ (recall $M^*([\alpha + n + 1]) = 1 \otimes [\alpha + n + 1] + [\alpha + n + 1] \otimes 1 + [-(\alpha + n + 1)] \otimes 1$). Write $\gamma_2 = \gamma_2' \otimes \gamma_2''$. Then $[\alpha + n + 1]$ in the support of τ_1 must come from γ_2', i.e. it must be in $\mathrm{supp}(\gamma_2')$. Now formula (2.4) implies that in this case $[\alpha - 1]$ will also be in $\mathrm{supp}(\gamma_2')$, and therefore also in the support of τ_1, which contradicts our assumption on the support of τ_1. Therefore, we have proved multiplicity one of γ in $\mu^*(\Pi)$ which implies $\pi_{m,n+1} \hookrightarrow [\alpha + n + 1] \rtimes \pi_{m,n}$.

The last relation and Frobenius reciprocity imply

$$(4.9) \qquad [\alpha + n + 1] \otimes \pi_{m,n} \leq \mu^*(\pi_{m,n+1}).$$

Observe that $\pi_{m,n} \leq L([\alpha - 1, \alpha + m]^{\mathbf{t}}) \rtimes \delta([\alpha, \alpha + n]; \sigma))$ and (2.6) imply

$$s_{GL}(\pi_{m,n}) \leq M^*_{GL}(L([\alpha - 1, \alpha + m]^{\mathbf{t}})) \times \delta([\alpha, \alpha + n]) \otimes \sigma,$$

which further implies that $\mu^*(\pi_{m,n})$ does not have an irreducible subquotient of the form $[\alpha + n + 1] \otimes -$ (use (2.5) and (2.6)). Now this, (4.9) and Lemma 2.3 imply

$$\mu^*_{\{[\alpha+n+1]\}}(\pi_{m,n+1}) = [\alpha + n + 1] \otimes \pi_{m,n}.$$

Further, (3) of Remark 2.4 and the inductive assumption imply $\pi^{\mathbf{t}}_{m,n+1} \hookrightarrow [-(\alpha + n + 1)] \rtimes \pi_{n,m}$, which implies $\pi^{\mathbf{t}}_{m,n+1} = \pi_{n+1,m}$ since $\pi_{n+1,m}$ embeds into $[-(\alpha + n + 1)] \rtimes \pi_{n,m}$ as the unique irreducible subrepresentation. This completes the proof of the proposition $\qquad \square$

4.2. **Definition of** $(\psi, \epsilon)_{k,l}$ **in the case** $\alpha > 1$. The assumption $\alpha > 1$ and 3.14 imply that $\psi_\sigma = \psi_- \oplus E^\rho_{2\alpha-3,1} \oplus E^\rho_{2\alpha-1,1}$ for some $\psi_- \in \Psi_{\mathrm{ele.}} \cap \Psi_{\mathrm{d.d.r.}}$ (this defines ψ_-). Denote in the sequel

$$\psi_{k,l} := \psi_- \oplus E^\rho_{k,1} \oplus E^\rho_{1,l},$$

where $k, l \geq 0$ will be always chosen to be of the same parity as $2\alpha - 1$ (and therefore $\psi_{k,l} \in \Psi_{\mathrm{g.p.}}$, which implies $\psi_{k,l} \in \Psi_{\mathrm{ele.}}$). In the sequel, we will always chose k and l such that $\psi_{k,l}$ is a multiplicity one representation. Clearly, $(\psi_{2\alpha-1,2\alpha-3})_d = (\psi_\sigma)_d$.

Further, denote by $\epsilon_{k,l}$ the character of the component group of $\psi_{k,l}$ which extends ϵ_σ on ψ_-, and which satisfies

$$\epsilon_{k,l}(E^\rho_{k,1}) = \epsilon_\sigma(E^\rho_{2\alpha-1,1}),$$

$$\epsilon_{k,l}(E^\rho_{1,l}) = \epsilon_\sigma(E^\rho_{2\alpha-3,1}) \text{ if } \alpha > \tfrac{3}{2}, \text{ and } \epsilon_{k,l}(E^\rho_{1,l}) = 1 \text{ if } \alpha = \tfrac{3}{2}.$$

We will work throughout in this section with pairs $(\psi_{k,l}, \epsilon_{k,l})$. Therefore, to shorten notation in the sequel, we denote such a pair by $(\psi, \epsilon)_{k,l}$.

4.3. **On corresponding A-packets.**

Theorem 4.2. *Let* $\alpha \geq \tfrac{3}{2}$ *and* $m, n \in \mathbb{Z}_{\geq 0}$. *Using the notation for A-parameters introduced above, we have*

(1) $L([\alpha - 1, \alpha + m]^{\mathbf{t}}; \delta([\alpha, \alpha + n]; \sigma)) \in \Pi_{\psi_{2\alpha+1+2n,2\alpha+1+2m}}$. *In particular,* $L([\alpha - 1, \alpha + m]^{\mathbf{t}}; \delta([\alpha, \alpha + n]; \sigma))$ *is unitarizable.*

(2) For $m \neq n$ we have

$$(4.10) \qquad \pi((\psi, \epsilon)_{2\alpha+1+2n,2\alpha+1+2m}) = L([\alpha - 1, \alpha + m]^{\mathbf{t}}; \delta([\alpha, \alpha + n]; \sigma)).$$

Proof. The proof goes through several steps.

4.3.1. *Case of $m = -2$.* Using induction, we first prove the (well-known) simple fact that

$$\pi((\psi, \epsilon)_{2\alpha+1+2n,2\alpha-3})) = \delta([\alpha, \alpha + n]; \sigma), \quad n \geq -1.$$

Observe first that $\sigma = \pi((\psi, \epsilon)_{2\alpha-1,2\alpha-3})$. Therefore, we have a basis of induction. Suppose $n \geq 0$ and that the above formula holds for $n - 1$. Consider now $(\psi, \epsilon)_{2\alpha+1+2n,2\alpha-3}$. Then

$$b_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha-3}} = 2\alpha - 3, \quad a_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha-3}} = 2\alpha + 1 + 2n,$$

$$\delta_{a_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha-3}}} = 1.$$

Now by 3.9 we know that $\pi((\psi, \epsilon)_{2\alpha+1+2n,2\alpha-3})$ is the unique irreducible subrepresentation of

$$[(\delta_{(\psi,\epsilon)_{2\alpha+1+2n,2\alpha-3}})^{\frac{(2\alpha+1+2n)-1}{2}}] \rtimes \pi((\psi, \epsilon)_{2\alpha-1+2(n-1),2\alpha-3}) = [\alpha + n] \rtimes \delta([\alpha, \alpha + n - 1]; \sigma),$$

which easily implies $\pi((\psi, \epsilon)_{2\alpha+1+2n,2\alpha-3}) = \delta([\alpha, \alpha + n]; \sigma)$, and completes the proof of the inductive step.

4.3.2. *Proof of (4.10) for $-2 \leq m < n$.* We have proved above that the claim holds for $m = -2$. We now fix some $m \geq -1$ (together with $n > m$), and assume that the formula (4.10) holds for $m - 1$. We will prove by induction that it holds for m. Now

$$b_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha+1+2m}} = 2\alpha - 5, \quad a_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha+1+2m}} = 2\alpha + 1 + 2m,$$

$$\delta_{a_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha+1+2m}}} = -1,$$

with the exception that for $\alpha = \frac{3}{2}$ we take $b_{(\psi,\epsilon)_{2\alpha+1+2n,2\alpha-1}} = 0$. By 3.9 we know that $\pi((\psi, \epsilon)_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha+1+2m}})$ is the unique irreducible subrepresentation of

$$(4.11) \quad [(\delta_{\rho,(\psi,\epsilon)_{2\alpha+1+2n,2\alpha+1+2m}})^{\frac{(2\alpha+1+2m)-1}{2}}] \rtimes \pi((\psi, \epsilon)_{(\psi,\epsilon)_{2\alpha+1+2n,2\alpha-1+2m}})$$

$$= [-(\alpha + m)] \rtimes L([\alpha - 1, \alpha + m - 1]^{\mathbf{t}}; \delta([\alpha, \alpha + n]; \sigma)),$$

which implies formula (4.10), and completes the proof of the inductive step.

4.3.3. *Reverse setting, $n = 0$.* We now repeat the previous construction in the reversed setting. Denote by $\epsilon'_{k,l}$ the character of the component group of $\psi_{k,l}$ which extends ϵ_σ on ψ_-, and which satisfies

$$\epsilon'_{k,l}(E^\rho_{k,1}) = \epsilon_\sigma(E^\rho_{2\alpha-3,1}) \text{ if } \alpha > \tfrac{3}{2}, \text{ and } \epsilon'_{k,l}(E^\rho_{1,l}) = 1 \text{ if } \alpha = \tfrac{3}{2},$$

$$\epsilon'_{k,l}(E^\rho_{1,l}) = \epsilon_\sigma(E^\rho_{2\alpha-1,1}).$$

We claim that

$$(4.12) \qquad \pi((\psi, \epsilon')_{2\alpha-3,2\alpha+1+2m}) = L([\alpha, \alpha + m]^{\mathbf{t}}; \sigma), \quad m \geq -1.$$

The proof goes by induction. Observe that $\sigma = \pi((\psi, \epsilon')_{2\alpha-3,2\alpha-1})$, which is the basis of the induction. Suppose $m \geq 0$ and that the formula holds for $m - 1$. Then

$$b_{\rho,(\psi,\epsilon')_{2\alpha-3,2\alpha+1}} = 2\alpha - 3, \quad a_{\rho,(\psi,\epsilon')_{\rho,2\alpha-3,2\alpha+1+2m}} = 2\alpha + 1 + 2m,$$

$$\delta_{a_{\rho,(\psi,\epsilon')_{2\alpha-3,2\alpha+1+2m}}} = -1.$$

By 3.9 we know that $\pi((\psi, \epsilon')_{2\alpha-3,2\alpha+1+2m})$ is the unique irreducible subrepresentation of

$$[(\delta_{(\psi,\epsilon')_{2\alpha-3,2\alpha+1+2m}})^{\frac{(2\alpha+1+2m)-1}{2}}] \rtimes \pi((\psi, \epsilon')_{2\alpha-3,2\alpha-1+2(m-1)}) = [-\alpha-m] \rtimes L([\alpha, \alpha+m-1]^{\mathbf{t}}; \sigma),$$

which directly implies formula 4.12. This completes the proof of the inductive step.

4.3.4. *Reverse setting,* $-1 \leq n < m$. With $\epsilon'_{k,l}$ introduced in 4.3.3, we now prove the formula

$$(4.13) \qquad \pi((\psi, \epsilon')_{2\alpha+1+2n,2\alpha+1+2m}) = L([\alpha-1, \alpha+m]^{\mathbf{t}}; \delta([\alpha, \alpha+n]; \sigma)), \quad m > n \geq -1$$

by induction with respect to n. For the basis of induction for $n = -1$, we need to consider $(\psi, \epsilon')_{2\alpha-1,2\alpha+1+2m}$. Then

$$b_{\rho,(\psi,\epsilon')_{2\alpha-1,2\alpha+1+2m}} = 2\alpha - 5, \quad a_{\rho,(\psi,\epsilon')_{2\alpha-1,2\alpha+1+2m}} = 2\alpha - 1,$$

$$\delta_{a_{\rho(\psi,\epsilon')_{2\alpha-1,2\alpha+1+2m}}} = 1,$$

with the exception that for $\alpha = \frac{3}{2}$ we take $b_{(\psi,\epsilon')_{2\alpha-1,2\alpha+1+2m}} = 0$. By 3.9 we know that $\pi((\psi, \epsilon')_{2\alpha-1,2\alpha+1+2m})$ is the unique irreducible subrepresentation of

$$[(\delta_{(\psi,\epsilon')_{2\alpha-1,2\alpha+1+2m}})^{\frac{(2\alpha-1)-1}{2}}] \rtimes \pi((\psi, \epsilon')_{2\alpha-3,2\alpha+1+2m}) = [\alpha-1] \rtimes L([\alpha, \alpha+m]^{\mathbf{t}}; \sigma)$$

(the above equality follows from 4.3.3). Therefore

$$\pi((\psi, \epsilon')_{2\alpha-1,2\alpha+1+2m}) \hookrightarrow [\alpha-1] \times L([-(\alpha+m), -\alpha]^{\mathbf{t}}) \rtimes \sigma \cong L([-(\alpha+m), -\alpha]^{\mathbf{t}}) \times [\alpha-1] \rtimes \sigma$$

$$\cong L([-(\alpha+m), -\alpha]^{\mathbf{t}}) \times [-(\alpha-1)] \rtimes \sigma.$$

This obviously implies (4.13) for $n = -1$.

We go now to the inductive step. Suppose $n \geq 0$ and that formula (4.13) holds for $n-1$. Here

$$b_{\rho,(\psi,\epsilon')_{2\alpha+1+2n,2\alpha+1+2m}} = 2\alpha - 5, \quad a_{\rho,(\psi,\epsilon')_{2\alpha+1+2n,2\alpha+1+2m}} = 2\alpha + 1 + 2n,$$

$$\delta_{a_{\rho,(\psi,\epsilon')_{2\alpha+1+2n,2\alpha+1+2m}}} = 1.$$

Then $\pi((\psi, \epsilon')_{2\alpha+1+2n,2\alpha+1+2m})$ is the unique irreducible subrepresentation of

$$[(\delta_{\rho,(\psi,\epsilon')_{2\alpha+1+2n,2\alpha+1+2m}})^{\frac{(2\alpha+1+2n)-1}{2}}] \rtimes \pi((\psi, \epsilon')_{2\alpha+1+2(n-1),2\alpha+1+2m})$$

$$= [\alpha+n] \rtimes L([\alpha-1, \alpha+m]^{\mathbf{t}}; \delta([\alpha, \alpha+n-1]; \sigma)).$$

The last representation embeds into

$$(4.14) \quad \Gamma := [\alpha+n] \times L([-(\alpha+m), -(\alpha-1)]^{\mathbf{t}}) \rtimes \delta([\alpha, \alpha+n-1]; \sigma)$$

$$\cong L([-(\alpha+m), -(\alpha-1)]^{\mathbf{t}}) \times [\alpha+n] \rtimes \delta([\alpha, \alpha+n-1]; \sigma)$$

We will show that the multiplicity of

$$\gamma := [\alpha + n] \times L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}) \otimes \delta([\alpha, \alpha + n - 1]; \sigma)$$

in $\mu^*(\Gamma)$ is one. To get γ as a subquotient, from formula

$$\mu^*(\Gamma) = M^*([\alpha + n]) \times M^*(L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}})) \rtimes \mu^*(\delta([\alpha, \alpha + n - 1]; \sigma))$$

we see that on the last term on the right hand side we must take $1 \otimes \delta([\alpha, \alpha + n - 1]; \sigma)$ (to see this, one can consider cuspidal supports). If we did not take $[\alpha + n] \otimes 1$ from $M^*([\alpha + n])$, formula (2.5) implies that we would have positive exponents different from $\alpha + n$ on the left hand side of $\otimes$. Therefore, we must take $[\alpha + n] \otimes 1$, which further implies that from $M^*(L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}))$ we must take $L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}) \otimes 1$. This implies multiplicity one.

Therefore Γ has a unique irreducible subrepresentation. Since $L([-(\alpha+m), -(\alpha-1)]^{\mathbf{t}}) \rtimes \delta([\alpha, \alpha + n]; \sigma) \hookrightarrow \Gamma$, we get that (4.13) holds. This completes the proof of the inductive step.

4.3.5. *Case $m = n \geq 0$.* Denote $\psi_{\gg} := \psi_{2\alpha+3+2m, 2\alpha+1+2m}$, $\psi := \psi_{2\alpha+1+2m, 2\alpha+1+2m}$. We fix any standard order on $\mathrm{Jord}_\rho(\psi_{\gg})$, and denote it by $>_{\psi_{\gg}}$. Then this is a natural order. Define a bijection $\mathrm{Jord}_\rho(\psi_{\gg}) \to \mathrm{Jord}_\rho(\psi)$ which carries

$$\varphi : (\rho, 2\alpha + 3 + 2m, 1) \mapsto (\rho, 2\alpha + 1 + 2m, 1),$$

and is equal to the identity on the remaining elements. Using the bijection φ, we define total order $>_\psi$ on $\mathrm{Jord}_\rho(\psi)$ (i.e. $\varphi(u) >_\psi \varphi(v) \iff u >_{\psi_{\gg}} v$). This is an admissible order on $\mathrm{Jord}_\rho(\psi)$ and φ preserves the order (by definition of $>_\psi$). In this way $\mathrm{Jord}(\psi_{\gg})$ dominates $\mathrm{Jord}(\psi)$ and by [Mœg11, 3.1.2] or [Xu17a, section 8], we can get all the elements of Π_ψ from the elements of $\Pi_{\psi_{\gg}}$ applying $\mathrm{Jac}_{\alpha+m+1}$ (each application of the operator $\mathrm{Jac}_{\alpha+m+1}$ will result in either an irreducible representation or 0). Observe that

(4.15) $L([\alpha - 1, \alpha + m]^{\mathbf{t}}; \delta([\alpha, \alpha + m + 1]; \sigma)) \hookrightarrow$

$$L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}) \rtimes \delta([\alpha, \alpha + m + 1]; \sigma)$$

$$\hookrightarrow L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}) \times [\alpha + m + 1] \rtimes \delta([\alpha, \alpha + m]; \sigma)$$

$$\cong [\alpha + m + 1] \times L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}) \rtimes \delta([\alpha, \alpha + m]; \sigma).$$

Obviously

(4.16) $[\alpha + m + 1] \rtimes L([\alpha - 1, \alpha + m]^{\mathbf{t}}; \delta([\alpha, \alpha + m]; \sigma))$

$$\hookrightarrow [\alpha + m + 1] \times L([-(\alpha + m), -(\alpha - 1)]^{\mathbf{t}}) \rtimes \delta([\alpha, \alpha + m]; \sigma).$$

One sees directly that the last representation has a unique irreducible subrepresentation (showing that the multiplicity of $[\alpha + m + 1] \otimes L([-(\alpha+m), -(\alpha - 1)]^{\mathbf{t}}) \otimes \delta([\alpha, \alpha + m]; \sigma)$ in the Jacquet module is one). This implies

(4.17) $L([\alpha - 1, \alpha + m]^{\mathbf{t}}; \delta([\alpha, \alpha + m + 1]; \sigma))$

$$\hookrightarrow [\alpha + m + 1] \rtimes L([\alpha - 1, \alpha + m]^{\mathbf{t}}; \delta([\alpha, \alpha + m]; \sigma)).$$

Now Frobenius reciprocity implies that

$$\mathrm{Jac}_{\alpha+m+1}(L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+m+1];\sigma)))=L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+m];\sigma)),$$

and therefore, $L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+m];\sigma))$ is in the A-packet of $\psi_{2\alpha+1+2m,2\alpha+1+2m}$.

4.3.6. *Case* $0\le m<n$. Observe that if we take any $\psi'\in\Psi_{\mathrm{ele.}}\cap\Psi_{\mathrm{d.d.r.}}$ such that $(\psi')_d=(\psi_-)_d$ instead of ψ_- in 4.2 and use ψ' (instead of ψ_-) to define $\psi_{k,l}$, $\epsilon_{k,l}$ and $\psi'_{k,l}$, we get exactly the same results as we have obtained in the proof so far.

Assume below $0\le m<n$. Now in 4.3.2 put $\psi'=(\psi_-)^{\mathbf{t}}$ and denote the objects that correspond to $\psi_{k,l}$ and $\epsilon_{k,l}$ for this ψ' by $\psi''_{k,l}$ and $\epsilon''_{k,l}$ (recall $\psi''_{2\alpha+1+2n,2\alpha+1+2m}=L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+n];\sigma)))$. Then

$$(\psi''_{2\alpha+1+2n,2\alpha+1+2m})^{\mathbf{t}}=\psi_{2\alpha+1+2m,2\alpha+1+2n},$$

and $\epsilon''_{2\alpha+1+2n,2\alpha+1+2m}$ give the same diagonal restriction as $\psi'_{2\alpha+1+2m,2\alpha+1+2n}$ (defined in 4.3.4). Now 3.4 and Proposition 4.1 imply

$$(4.18)\quad \pi((\psi,\epsilon)_{2\alpha+1+2m,2\alpha+1+2n})=\pi((\psi''_{2\alpha+1+2n,2\alpha+1+2m})^{\mathbf{t}},\epsilon''_{2\alpha+1+2n,2\alpha+1+2m})$$
$$=\pi((\psi,\epsilon)_{2\alpha+1+2n,2\alpha+1+2m})^{\mathbf{t}}=L([\alpha-1,\alpha+m]^{\mathbf{t}};\delta([\alpha,\alpha+n];\sigma))^{\mathbf{t}}$$
$$=L([\alpha-1,\alpha+n]^{\mathbf{t}};\delta([\alpha,\alpha+m];\sigma)).$$

$\square$

Note that in the proof of the above theorem we have also proved what happens with the few additional cases where $m\ge-2$ and $n\ge-1$. We comment these mostly well known cases briefly in the following

Corollary 4.3. *Let* $m\ge-2$ *and* $n\ge-1$.
  (1) *The representation* $\delta([\alpha,\alpha+n];\sigma)$ *(resp.* $L([\alpha,\alpha+n]^{\mathbf{t}};\sigma)$*) is in* Π_ψ *for* $\psi=\psi_-\oplus E^\rho_{2\alpha-3,1}\oplus E^\rho_{2\alpha+1+2n,1}$ *(resp.* $\psi=\psi_-\oplus E^\rho_{1,2\alpha-3}\oplus E^\rho_{1,2\alpha+1+2n}$*).*
  (2) *For* $m,n\ge-1$, *the representations* $L([\alpha-1];\delta([\alpha,\alpha+n];\sigma))$ *and* $L([\alpha-1,\alpha+m]^{\mathbf{t}};\sigma)$ *are in A-packets. These representations are at the end of complementary series if* $n\ge0$ *(resp* $m\ge0$*).*
  (3) $\delta([\alpha,\alpha+n];\sigma)^{\mathbf{t}}=L([\alpha,\alpha+n]^{\mathbf{t}};\sigma)$.
  (4) $L([\alpha-1];\delta([\alpha,\alpha+n];\sigma))^{\mathbf{t}}=L([\alpha-1,\alpha+n]^{\mathbf{t}};\sigma)$.

Proof. The first three claims are proved in the previous theorem. It remains to consider only (4). This is very simple to prove by similar methods as these used in the proof of Proposition 4.1, and therefore we omit them. $\square$

5. Case of reducibility 0

In this and the following two sections we will handle the remaining reducibilities, i.e. $\alpha=0,\frac{1}{2}$ and 1, and write down A-packets and representations in them which can be considered as analogous cases for these reducibilities. It is very easy to get that they are in A-packets. We will also give some additional information about them (formulas for the

Aubert involution, and to which characters of the component groups they correspond in the case of discrete parameter).

In this section ρ, σ and α are as in 3.14, and we assume that $\alpha = 0$. We fix a decomposition of $\rho \rtimes \sigma$ in (2.7). We denote by ψ_σ the tempered elementary discrete parameter such that $\sigma \in \Pi_{\psi_\sigma}$. Applying [Mœg11, Proposition 6.0.3] to the parameter $\psi_\sigma \oplus E^\rho_{2n+1,1} \oplus E^\rho_{1,2m+1}$ (for which we know by 3.3 that it is again an A-parameter) we get directly that if $m, n \geq 0$, then

$$L([1,m]^{\mathbf{t}}; \delta([0,n]_\pm; \sigma)) \in \Pi_{\psi_\sigma \oplus E^\rho_{2n+1,1} \oplus E^\rho_{1,2m+1}}.$$

In the following theorem, we give additional information about elements of these packets in the case $m \neq n$.

5.1. Definition of $(\psi, \epsilon^\pm)_{k,l}$ in the case $\alpha = 0$. Denote in this section

$$\psi := \psi_\sigma, \qquad \psi_{k,l} := E^\rho_{k,1} \oplus E^\rho_{1,l},$$

where $k, l \geq 0$ will always be chosen to be of odd parity, and denote by $\epsilon^\pm_{k,l}$ the character of the component group of $\psi_{k,l}$ which extends ϵ_σ on ψ, and which is equal to ± 1 on remaining two elements. Similarly as before, we denote a pair $(\psi_{k,l}, \epsilon^\pm_{k,l})$ by $(\psi, \epsilon^\pm)_{k,l}$.

5.2. On corresponding A-packets and involution.

Theorem 5.1. *Let $n, m \geq 0$. Then*

(1) *$L([1,m]^{\mathbf{t}}; \delta([0,n]_\pm; \sigma)) \in \Pi_{\psi_{2n+1,2m+1}}$. In particular, $L([1,m]^{\mathbf{t}}; \delta([0,n]_\pm; \sigma))$ are unitarizable.*

(2) *For $m \neq n$ we have $\pi((\psi, \epsilon^\xi)_{2n+1,2m+1}) = L([1,m]^{\mathbf{t}}; \delta([0,n]_{sign(n-m)\xi}; \sigma))$, $\xi \in \{\pm\}$.*

Proposition 5.2. *Let $\alpha = 0$ and $m, n \in \mathbb{Z}_{\geq 0}$. Denote*

$$\pi^\pm_{m,n} := L([1,m]^{\mathbf{t}}; \delta([0,n]_\pm; \sigma)).$$

Then

$$(\pi^\pm_{n,m})^{\mathbf{t}} = \pi^\mp_{m,n}.$$

Remark 5.3. *Elements of an A-packet ψ of good parity which are not discrete are obtained from some suitable discrete A-packet $\psi_\gg$ of a bigger group by a procedure described in [Mœg11, 3.1.2] (see also [Xu17a, section 8]). Here one applies Jacquet module operators to representations $\pi(\psi, \mathbf{t}, \boldsymbol{\eta})$ to get elements of Π_ψ (the result can also be 0). The result can depend on an admissible order that one fixes on $\mathrm{Jord}_\rho(\psi)$. We comment below an example where one gets different results for A-packets corresponding to $m = n > 1$ for an admissible order which satisfies $(\rho, m, 1) >_\psi (\rho, 1, m)$ and admissible order which satisfies $(\rho, 1, m) >_\psi (\rho, m, 1)$ (in our case $\mathbf{t}$ is the zero function, and $\boldsymbol{\eta} = \epsilon^\pm$). The reason is that on $\mathrm{Jord}(\psi_\gg)$ we need to take a natural order. Therefore, in the case of the first admissible order, one gets*

$$\mathrm{Jac}_{m+1}(L([1,m]^{\mathbf{t}}; \delta([0,m+1]_\pm; \sigma))) = L([1,m]^{\mathbf{t}}; \delta([0,m]_\pm; \sigma)),$$

while in the case of the second admissible order, one gets

$$\mathrm{Jac}_{-(m+1)}(L([1,m+1]^{\mathbf{t}}; \delta([0,m]_\mp; \sigma))) = L([1,m]^{\mathbf{t}}; \delta([0,m]_\mp; \sigma)).$$

Note that if we change admissible orders as above in the setting of Theorem 4.2 for the case $m = n \geq 0$, the results there remain unchanged. The same holds for settings of Theorem 6.2.

6. Case of reducibility $\frac{1}{2}$

As in the previous sections, ρ, σ and α are as in 3.14, and we assume in this section that $\alpha = \frac{1}{2}$. By ψ_σ we denoted the tempered elementary discrete A-parameter such that $\sigma \in \Pi_{\psi_\sigma}$. Applying [Mœg11, Proposition 6.0.3] to $\psi_\sigma \oplus E^\rho_{2n,1} \oplus E^\rho_{1,2m}$ (which is also an A-parameter by 3.3), we get immediately that for $m, n \geq 0$,

$$L([\tfrac{1}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([\tfrac{1}{2}, \tfrac{2n-1}{2}]; \sigma)) \in \Pi_{\psi_\sigma \oplus E^\rho_{2n,1} \oplus E^\rho_{1,2m}}.$$

Later we will give additional information regarding these packets.

6.1. **Involution.** First we will see how these representations transform under the Aubert involution.

Proposition 6.1. *For $m, n \geq 1$ denote*

$$(6.1) \qquad \pi^+_{m,n} := L([\tfrac{1}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([\tfrac{1}{2}, \tfrac{2n-1}{2}]; \sigma)), \quad \pi^-_{m,n} := L([\tfrac{3}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([-\tfrac{1}{2}, \tfrac{2n-1}{2}]_-; \sigma)).$$

Then

$$(6.2) \qquad\qquad\qquad\qquad\qquad (\pi^+_{m,n})^{\mathbf{t}} = \pi^-_{n,m}.$$

6.2. **Definition of $(\psi, \epsilon^\pm)_{k,l}$ in the case $\alpha = \frac{1}{2}$.** Denote by

$$\psi := \psi_\sigma, \qquad \psi_{k,l} := E^\rho_{k,1} \oplus E^\rho_{1,l},$$

where $k, l \geq 0$ will be always chosen to be of even parity, and denote by $\epsilon^\pm_{k,l}$ the character of the component group of $\psi_{k,l}$ which extends ϵ_σ on ψ, and which is equal to ± 1 on the remaining two elements. As before, we denote a pair $(\psi_{k,l}, \epsilon^\pm_{k,l})$ by $(\psi, \epsilon^\pm)_{k,l}$.

6.3. **On corresponding A-packets.** With the above notation (and $\pi^\pm_{m,n}$ introduced in Proposition 6.1), we have the following

Theorem 6.2. *Let $m, n \geq 1$. Then the following holds:*

(1)

$$L([\tfrac{1}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([\tfrac{1}{2}, \tfrac{2n-1}{2}]; \sigma)), \ \ L([\tfrac{3}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([-\tfrac{1}{2}, \tfrac{2n-1}{2}]_-; \sigma)) \in \Pi_{\psi_{2n,2m}}.$$

In other words, $\pi^\pm_{m,n} \in \Pi_{\psi_{2n,2m}}$. In particular, representations $\pi^\pm_{m,n}$ are unitarizable.

(2)

$$(6.3) \qquad \pi((\psi, \epsilon^+)_{2n,2m}) = \begin{cases} L([\tfrac{1}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([\tfrac{1}{2}, \tfrac{2n-1}{2}]; \sigma)) = \pi^+_{m,n}, & m < n, \\ L([\tfrac{3}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([-\tfrac{1}{2}, \tfrac{2n-1}{2}]_-; \sigma)) = \pi^-_{m,n}, & n < m. \end{cases}$$

(3)

$$(6.4) \qquad \pi((\psi, \epsilon^-)_{2n,2m}) = \begin{cases} L([\tfrac{3}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}; \delta([-\tfrac{1}{2}, \tfrac{2n-1}{2}]_-; \sigma)) = \pi^-_{m,n}, & m < n. \\ L([\tfrac{1}{2}, \tfrac{2m-1}{2}]^{\mathbf{t}}, \delta([\tfrac{1}{2}, \tfrac{2n-1}{2}]; \sigma)) = \pi^+_{m,n}, & n < m. \end{cases}$$

7. Case of reducibility at 1

Again in this section ρ, σ and α are as in 3.14, and we assume that $\alpha = 1$. Denote by ψ_σ the tempered elementary discrete parameter such that $\sigma \in \Pi_{\psi_\sigma}$. Applying Proposition 6.0.3 of [Mœg11] to $\psi_\sigma \oplus E^\rho_{2n+1,1} \oplus E^\rho_{1,2m+1}$ we get that for $m, n \geq 1$,

$$L([1,m]^{\mathbf{t}}; \tau([0]_\pm; \delta([1,n]; \sigma))) \in \Pi_{\psi_\sigma \oplus E^\rho_{2n+1,1} \oplus E^\rho_{1,2m+1}}.$$

Before we give more information about these packets, we calculate the Aubert involutions of the above representations.

7.1. **Involution.** We start with the following

Lemma 7.1. *For $n \geq 1$ we have*

$$(7.1) \qquad \tau([0]_x; \delta([1,n]; \sigma))^{\mathbf{t}} = \begin{cases} L([1,n]^{\mathbf{t}}; [0] \rtimes \sigma), & x = +, \\ L([0,1],[2,n]^{\mathbf{t}}; \sigma), & x = -. \end{cases}$$

The above representations are unitarizable.

Now we have the following

Proposition 7.2. *Let $m, n \geq 1$. Denote*

$$\pi^\pm_{m,n} := L([1,m]^{\mathbf{t}}; \tau([0]_\pm; \delta([1,n]; \sigma))), \quad \tau^-_{m,n} := L([2,m]^{\mathbf{t}}; \delta([-1,n]_-; \sigma)).$$

Then

$$(7.2) \qquad (\pi^+_{m,n})^{\mathbf{t}} = \pi^+_{n,m}, \qquad (\pi^-_{m,n})^{\mathbf{t}} = \tau^-_{n,m}.$$

7.2. **Definition of $(\psi, \epsilon^\pm)_{k,l}$ and $\epsilon^{+;-,-}_{k,l}$ in the case $\alpha = 1$.** Denote in the rest of this section

$$\psi_{k,l} := \psi \oplus E^\rho_{k,1} \oplus E^\rho_{1,l},$$

where $k, l \geq 0$ will be always chosen to be of odd parity. Set

$$\xi = \epsilon_\sigma(\rho, 1, 1).$$

Next we define characters $\epsilon^\pm_{k,l}$ of the component group of $\psi^\pm_{k,l}$ when k and l are different odd integers > 1. They coincide with ψ_σ on $\mathrm{Jord}(\psi_\sigma) - ((\rho, 1, 1))$ and satisfy

$$\epsilon^\pm_{k,l}(\rho, 1, 1) = \epsilon^\pm_{k,l}(\rho, \min(k,l), \delta_{\min(k,l)}) = \pm\xi,$$

$$\epsilon^\pm_{k,l}(\rho, \max(k,l), \delta_{\min(k,l)}) = \xi$$

(we need to assume that $\epsilon^\pm_{k,l}$ is equal on the pair of blocks for which $E^\rho_{k',l'} = E^\rho_{k'',l''}$). As before, we denote such a pair $(\psi_{k,l}, \epsilon^\pm_{k,l})$ by $(\psi, \epsilon^\pm)_{k,l}$. Denote

$$\epsilon^{+;-,-}_{k,l}$$

$$\epsilon^{+;-,-}_{k,l}(\rho, 1, 1) = \xi,$$

$$\epsilon^{+;-,-}_{k,l}(\rho, \min(k,l), \delta_{\min(k,l)}) = \epsilon^{+;-,-}_{k,l}(\rho, \max(k,l), \delta_{\min(k,l)}) = -\xi.$$

7.3. **On corresponding A-packets.** With the above notation we have the following

Theorem 7.3. *Let* $m, n \geq 1$. *Then the following holds:*

(1)

$$L([1, m]^{\mathbf{t}}; \tau([0]_{\pm}; \delta([1, n]; \sigma))), L([2, m]^{\mathbf{t}}; \delta([-1, n]_{-}; \sigma)) \in \Pi_{\psi_{\sigma} \oplus E^{\rho}_{2n+1,1} \oplus E^{\rho}_{1,2m+1}}.$$

In other words, $\pi^{\pm}_{m,n}, \tau^{-}_{m,n} \in \Pi_{\psi_{2n+1,2m+1}}$. *In particular, representations* $\pi^{\pm}_{m,n}$ *and* $\tau^{-}_{m,n}$ *are unitarizable.*

(2)

$$(7.3) \qquad \pi((\psi, \epsilon^+)_{2n+1,2m+1}) = \begin{cases} L([1, m]^{\mathbf{t}}; \tau([0]_{-}; \delta([1, n]; \sigma))) = \pi^{-}_{m,n}, & m < n, \\ L([2, m]^{\mathbf{t}}; \delta([-1, n]_{-}; \sigma)) = \tau^{-}_{m,n}, & n < m. \end{cases}$$

(3)

$$(7.4) \qquad \pi((\psi, \epsilon^-)_{2n+1,2m+1}) = L([1, m]^{\mathbf{t}}; \tau([0]_{+}; \delta([1, n]; \sigma))) = \pi^{+}_{m,n}, \quad m \neq n.$$

(4)

$$(7.5) \qquad \pi((\psi, \epsilon^{+--})_{2n+1,2m+1}) = \begin{cases} L([2, m]^{\mathbf{t}}; \delta([-1, n]_{-}; \sigma)) = \tau^{-}_{m,n}, & m < n, \\ L([1, m]^{\mathbf{t}}; \tau([0]_{-}; \delta([1, n]; \sigma))) = \pi^{-}_{m,n}, & n < m. \end{cases}$$

8. ON IRREDUCIBLE UNITARIZABLE SUBQUOTIENTS AT CRITICAL POINTS

Definition 8.1. *Let* $\rho_1, \ldots, \rho_k \in \mathcal{C}$ *and let* σ *be an irreducible cuspidal representation of a classical group. Assume that for any* i *we have*

(1) $\rho^u_i \cong (\rho^u_i)\check{}$;
(2) *the set* $\{e(\rho_j) : \rho^u_j \cong \rho^u_i\}$ *is a* $\mathbb{Z}$-*segment in* $\frac{1}{2}\mathbb{Z}$ *(possibly with multiplicities);*
(3) *the* $\mathbb{Z}$-*segment in (2) contains the reducibility exponent* $\alpha_{\rho^u_i, \sigma}$.

Then, we say that the representation $\rho_1 \times \cdots \times \rho_k \rtimes \sigma$ *is of critical type. If additionally* π *is an irreducible subquotient of* $\rho_1 \times \cdots \times \rho_k \rtimes \sigma$, *then we also say that* π *is of critical type.*

The aim of this section is to prove the following

Theorem 8.2. *Let* π *be an irreducible unitarizable subquotient of a representation*

$$\rho_1 \times \cdots \times \rho_k \rtimes \sigma, \qquad k \leq 3$$

of critical type. Then π *is contained in an A-packet.*

Proof. First we recall some simple general facts which will considerably shorten the proof of the theorem.

8.0.1. *Some simple remarks about A-packets.*

   (1) Each irreducible tempered representation is an element of some A-packet (with tempered A-parameter).
   (2) If π is an element of an elementary discrete A-packet, then $\pi^{\mathbf{t}}$ is also an element of an elementary discrete A-packet.
   (3) Each irreducible cotempered representation is contained in an A-packet (with cotempered A-parameter).

For coranks 0 and 1 the theorem follows directly from remarks in 8.0.1. It remains to consider coranks 2 and 3. We will consider below only the cases which are not covered by remark 8.0.1. We will also prove the theorem in the case when all ρ_i^u are the same, denoted by ρ (the proof in the other case is very simple, and we omit it here). We fix an irreducible cuspidal representation σ of a classical group. We assume that $\sigma = \pi(\psi_\sigma, \epsilon_\sigma)$ for some $\psi_\sigma \in \Psi_{\text{ele.}} \cap \Psi_{\text{d.d.r.}}$. Denote $\alpha = \alpha_{\rho,\sigma}$ (as usual). If we have some $\psi \in \Psi_{\text{ele.}}$, and write $\text{Jord}_\rho(\psi) = ((a_1, b_1), \ldots, (a_k, b_k))$, then we will always assume that the enumeration satisfies $\max(a_1, b_1) \leq \cdots \leq \max(a_k, b_k)$.

Below we will consider exponents $(x_1, \ldots, x_k)$, $k = 2$ or 3, the representation $\nu^{x_k}\rho \times \cdots \times \nu^{x_1}\rho \rtimes \sigma$ of critical type, and irreducible unitarizable subquotients of it. We will give precise references about where these representations were considered in [Tad20], and denote their irreducible unitarizable subquotients in the same way as in [Tad20] (therefore, we will not recall here this notation).

The arguments below are usually simple (and we have already used them in the previous part of the paper). Therefore, we will only sketch them very briefly below.

When we have a parameter (ψ', ϵ') as below, and when we get a new parameter (ψ'', ϵ'') by replacing $(\rho, a', b') \in \psi'$ by (ρ, a'', b''), then we will always assume that $\epsilon'(\rho, a, b)' = \epsilon''(\rho, a'', b'')$ and that ϵ' and ϵ'' coincide on remaining blocks.

Also if we get (ψ'', ϵ'') from (ψ', ϵ') by replacing some elements (ρ, a, b) with (ρ, b, a), then we will assume $\epsilon''(\rho, a, b) = \epsilon'(\rho, c, d)$ if $\max(a, b) = \max(c, d)$.

8.1. **Corank 2.**

8.1.1. *Case* $(\alpha - 1, \alpha), \alpha > 1$ (3.4.3 of [Tad20]). Here all 4 irreducible subquotients are unitarizable. One is square integrable, and another is its Aubert involution. Therefore, we need to consider only representations

$$\pi_2 := L([\alpha - 1]; \delta([\alpha]; \sigma)), \quad \pi_3 := L([\alpha - 1], [\alpha]; \sigma),$$

where $\pi_2^{\mathbf{t}} = \pi_3$. Both above representations are contained in A-packets by (2) of Corollary 4.3 (in the corollary, consider the case of $n = 0$ and $m = 0$, respectively).

8.1.2. *Case* $(0.1), \alpha = 0$ (3.4.6 of [Tad20]). Here all 5 irreducible subquotients are unitarizable. Two of them are square integrable, and another two are their Aubert duals. Therefore we need to consider only

$$\pi_2 := L([0, 1]; \sigma).$$

Let $\psi := \psi_\sigma \oplus E_{2,2}^\rho$. Then $\pi_2 \in \Pi_\psi$ by Proposition 6.0.3 of [Mœg11] (construction "L-packet inside A-packet").

8.2. Corank 3.

8.2.1. *Case* $(\alpha - 1, \alpha, \alpha + 1), \alpha > 1$ (4.5 of [Tad20]). Here we have 4 irreducible unitarizable subquotients. One of them is square integrable, and another is its Aubert involution. Therefore, we need to consider the following representations

$$\pi_3 := L([\alpha - 1]; \delta([\alpha, \alpha + 1]; \sigma)), \quad \pi_4 := L([\alpha + 1], [\alpha], [\alpha - 1]; \sigma).$$

Both above representations are contained in A-packets by (2) of Corollary 4.3 (in the corollary, consider the case of $n = 1$ and $m = 1$, respectively).

8.2.2. $(\alpha - 1, \alpha, \alpha), \alpha > 1$ (4.6 of [Tad20]). Here only one irreducible subquotient is unitarizable:

$$\pi_0 := L([\alpha - 1], [\alpha]; \delta([\alpha]; \sigma)).$$

The above representation is contained in an A-packet by (1) of Theorem 4.2 (in the theorem, consider the case of $n = 0$ and $m = 0$; recall that C. Mœglin has shown that this representation is in an A-packet in Appendix A of [Tad20]).

8.2.3. *Case* $(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}), \alpha = \frac{3}{2}$ (4.7.2 of [Tad20]). Here all 8 irreducible subquotients are unitarizable. Two of them are tempered, while another two are cotempered. Therefore it remains to consider representations

$$\pi_5 := L([\tfrac{3}{2}]; \delta([-\tfrac{1}{2}, \tfrac{1}{2}]) \rtimes \sigma), \quad \pi_6 := L([\tfrac{1}{2}], [\tfrac{1}{2}]; \delta([\tfrac{3}{2}]; \sigma)),$$
$$\pi_7 := L([-\tfrac{1}{2}, \tfrac{3}{2}]; \sigma), \quad \pi_8 := L([\tfrac{1}{2}]; \delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}]; \sigma)).$$

For π_5, consider ψ_σ', which we get from ψ_σ by replacing $(2, 1)$ with $(1, 2)$ in $\text{Jord}_\rho(\psi)$. Now increase $(1, 2)$ to $(1, 4)$ and denote new parameter by ψ'. We get $L([\tfrac{3}{2}]; \sigma)$ in the packet of ψ'. Now add $(2, 1), (2, 1)$ to $\text{Jord}_\rho(\psi')$. Now applying [Tad09, Proposition 5.3] we get that π_5 is in this new packet.

For π_6, increase $(2, 1)$ to $(4, 1)$ in $\text{Jord}_\rho(\sigma)$ and denote this packet b ψ'. Then $\delta([\tfrac{3}{2}]; \sigma)$ is in the new packet. Now add $(1, 2), (1, 2)$ to $\text{Jord}_\rho(\psi')$, and we get π_8 in the packet of this new parameter.

Observe that $\pi_7 \in \Pi_\psi$, where $\psi := \psi_\sigma \oplus E_{3,2}^\rho$.

For π_8, recall that $(2, 1) \in \text{Jord}_\rho(\psi_\sigma)$. Then increasing $(2, 1)$ to $(6, 1)$ we get $\delta([\tfrac{3}{2}, \tfrac{5}{2}]; \sigma)$ in the packet. Adding $(2, 1)$ and then replacing it with $(4, 1)$, we get (in two steps) that $\delta_{\text{s.p.}}([\tfrac{1}{2}, \tfrac{3}{2}], [\tfrac{3}{2}, \tfrac{5}{2}]; \sigma)$. Adding $(1, 2)$ to the previous packet, we get $L([\tfrac{1}{2}]; \delta_{\text{s.p.}}([\tfrac{1}{2}, \tfrac{3}{2}], [\tfrac{3}{2}, \tfrac{5}{2}]; \sigma))$ in the packet. This representation is (by our construction) in Π_ψ, where $\text{Jord}_\rho(\psi) = ((6, 1), (4, 1), (1, 2))$. Put a standard order on $\text{Jord}_\rho(\psi)$. Denote by ψ' the A-parameter obtained from ψ by changing $\text{Jord}_\rho(\psi)$ to $\text{Jord}_\rho(\psi') = (4, 1), (2, 1), (1, 2)$. Consider a standard order on $\text{Jord}_\rho(\psi')$ satisfying $(2, 1) >_{\psi'} (1, 2)$, and let $\varphi : \text{Jord}_\rho(\psi) \to \text{Jord}_\rho(\psi')$ be a standard bijection which preserves order. Then $\text{Jord}_\rho(\psi)$ dominates $\text{Jord}_\rho(\psi')$ with respect to $>_{\psi'}$. By [Mœg11, 3.1.2] or [Xu17a, section 8], $\text{Jac}_{[\frac{5}{2}]} \circ \text{Jac}_{[\frac{3}{2}]}(L([\tfrac{1}{2}]; \delta_{\text{s.p.}}([\tfrac{1}{2}, \tfrac{3}{2}], [\tfrac{3}{2}, \tfrac{5}{2}]; \sigma)))$,

we get an element of the packet of ψ' or 0. To compute the last representation, observe that

$$L([\tfrac{1}{2}]; \delta_{\text{s.p.}}([\tfrac{1}{2}, \tfrac{3}{2}], [\tfrac{3}{2}, \tfrac{5}{2}]; \sigma)) \hookrightarrow [\tfrac{3}{2}] \times [-\tfrac{1}{2}] \rtimes \delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}, \tfrac{5}{2}]; \sigma),$$

and that the last representation has a unique irreducible subrepresentation. This implies

$$L([\tfrac{1}{2}]; \delta_{\text{s.p.}}([\tfrac{1}{2}, \tfrac{3}{2}], [\tfrac{3}{2}, \tfrac{5}{2}]; \sigma)) \hookrightarrow [\tfrac{3}{2}] \rtimes L([\tfrac{1}{2}]; \delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}, \tfrac{5}{2}]; \sigma)),$$

which easily implies

$$\text{Jac}_{[\tfrac{3}{2}]}(L([\tfrac{1}{2}]; \delta_{\text{s.p.}}([\tfrac{1}{2}, \tfrac{3}{2}], [\tfrac{3}{2}, \tfrac{5}{2}]; \sigma))) = L([\tfrac{1}{2}]; \delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}, \tfrac{5}{2}]; \sigma)).$$

Observe that

$$\delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}, \tfrac{5}{2}]; \sigma)) \hookrightarrow [\tfrac{1}{2}] \times [\tfrac{5}{2}] \rtimes \delta([\tfrac{3}{2}]; \sigma) \cong [\tfrac{5}{2}] \times [\tfrac{1}{2}] \rtimes \delta([\tfrac{3}{2}]; \sigma).$$

Since the last representation has a unique irreducible subrepresentation, we get

$$\delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}, \tfrac{5}{2}]; \sigma)) \hookrightarrow [\tfrac{5}{2}] \rtimes \delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}]; \sigma).$$

Now

$$L([\tfrac{1}{2}]; \delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}, \tfrac{5}{2}]; \sigma)) \hookrightarrow [-\tfrac{1}{2}] \times [\tfrac{5}{2}] \rtimes \delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}]; \sigma) \cong [\tfrac{5}{2}] \times [-\tfrac{1}{2}] \rtimes \delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}]; \sigma).$$

Since the last representation has a unique irreducible subrepresentation, we get

$$L([\tfrac{1}{2}]; \delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}, \tfrac{5}{2}]; \sigma)) \hookrightarrow [\tfrac{5}{2}] \rtimes L([\tfrac{1}{2}]; \delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}]; \sigma)).$$

This implies $\text{Jac}_{[\tfrac{5}{2}]}(L([\tfrac{1}{2}]; \delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}, \tfrac{5}{2}]; \sigma))) = L([\tfrac{1}{2}]; \delta_{\text{s.p.}}([\tfrac{1}{2}], [\tfrac{3}{2}]; \sigma))$, and completes the proof that π_8 is in an A-packet.

8.2.4. $(\alpha - 2, \alpha - 1, \alpha), \alpha > 2$ (4.8.1 of [Tad20]). Here all 8 irreducible subquotients are unitarizable. They are

$$\pi_1 = \delta_{\text{s.p.}}([\alpha - 2], [\alpha - 1], [\alpha]; \sigma), \quad \pi_2 = L([\alpha - 2]; \delta_{\text{s.p.}}([\alpha - 1], [\alpha]; \sigma)),$$
$$\pi_3 := L([\alpha - 1], [\alpha - 2]; \delta([\alpha]; \sigma)), \quad \pi_4 := L([\alpha - 2, \alpha - 1]; \delta([\alpha]; \sigma)),$$
$$\pi_5 := L([\alpha], [\alpha - 1], [\alpha - 2]; \sigma), \quad \pi_6 := L([\alpha], [\alpha - 2, \alpha - 1]; \sigma),$$
$$\pi_7 := L([\alpha - 1, \alpha], [\alpha - 2]; \sigma), \quad \pi_8 := L([\alpha - 2, \alpha]; \sigma).$$

We have $\pi_1^t = \pi_8$, $\pi_2^t = \pi_7$, $\pi_3^t = \pi_6$, $\pi_4^t = \pi_5$. Since π_1 is tempered, and π_8 cotempered, it remains to consider 6 representations.

For π_7 observe that $\delta_{\text{s.p.}}([\alpha - 1], [\alpha]; \sigma)$ is in the A-packet corresponding to ψ_1, where we get ψ_1 from ψ_σ by replacing $(2\alpha - 3, 1), (2\alpha - 1, 1)$ with $(2\alpha - 1, 1), (2\alpha + 1)$ in $\text{Jord}_\rho(\psi_\sigma)$. Note that $\text{Jord}_\rho(\psi_1)$ ends with $(2\alpha - 5, 1), (2\alpha - 1, 1), (2\alpha + 1, 1)$. Now $L([\alpha - 1, \alpha]; \sigma)$ (which is the Aubert dual of previous discrete series by (3) of Proposition 3.7 in [Tad20]) is in the A-packet of ψ_1^t and $\text{Jord}_\rho(\psi_1^t)$ ends with $(1, 2\alpha - 5), (1, 2\alpha - 1), (1, 2\alpha + 1)$. Increasing $(1, 2\alpha - 5)$ to $(1, 2\alpha - 3)$, we get π_7 in the new A-packet. Since the last A-packet is discrete and elementary, π_2 is also in an A-packet.

For π_5, first observe that $\sigma \in \Pi_{\psi'_\sigma}$, where one gets ψ'_σ from ψ_σ by replacing $(2\alpha - 3, 1)$ with $(1, 2\alpha - 3)$ in $\text{Jord}_\rho(\psi)$, i.e. $\text{Jord}_\rho(\psi'_\sigma) = \{\ldots, (1, 2\alpha - 3), (2\alpha - 1, 1)\}$. One defines a new A-parameter ψ by increasing the last block by 2, and then the previous block also by 2 (now $\text{Jord}_\rho(\psi)$ ends with $(2\alpha - 5, 1), (1, 2\alpha - 1), (2\alpha + 1, 1)$), and gets $L([\alpha - 1]; \delta([\alpha]; \sigma)) \in \Pi_\psi$.

This is an elementary discrete packet. Therefore $L([\alpha-1],[\alpha];\sigma) = L([\alpha-1];\delta([\alpha];\sigma))^{\mathbf{t}}$ is in an elementary discrete packet of $\psi^{\mathbf{t}}$, and $\psi^{\mathbf{t}}$ ends with $(1, 2\alpha-5), (2\alpha-1,1), (1, 2\alpha+1)$. Replace $(1, 2\alpha-5)$ with $(1, 2\alpha-3)$ in $\psi^{\mathbf{t}}$. Then in this new packet we have the unique irreducible subrepresentation of $[-(\alpha-2)] \rtimes L([\alpha-1],[\alpha];\sigma)$. It is easy to show that this unique irreducible subrepresentation is π_5. Therefore, π_5 is in an A-packet. Further π_4 is in an A-packet since π_5 is in an elementary discrete A-packet (and $\pi_4^{\mathbf{t}} = \pi_5$).

For π_3 consider ψ_σ' which we get from ψ_σ by replacing $(2\alpha-5,1), (2\alpha-3,1)$ with $(1, 2\alpha-5), (1, 2\alpha-3)$ in $\mathrm{Jord}_\rho(\psi_\sigma)$. Then $\mathrm{Jord}_\rho(\psi_\sigma')$ ends with $(1, 2\alpha-5), (1, 2\alpha-3), (2\alpha-1,1)$. Now we proceed in the usual way (increasing each of these blocks by 2), and we get π_3 in the packet. Further π_6 is in an A-packet since π_3 is in an elementary discrete A-packet (and $\pi_3^{\mathbf{t}} = \pi_6$).

8.2.5. *Case* $(0, 1, 2), \alpha = 2$ (4.8.2 of [Tad20]). Here all 8 irreducible subquotients are unitarizable. Two of them are tempered, while another two are cotempered. Therefore it remains to consider representations

$$\pi_5 = L([1]; [0] \rtimes \delta([2];\sigma)), \quad \pi_6 = L([2], [0,1]; \sigma),$$
$$\pi_7 = L([0,1]; \delta([2];\sigma)), \quad \pi_8 = L([2], [1]; [0] \rtimes \sigma),$$

where $\pi_5^{\mathbf{t}} = \pi_6$ and $\pi_7^{\mathbf{t}} = \pi_8$.

For π_5 and π_7, recall that by 8.1.1, $L([1]; \delta([2];\sigma))$ is in Π_ψ for some A-parameter ψ. Now each irreducible subquotient of $[0] \rtimes L([1]; \delta([2];\sigma))$ (it is also a subrepresentation) is in the packet of $\psi \oplus E_{1,1}^\rho \oplus E_{1,1}^\rho$. One of them is π_5 (apply [Tad09, Proposition 5.3]). For another one, observe that $([0] \rtimes L([1]; \delta([2];\sigma)))^{\mathbf{t}} = [0] \rtimes L([1]; \delta([2];\sigma))^{\mathbf{t}} = [0] \rtimes L([2], [1]; \sigma)$, and that here π_8 is a subquotient (again apply [Tad09, Proposition 5.3]). Then $\pi_7 = \pi_8^{\mathbf{t}}$ is a subquotient of $[0] \rtimes L([1]; \delta([2];\sigma))$. Therefore, π_7 is also in an A-packet, as well as π_5.

For π_6 and π_8, recall that by 8.1.1, $L([2], [1]; \sigma)$ is in Π_ψ for some A-parameter ψ. Now each irreducible subquotient of $[0] \rtimes L([2], [1]; \sigma)$ is in the packet of $\psi \oplus E_{1,1}^\rho \oplus E_{1,1}^\rho$. One of them is π_8 (by [Tad09, Proposition 5.3]). For other one, observe that $([0] \rtimes L([2], [1]; \sigma))^{\mathbf{t}} = [0] \rtimes L([2], [1]; \sigma)^{\mathbf{t}} = [0] \rtimes L([1]; \delta([2];\sigma))$, and that here π_5 is subquotient (by [Tad09, Proposition 5.3]). Then $\pi_6 = \pi_5^{\mathbf{t}}$ is a subquotient of $[0] \rtimes L([2], [1]; \sigma)$. Therefore, π_6 is also in an A-packet, as well as π_8.

8.2.6. *Case* $(0, 1, 1), \alpha = 1$ (5.2 of [Tad20]). Here all 7 irreducible subquotients are unitarizable. Two of them are tempered, while another two are cotempered. Therefore it remains to consider representations

$$\pi_1 = L([0,1], [1]; \sigma), \ \pi_3 = L([0,1]; \delta([1];\sigma)),$$
$$\pi_4^+ = L([1]; \tau([0]_+; \delta([1];\sigma))),$$

where π_1 and π_3 are dual.

For π_1 (resp. π_3) consider ψ obtained from ψ_σ by replacing $(1,1)$ with the pair $(1,3), (2,2)$ (resp. $(3,1), (2,2)$) in $\mathrm{Jord}_\rho(\psi_\sigma)$. Now π_1 (resp. π_3) is in the L-packet inside Π_ψ (by Proposition 6.0.3) of [Mœg11]).

For π_4^+ consider $\psi := \psi_\sigma \oplus E_{3,1}^\rho \oplus E_{1,3}^\rho$. One easily shows that π_4 is in the the L-packet inside Π_ψ .

8.2.7. *Case* $(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}), \alpha = \frac{1}{2}$ (5.4 of [Tad20]). Here we have 8 irreducible unitarizable sub-quotients (and two non-unitarizable). Two of them are square integrable, and another two cotempered. Therefore, we need to consider the following

$$\pi_3 = L([-\tfrac{1}{2}, \tfrac{3}{2}]; \sigma), \quad \pi_4 = L([\tfrac{1}{2}, \tfrac{3}{2}]; \delta([\tfrac{1}{2}]; \sigma)),$$
$$\pi_7 = L([\tfrac{1}{2}]; \delta([\tfrac{1}{2}, \tfrac{3}{2}]; \sigma)), \quad \pi_8 = L([\tfrac{3}{2}]; \delta([-\tfrac{1}{2}, \tfrac{1}{2}]_-; \sigma)),$$

where $\pi_3^{\mathbf{t}} = \pi_4$ and $\pi_7^{\mathbf{t}} = \pi_8$.

Theorem 6.2 implies that π_7 and π_8 are in A-packets. For π_3 (resp. π_4) consider $\psi := \psi_\sigma \oplus E_{3,2}^\rho$ (resp. $\psi := \psi_\sigma \oplus E_{2,3}^\rho$). One directly sees that π_3 (resp. π_4) is in the L-packet inside the A-packet Π_ψ .

8.2.8. *Case* $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \alpha = \frac{1}{2}$ (5.5 of [Tad20]). Here all 5 irreducible subquotients are unitarizable. One of them is tempered, and another one cotempered. Therefore, we need to consider the following representations

$$\pi_2 = [\tfrac{1}{2}] \rtimes \delta([-\tfrac{1}{2}, \tfrac{1}{2}]_-; \sigma), \pi_3 = [\tfrac{1}{2}] \rtimes L([\tfrac{1}{2}]; \delta([\tfrac{1}{2}]; \sigma)),$$
$$\pi_5 = L([\tfrac{1}{2}]; \delta([-\tfrac{1}{2}, \tfrac{1}{2}]_+; \sigma)),$$

where π_2 and π_3 are dual.

Representations π_2 and π_5 are in the L-packet inside the A-packet of $\psi_\sigma \oplus E_{1,2}^\rho \oplus E_{2,1}^\rho \oplus E_{2,1}^\rho$. The representation π_3 is in the L-packet inside the A-packet of $\psi_\sigma \oplus E_{2,1}^\rho \oplus E_{1,2}^\rho \oplus E_{1,2}^\rho$.

8.2.9. *Case* $(0, 1, 1), \alpha = 0$ (6.2 of [Tad20]). Here 6 irreducible subquotients are unitarizable. Two of them are tempered, and another two cotempered. Therefore, we need to consider the following representations

$$\pi_3^\pm = L([1]; \delta([0, 1]_\pm; \sigma))$$

(here $(\pi_3^+)^{\mathbf{t}} = \pi_3^-$). The above representation is contained in an A-packet by (1) of Theorem 5.1 (consider the case of $m = n = 1$ in the corollary).

8.2.10. *Case* $(0, 0, 1), \alpha = 0$ (6.3 of [Tad20]). Here all 6 irreducible subquotients are unitarizable. Two of them are tempered, and another two cotempered. Therefore, we need to consider the following representations

$$\pi_2^\pm = L([0, 1]; \delta([0]_\pm; \sigma)).$$

Here $(\pi_2^+)^{\mathbf{t}} = \pi_2^-$.

Representations $\pi_2^\pm$ are in the L-packet inside the A-packet of $\psi_\sigma \oplus E_{2,2}^\rho \oplus E_{1,1}^\rho \oplus E_{1,1}^\rho$. $\square$

Appendix: Some complementary series of A-class

Complementary series form a considerable part of unitary duals of reductive groups. Among them, the simplest ones are one-dimensional complementary series, which in the case of classical groups are of the form

$$(8.1) \qquad \nu^x \sigma \rtimes \pi, \quad 0 < x < \beta,$$

where σ and π are irreducible unitarizable representations of a general linear and a classical group, respectively, such that all representations $\nu^x \sigma \rtimes \pi, 0 \le x < \beta$, are irreducible, and such that $\nu^\beta \sigma \rtimes \pi$ is reducible.

Observe that for parameterising the continuous family of complementary series (8.1), it is enough to know lower and upper bounds of the complementary series, i.e. σ and π (such that $\sigma \rtimes \pi$ is irreducible), and further, the first reducibility exponent β.

C. Mœglin mentioned to us that it is possible that some complementary series representations can be of A-class. We present below an example of this type. Below ρ, σ and α are as in section 3.14.

Lemma 8.3. *Let $\alpha \ge 1$, $x \ge 0$ and $\alpha - x \in \mathbb{Z}_{>0}$. Then $[x] \rtimes \sigma$ is in an A-packet (we already know that for $x = \alpha$, both irreducible subquotients are in A-packets).*

Proof. If $x = 0$, then we are in the tempered situation, and the claim obviously holds (the A-parameter is $\psi_\sigma \oplus E_{1,1}^\rho \oplus E_{1,1}^\rho$). Therefore, we suppose $x > 0$, which implies $\alpha > 1$.

First we show that $[\alpha - 1] \rtimes \sigma$ is in an A-packet if $\alpha > 1$. Denote by $(\psi'_\sigma, \epsilon'_\sigma)$ the parameter obtained from $(\psi_\sigma, \epsilon_\sigma)$ deforming $E_{2\alpha-3,1}^\rho$ to $E_{1,2\alpha-3}^\rho$ (then $\mathrm{Jord}_\rho(\psi'_\sigma)$ ends with $(1, 2\alpha - 3), (2\alpha - 1, 1)$). We have $\sigma = \pi(\psi'_\sigma, \epsilon'_\sigma)$.

Denote by (ψ_1, ϵ_1) the parameter obtained from $(\psi'_\sigma, \epsilon'_\sigma)$ by deforming $E_{2\alpha-1,1}^\rho$ to $E_{2\alpha+1,1}^\rho$ (now $\mathrm{Jord}_\rho(\psi_1)$ ends with $(1, 2\alpha - 3), (2\alpha + 1, 1)$). Then $\pi(\psi_1, \epsilon_1) = \delta([\alpha]; \sigma)$.

Let (ψ_2, ϵ_2) be obtained from (ψ_1, ϵ_1) by deforming $E_{1,2\alpha-3}^\rho$ to $E_{1,2\alpha-1}^\rho$ (now $\mathrm{Jord}_\rho(\psi_1)$ ends with $(1, 2\alpha - 1), (2\alpha + 1, 1)$). Then $\pi(\psi_2, \epsilon_2) = L([\alpha - 1]; \delta([\alpha]; \sigma))$. Denote by $>_{\psi_2}$ the standard order on $\mathrm{Jord}_\rho(\psi_2)$.

Let ψ_3 be the A-parameter obtained from ψ_2 by replacing $(1, 2\alpha + 1)$ with $(1, 2\alpha - 1)$ (now $\mathrm{Jord}_\rho(\psi_3)$ ends with $(2\alpha - 1, 1), (1, 2\alpha - 1)$). Denote by $>_{\psi_3}$ on $\mathrm{Jord}_\rho(\psi_3)$ standard order which satisfies

$$(2\alpha - 1, 1) >_{\psi_3} (1, 2\alpha - 1)$$

($>_{\psi_3}$ is an admissible order, but not natural; ψ_3 is a multiplicity one parameter, but not discrete).

We denote by $\varphi : \mathrm{Jord}_\rho(\psi_2) \to \mathrm{Jord}_\rho(\psi_3)$ the standard bijection which preserves order. This implies that it carries

$$(2\alpha + 1, 1) \mapsto (2\alpha - 1, 1)$$

(on the remaining elements it is the identity). Now $\mathrm{Jord}(\psi_2)$ dominates $\mathrm{Jord}(\psi_3)$ with respect to $>_{\psi_3}$. Here we need to consider the matrix $X_{(\rho, A, B, \zeta_{a,b})}^{\gg}$ (defined in section 5 of [Xu17a]), which is in our case a 1×1 matrix $X_{(\rho, \alpha-1, 0, 1)}^{\gg} = [\alpha]$. We get the elements of

Π_{ψ_3} from $\mathrm{Jord}(\psi_2)$ applying Jac_α to each element of Π_{ψ_2} (the result is always either an irreducible representation or 0). Observe that

$$(8.2) \quad L([\alpha-1]; \delta([\alpha]; \sigma)) \hookrightarrow [-(\alpha-1)] \rtimes \delta([\alpha]; \sigma)$$
$$\hookrightarrow [-(\alpha-1)] \times [\alpha] \rtimes \sigma \cong [\alpha] \times [-(\alpha-1)] \rtimes \sigma \cong [\alpha] \times [\alpha-1] \rtimes \sigma.$$

Now Frobenius reciprocity implies that $\mathrm{Jac}_\alpha(L([\alpha-1]; \delta([\alpha]; \sigma))) = [\alpha-1] \rtimes \sigma$. Therefore, $[\alpha-1] \rtimes \sigma$ is in the A-packet of ψ_3.

In a similar way we show next that $[\alpha-2] \rtimes \sigma$ is in an A-packet if $\alpha > 2$. Denote now by $(\psi'_\sigma, \epsilon'_\sigma)$ the parameter obtained from $(\psi_\sigma, \epsilon_\sigma)$ by deforming $E^\rho_{2\alpha-5,1}$ to $E^\rho_{1,2\alpha-5}$ (then $\mathrm{Jord}_\rho(\psi'_\sigma)$ ends with $(1, 2\alpha-5), (2\alpha-3, 1), (2\alpha-1, 1)$). We have $\sigma = \pi(\psi'_\sigma, \epsilon'_\sigma)$.

Denote by (ψ_1, ϵ_1) the parameter obtained from $(\psi'_\sigma, \epsilon'_\sigma)$ by deforming $E^\rho_{2\alpha-1,1}$ to $E^\rho_{2\alpha+1,1}$ and then $E^\rho_{2\alpha-3,1}$ to $E^\rho_{2\alpha11,1}$ (now $\mathrm{Jord}_\rho(\psi_1)$ ends with $(1, 2\alpha-5), (2\alpha-1, 1), (2\alpha+1, 1)$). We get directly that $\pi(\psi_1, \epsilon_1) = \delta_{\mathrm{s.p.}}([\alpha-1], [\alpha]; \sigma)$.

Let (ψ_2, ϵ_2) be obtained from (ψ_1, ϵ_1) by deforming $E^\rho_{1,2\alpha-5}$ to $E^\rho_{1,2\alpha-3}$ (now $\mathrm{Jord}_\rho(\psi_1)$ ends with $(1, 2\alpha-3), (2\alpha-1, 1), (2\alpha+1, 1)$). Then $\pi(\psi_2, \epsilon_2) = L([\alpha-2]; \delta_{\mathrm{s.p.}}([\alpha-1], [\alpha]; \sigma))$. Denote by $>_{\psi_2}$ the standard order on $\mathrm{Jord}_\rho(\psi_2)$.

Denote by ψ_3 A-parameter obtained from ψ_2 by replacing $(2\alpha-1, 1)$ by $(2\alpha-3, 1)$ and then $(2\alpha+1, 1)$ by $(2\alpha-1, 1)$ (now $\mathrm{Jord}_\rho(\psi_3)$ ends with $(2\alpha-3, 1), (1, 2\alpha-3), (2\alpha-1, 1)$). Denote by $>_{\psi_3}$ the standard order on $\mathrm{Jord}_\rho(\psi_3)$ which satisfies

$$(2\alpha-3, 1) >_{\psi_3} (1, 2\alpha-3).$$

Let $\varphi : \mathrm{Jord}_\rho(\psi_2) \to \mathrm{Jord}_\rho(\psi_3)$ be the standard bijection which preserves order. This implies that it carries

$$(2\alpha-1, 1) \mapsto (2\alpha-3, 1), \qquad (2\alpha+1, 1) \mapsto (2\alpha-1, 1),$$

(on the remaining elements it is the identity). Now $\mathrm{Jord}(\psi_2)$ dominates $\mathrm{Jord}(\psi_3)$ with respect to $>_{\psi_3}$. Here we need to consider the matrices $X^{\gg}_{(\rho,A,B,\zeta_{a,b})}$, which in our case are 1×1 matrices $X^{\gg}_{(\rho,\alpha-2,0,1)} = [\alpha-1]$ and $X^{\gg}_{(\rho,\alpha-1,0,1)} = [\alpha]$. We need to apply them in descending order to $L([\alpha-2]; \delta_{\mathrm{s.p.}}([\alpha-1], [\alpha]; \sigma))$, i.e. we need to apply $\mathrm{Jac}_\alpha \circ \mathrm{Jac}_{\alpha-1}$ to the last representation (and we will get either 0 or an element of Π_{ψ_3}).

Now we will compute the action of the above operator on $L([\alpha-2]; \delta_{\mathrm{s.p.}}([\alpha-1], [\alpha]; \sigma)$. Observe that

$$(8.3) \quad L([\alpha-2]; \delta_{\mathrm{s.p.}}([\alpha-1], [\alpha]; \sigma) \hookrightarrow [-(\alpha-2)] \rtimes \delta_{\mathrm{s.p.}}([\alpha-1], [\alpha]; \sigma)$$
$$\hookrightarrow [-(\alpha-2)] \times [\alpha-1] \rtimes \delta([\alpha]; \sigma) \cong [\alpha-1] \times [-(\alpha-2)] \rtimes \delta([\alpha]; \sigma).$$

Note that $[-(\alpha-2)] \rtimes \delta([\alpha]; \sigma)$ is irreducible. Now Frobenius reciprocity implies that

$$\mathrm{Jac}_{\alpha-1}(L([\alpha-2]; \delta_{\mathrm{s.p.}}([\alpha-1], [\alpha]; \sigma))) = [-(\alpha-2)] \rtimes \delta([\alpha]; \sigma).$$

Further

$$[-(\alpha-2)] \rtimes \delta([\alpha]; \sigma) \hookrightarrow [-(\alpha-2)] \times [\alpha] \times \sigma \cong [\alpha] \times [-(\alpha-2)] \times \sigma \cong [\alpha] \times [\alpha-2] \times \sigma,$$

and one directly concludes that $\mathrm{Jac}_\alpha([-(\alpha-2)] \rtimes \delta([\alpha]; \sigma)) = [\alpha-2] \times \sigma$. Therefore, $[\alpha-2] \rtimes \sigma$ is in the A-packet of ψ_3.

Continuing this procedure, we complete the proof of the lemma. $\square$

Definition 8.4. *Suppose that an irreducible representation π of a classical group S_n is in an A-packet, and that there do not exist Speh representations $\tau_1, \ldots, \tau_k$ and an irreducible representation π_0 of a classical group S_m with $m < n$, contained in some A-packet, such that*

$$\pi \hookrightarrow \tau_1 \times \cdots \times \tau_k \rtimes \pi_0.$$

Then π_0 will be called a primitive representation of A-type.

We will very briefly recall the notion of automorphic dual, which L. Clozel introduced in [Clo07] (one can find more details and further references in the original Clozel paper).

We first recall of notion of the support of a unitary representation Π of a locally compact group **G**. An irreducible unitary representation π of **G** is weakly contained in Π if each diagonal matrix coefficient of π on each compact subset of **G** can be approximated by finite sums of diagonal matrix coefficients of Π (i.e. each diagonal matrix coefficient of π on each compact subset of **G** is the limit of sums of diagonal matrix coefficients of Π). The support of Π is the set of equivalence classes of all irreducible unitary representations π of **G** which are weakly contained in Π.

Let G be a reductive group defined over an algebraic number field k (or more generally, over a global field k). Fix any place v of k and denote by k_v the completion of k at v. The automorphic dual $\widehat{G}_{v,aut}$ is the support of the representations of the group $G(k_v)$ of k_v-rational points of G in the space of square integrable automorphic forms $L^2(G(k)\backslash H(\mathbb{A}_k))$ (by right translations). We denote $F = k_v$.

Motivated by [Tad10], we ask the following

Question 9.3. Is each primitive representation of A-type isolated in the automorphic dual?

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