ON TEMPERED AND SQUARE INTEGRABLE REPRESENTATIONS OF CLASSICAL p-ADIC GROUPS

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Abstract. This paper has two aims. The first is to give a description of irreducible tempered representations of classical p-adic groups which follows naturally the classification of irreducible square integrable representations modulo cuspidal data obtained in [15] and [19]. The second aim of the paper is to give description of an invariant (partially defined function) of irreducible square integrable representation of a classical p-adic group (defined by C. Mœglin using embeddings) in terms of subquotients of Jacquet modules. As an application, we describe behavior of partially defined function in one construction of square integrable representations of a bigger group from such representations of a smaller group (which is related to deformation of Jordan blocks of representations).

Contents

Date: June 30, 2013.

1991 *Mathematics Subject Classification.* Primary: 22E50, 22E55, Secondary: 11F70, 11S37.

Key words and phrases. non-archimedean local fields, classical groups, square integrable representations, tempered representations.

The author was partly supported by Croatian Ministry of Science, Education and Sports grant #037-0372794-2804.

1. Introduction

In this paper we shall fix a non-archimedean local field F and consider irreducible tempered and square integrable representations of classical groups over F.

First we shall describe parameterization of tempered representations obtained in this paper. These representations are important for a number of reasons (Plancherel measure, non-unitary dual, orbital integrals etc.).

At the beginning, we shall recall a fundamental result of D. Goldberg on tempered representations of a classical group G over F ([8]). Levi factor M of a proper parabolic subgroup P of G is isomorphic to a direct product $GL(n_1, F) \times \ldots \times GL(n_k, F) \times G'$, where G' is a classical group from the same series as G, whose split rank is smaller then the split rank of G (see section 2 for more details).

Theorem 1.1. (D. Goldberg) *Take irreducible square integrable (modulo center) representations δ_i of $GL(n_i, F)$, $i = 1, \ldots, k$, and an irreducible square integrable representation π of G'. Denote by l the number of non-isomorphic δ_i's such that the parabolically induced representation*

$$Ind^{G_i}(\delta_i \otimes \pi)$$

of the appropriate classical group G_i reduces. Then the parabolically induced representation

$$Ind_P^G(\delta_1 \otimes \ldots \otimes \delta_k \otimes \pi) \tag{1.1}$$

is a multiplicity one representation of length 2^l. Further, if τ is (equivalent to) an irreducible subrepresentation of some representation (1.1) *as above, then τ determines (equivalence class of) π, and it determines (equivalence classes of) $\delta_1, \ldots, \delta_k$ up to a permutation and taking contragredients*[1].

This result reduces the problem of description of irreducible tempered representations to square integrable representations and tempered reducibilities in the generalized rank one case. The rank one reducibilities are part of the classification of square integrable representations of classical groups modulo cuspidal data in [15] and [19] (we shall say later more regarding this). The theory of R-groups gives a parameterization of irreducible pieces of $\mathrm{Ind}_P^G(\delta_1 \otimes \ldots \otimes \delta_k \otimes \sigma)$ in terms of characters of R-groups. In this paper we shall give description of irreducible pieces by parameters coming from the parameters of square integrable representations of the classification in [15] and [19].

[1] In the case of unitary groups, one needs to consider Hermitian contragredients

First observe that for parameterizing irreducible pieces of $\mathrm{Ind}_P^G(\delta_1 \otimes \ldots \otimes \delta_k \otimes \sigma)$, it is enough to know to parameterize them in the case $l = k$ (further tempered parabolical induction is irreducible). Therefore, we shall assume $l = k$ in what follows. For this case, we have the following simple reduction to the generalized rank one case.

Each representation $\mathrm{Ind}^{G_i}(\delta_i \otimes \pi)$ splits into two irreducible non-isomorphic representations. Denote these pieces by π_{δ_i} and $\pi_{-\delta_i}$, i.e.

$$\mathrm{Ind}^{G_i}(\delta_i \otimes \pi) = \pi_{\delta_i} \oplus \pi_{-\delta_i} \tag{1.2}$$

(later on, we shall come to the problem of parameterizing irreducible pieces of $\mathrm{Ind}^{G_i}(\delta_i \otimes \pi)$). Let $j_1, \ldots, j_k \in \{\pm\}$. Then there exists a unique irreducible subrepresentation τ of $\mathrm{Ind}_P^G(\delta_1 \otimes \ldots \otimes \delta_k \otimes \pi)$ such that τ is a subrepresentation of

$$\mathrm{Ind}_P^G(\delta_1 \otimes \ldots \delta_{i-1} \otimes \delta_{i+1} \otimes \ldots \delta_k \otimes \pi_{j_i\delta_i}),$$

for each $i = 1, \ldots, k$. We denote such τ by

$$\pi_{j_1\delta_1,\ldots,j_k\delta_k}.$$

Therefore, to get a parameterization of irreducible tempered representations of classical groups, it remains to determine in (1.2) which irreducible subrepresentation will be denote by π_{δ_i} (we have two choices; the other irreducible subrepresentation is then denoted by $\pi_{-\delta_i}$). To describe which subrepresentation will be denoted by π_{δ_i}, we shall briefly recall the notion of Jordan blocks attached to an irreducible square integrable representation of a classical group (Jordan blocks $Jord(\pi)$ attached to an irreducible square integrable representation π of a classical group is one of three invariants which classify irreducible square integrable representations of a classical groups modulo cuspidal data, and a natural assumption).

Before we recall the definition of Jordan blocks, we shall recall some notation for general linear groups. Let ρ be an irreducible cuspidal representation of $GL(p, F)$ and let n be a positive integer (we consider only smooth representations in this paper). Let

$$[\,\rho, |\det|_F^n\rho\,] \ := \ \{\rho, |\det|_F\rho, |\det|_F^2\rho, \ldots, |\det|_F^n\rho\}$$

($|\ |_F$ denotes the normalized absolute value on F). The parabolically induced representation

$$\mathrm{Ind}^{GL(np,F)}(|\det|_F^n\rho \otimes |\det|_F^{n-1}\rho \otimes \cdots \otimes |\det|_F\rho \otimes \rho),$$

induced from the appropriate parabolic subgroup which is standard with respect to the minimal parabolic subgroup of all upper triangular matrices in the group, contains a unique irreducible subrepresentation. This subrepresentation is denoted by

$$\delta([\,\rho, |\det|_F^n\rho\,])$$

(the parabolic induction that we consider in this paper is normalized). Then the representation $\delta([\rho, |\det|_F^n \rho])$ is an essentially square integrable representation. Denote

$$\delta(\rho, n) := \delta([\,|\det|_F^{-\frac{n-1}{2}}\rho, |\det|_F^{\frac{n-1}{2}}\rho\,]).$$

For simplicity, in the introduction we shall only deal with symplectic and split special odd-orthogonal groups (for the definition of these groups see section 2). We shall fix one of these series of groups, and denote the group of (split) rank n in the series by S_n.

Let π be an irreducible square integrable representation of S_q. In what follows, we shall assume that a natural hypothesis, called basic assumption, holds (this is (BA) in section 2). Fix an irreducible selfdual representation ρ of a general linear group (selfdual means that the contragredient representation $\tilde{\rho}$ of ρ is isomorphic to ρ). Consider representations

$$\mathrm{Ind}^{S_{np+q}}(\delta(\rho, n) \otimes \pi), \tag{1.3}$$

parabolically induced from appropriate parabolic subgroups. Then for one parity of n in $\mathbb{Z}_{>0}$, the corresponding representations (1.3) are always irreducible, while for the other parity we have always reducibility, except for finitely many n. All the exceptions n are denote by

$$\mathrm{Jord}_\rho(\pi).$$

Then the Jordan blocks of π are defined by

$$\mathrm{Jord}(\pi) = \bigcup_{\rho} \{\rho\} \times \mathrm{Jord}_\rho(\pi),$$

when ρ runs over all equivalence classes of irreducible selfdual cuspidal representations of general linear groups (see section 2 for more details).

Let us recall that Jordan blocks are one of the invariants that C. Moeglin has attached in [15] to an irreducible square integrable representation π of a classical group over F. To such π, she has also attached invarinat

$$\epsilon_\pi,$$

called the partially defined function of π , and an irreducible cuspidal representation

$$\pi_{cusp}$$

of a classical group, called the partial cuspidal support of π. The importance of these invarinats comes from the fact that triples

$$(\mathrm{Jord}(\pi), \epsilon_\pi, \pi_{cusp})$$

classify irreducible square integrable representations of classical groups modulo cuspidal data (and a natural assumption; see [19]). The definition of the partial cuspidal support will be recalled later in the introduction.

Since the above invariants classify irreducible square integrable representations, it is important to know if their definition is canonical. This is the case for $\mathrm{Jord}(\pi)$ and π_{cusp}.

The definition of ϵ_π in [15] is given in terms of embeddings in some cases, and in terms of normalized standard integral intertwining operators in the other cases[2]. The part of the definition given by embeddings is also canonical. Only the part relaying on normalized standard intertwining operators is not canonical, since it depends on the choice of normalization of the operators that one uses in the definition. This non-canonical case of the definition occurs precisely when $Jord_\rho(\pi)$ is a non-empty subset of odd integers, while $Jord_\rho(\pi_{cusp}) = \emptyset$. The last condition is equivalent to the fact that

$$\mathrm{Ind}^{S_{p+q'}}(\rho \otimes \pi_{cusp}) \tag{1.4}$$

reduces.

Let us briefly explain how one can fix a normalization as above. We suppose that (1.4) reduces (as was the case above). Then (1.4) reduces into two nonequivalent irreducible pieces:

$$\mathrm{Ind}^{S_{p+q'}}(\rho \otimes \pi_{cusp}) = \tau_1 \oplus \tau_{-1} \tag{1.5}$$

(it would be more precise to denote these pieces by $\tau_1^{(\rho,\pi_{cusp})}$ and $\tau_{-1}^{(\rho,\pi_{cusp})}$, but to simplify notation, we drop the superscripts (ρ, π_{cusp})). J.-L. Waldspurger observed that the normalization of standard intertwining operators which uses C. Mœglin in her definition of $\epsilon_\pi((\rho, a))$ is determined by the choice of the signs that one attaches to the irreducible pieces in (1.5).

In this paper we work with the classification obtained in [19]. Therefore, we assume that the normalization of standard intertwining operators in [15] is fixed. This implies that the choice of indexes in (1.5) is always fixed, when we have situation as above. We shall use this choice of indexes several times in what follows.

Let us go back to irreducible square integrable representations π of a classical and δ of a general linear group. We shall assume below that

$$\mathrm{Ind}^G(\delta \otimes \pi)$$

reduces. Then δ is selfdual. Therefore, we can find a selfdual irreducible cuspidal representation ρ of a general linear group and $b \in \mathbb{Z}_{>0}$ such that

$$\delta \cong \delta(\rho, b).$$

Now we shall define tempered representation

$$\pi_\delta.$$

The parabolically induced representations that we consider in the introduction will be again assumed to be induced from the appropriate parabolic subgroup, which is standard with respect to the minimal parabolic subgroup of all upper triangular matrices in the group that we consider.

[2] One can find in [36] the definition which does not use the normalized standard intertwining operators

Below, we denote by

$$\pi_{cusp}$$

the partial cuspidal support of π (π_{cusp} is the unique irreducible cuspidal representation of a classical group for which there exists an irreducible representation θ of a general linear group such that $\pi \hookrightarrow \mathrm{Ind}(\theta \otimes \pi_{cusp})$).

Theorem 1.2. (Definition of π_δ) *Let π be an irreducible square integrable representation of a classical group, let ρ be an irreducible selfdual representation of a general linear group and let b be a positive integer. Denote*

$$\delta = \delta(\rho, b).$$

Assume that

$$Ind^G(\delta \otimes \pi)$$

reduces.

(1) *Suppose*

$$\mathrm{Jord}_\rho(\pi) \cap [1, b] \neq \emptyset.$$

Denote

$$a = \max(\mathrm{Jord}_\rho(\pi) \cap [1, b]).$$

Then there exists a unique irreducible subrepresentation of $Ind^G(\delta \otimes \pi)$, denoted by π_δ, satisfying

$$\pi_\delta \hookrightarrow Ind^G(\delta([\nu^{(a-1)/2+1}\rho, \nu^{(b-1)/2}\rho])^2 \otimes \lambda)$$

for some irreducible representation λ of a classical group.

(2) *Suppose*

$$\mathrm{Jord}_\rho(\pi) \cap [1, b] = \emptyset.$$

(a) *Let b be even. Then there exists a unique irreducible subrepresentation of $Ind^G(\delta \otimes \pi)$, denoted by π_δ, satisfying*

$$\pi_\delta \hookrightarrow Ind^G(\delta([\nu^{1/2}\rho, \nu^{(b-1)/2}\rho])^2 \otimes \lambda)$$

for some irreducible representation λ of a classical group.

(b) *Let b be odd.*

(i) *Suppose*

$$\mathrm{Jord}_\rho(\pi) \neq \emptyset.$$

Denote

$$a := \min(\mathrm{Jord}_\rho(\pi)).$$

Then there exists a unique irreducible subrepresentation of $Ind^G(\delta \otimes \pi)$, denoted by π_δ, satisfying

$$\pi_\delta \hookrightarrow Ind^G(\delta([\nu\rho, \nu^{(b-1)/2}\rho])^2 \otimes \delta([\nu\rho, \nu^{(a-1)/2}\rho]) \otimes \lambda)$$

for some irreducible representation λ of a classical group.

(ii) *Suppose*

$$\mathrm{Jord}_\rho(\pi) = \emptyset.$$

Then $\rho \rtimes \pi_{cusp}$ *reduces. Decompose*

(1.6)
$$\rho \rtimes \pi_{cusp} = \tau_1 \oplus \tau_{-1}$$

into the sum of irreducible (tempered) subrepresentations. Then there exists a unique irreducible subrepresentation of $Ind^G(\delta \otimes \pi)$*, denoted by* π_δ*, satisfying*

$$\pi_\delta \hookrightarrow Ind^G(\theta \otimes \tau_1).$$

for some irreducible representation θ *of a general linear group. (Analogously we can define* $\pi_{-\delta}$ *using* τ_{-1} *instead* τ_1*.)*

One can find Jacquet module definition of representations π_δ in section 4. Let us note that for the above parameterization of tempered representations, we did not need to make any new choice besides the choices that we needed to make for the classification of square integrable representations of classical groups (for square integrable representations we needed to make choices of τ_1 in (1.6), which in general are not canonical).

Now we shall describe the second aim of this paper. The partially defined function ϵ_π (as well as the partial cuspidal support π_{cusp}) is defined using embeddings. In general, if we have an irreducible representation σ of a reductive group G and an irreducible representation τ of a Levi factor M of a parabolic subgroup P, the fact that σ embeds into $\mathrm{Ind}_P^G(\tau)$, i.e.

$$\sigma \hookrightarrow \mathrm{Ind}_P^G(\tau),$$

implies by Frobenius reciprocity that τ is a quotient of the corresponding Jacquet module of σ with respect to P (the converse also holds). Then, in particular, τ is a subquotient of the corresponding Jacquet module of σ. On the other side, the fact that τ is a subquotient of the corresponding Jacquet module of σ, does not imply in general the existence of embedding $\sigma \hookrightarrow \mathrm{Ind}_P^G(\tau)$ (one can see such examples in Remark 7.6).

Let us recall that we have fairly good control of subqutients of Jacquet modules of parabolically induced representations (through Geometric Lemma of [6]). The question of exact structure of Jacquet module is usually much more delicate (see [7] already for the case of $SL(2, F)$). Therefore, it would be much more convenient to have characterization of partially defined function in terms of subqutients of Jacquet modules, instead of quotients.

In this paper we show that in the definition of the partially defined function ϵ_π, it is enough to require only the subquotient condition instead of the quotient condition of the corresponding Jacquet module (actually, we shall show more; see section 7). We shall explain this in more detail below.

Regarding partial cuspidal support π_{cusp} of an irreducible (square integrable) representation π of a classical group over F, it is easy to show (and it is well-known) that one can define π_{cusp} requiring only subquotient (instead of quotient) condition (see Proposition 7.1).

To define partially defined function attached to an irreducible square integrable representation π of a classical group over F (defined in [15]; see also [36]), it is enough to consider three cases of the following theorem (the Jacquet modules that we consider below are all with respect to the standard parabolic subgroups; see section 2 for more details).

Theorem 1.3. (1) *Suppose that* $\mathrm{Jord}_\rho(\pi)$ *has at least two elements. Take any* $a_-, a \in \mathrm{Jord}_\rho(\pi)$ *such that* $a_- < a$ *and* $\{b \in \mathrm{Jord}_\rho(\pi); a_- < b < a\} = \emptyset$. *Then*

$$\delta([\nu^{(a_- -1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \otimes \sigma$$

is a subquotient of the appropriate Jacquet module of π *for some irreducible representation* σ, *if and only if* $\epsilon_\pi((\rho,a))\epsilon_\pi((\rho,a_-))^{-1} = 1$[3].

(2) *Suppose* $\mathrm{Jord}_\rho(\pi) \cap 2\mathbb{Z} \neq \emptyset$. *Denote*

$$a_{\pi,\min,\rho} = \min(\mathrm{Jord}_\rho(\pi)).$$

Then $\epsilon_\pi((\rho, a_{\pi,\min,\rho}))$ *is defined, and it is 1 if and only if some irreducible representation of the form* $\delta([\nu^{1/2}\rho, \nu^{(a_{\pi,\min,\rho}-1)/2}\rho]) \otimes \sigma$ *is a subquotient of the corresponding Jacquet module of* π[4].

(3) *Suppose* $\mathrm{Jord}_\rho(\pi) \cap (1+2\mathbb{Z}) \neq \emptyset$ *and* $Jord_\rho(\pi_{cusp}) = \emptyset$ *(the last condition is equivalent to the fact that* $\rho \rtimes \pi_{cusp}$ *reduces). Then* $\rho \rtimes \pi_{cusp}$ *reduces into two irreducible nonequivalent representations:* $\rho \rtimes \pi_{cusp} = \tau_1 \oplus \tau_{-1}$. *Then for any* $k \in \mathbb{Z}_{>0}$ *the representation* $Ind(\delta([\nu\rho, \nu^k\rho]) \otimes \tau_i)$, $i \in \{\pm 1\}$, *has the unique irreducible subrepresentation, denoted by*

$$\delta([\nu\rho, \nu^k\rho]_{\tau_i}; \pi_{cusp}).$$

This subrepresentation is square integrable.

Denote

$$a_{\pi,\max,\rho} = \max(\mathrm{Jord}_\rho(\pi)).$$

Then $\epsilon_\pi((\rho, a_{\pi,\max,\rho}))$ *is defined and* $\epsilon_\pi((\rho, a_{\pi,\max,\rho})) = i$ *if and only if an irreducible representation of the form* $\theta \otimes \delta([\nu\rho, \nu^{(a_{\pi,\max,\rho}-1)/2}\rho]_{\tau_i}; \pi_{cusp})$ *is a subquotient of the Jacquet module of* π[5].

Some other useful descriptions in terms of Jacquet modules of partially defined functions, are also given in section 7. They are related to the infinitesimal characters.

[3]By the original definition, $\epsilon_\pi((\rho,a))\epsilon_\pi((\rho,a_-))^{-1} = 1$ if and only if there exists a representation σ' of a classical group such that $\pi \hookrightarrow \mathrm{Ind}(\delta([\nu^{(a_- -1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \otimes \sigma')$.

[4]By the original definition, $\epsilon_\pi((\rho, a_{\pi,\min,\rho})) = 1$ if and only if $\pi \hookrightarrow \mathrm{Ind}(\delta([\nu^{1/2}\rho, \nu^{(a_{\pi,\min,\rho}-1)/2}\rho]) \otimes \sigma')$ for some irreducible representation σ'.

[5]By [36], $\epsilon_\pi((\rho, a_{\pi,\max,\rho})) = i$ if and only if $\pi \hookrightarrow \mathrm{Ind}(\theta' \otimes \delta([\nu\rho, \nu^{(a_{\pi,\max,\rho}-1)/2}\rho]_{\tau_i}; \pi_{cusp}))$ for some representation θ'.

After discussion of the role of Jacquet modules in the definition of the invariant ϵ_π (and π_{cusp}), we shall very briefly discuss the role of Jacquet modules in determining the invariant Jord(π). Jacquet modules in this case do not give complete answers, as they do for the other two invariants. For example, we can have an irreducible cuspidal representation π of a classical group which has many Jordan blocks, but we certainly can not detect the Jordan blocks from the Jacquet modules of π (since all the proper Jacquet modules of a cuspidal representation are trivial, i.e. zero spaces). Nevertheless, from non-trivial Jacquet modules we can get some information about Jordan blocks. An example is Proposition 3.6 (there are possible further results in that direction, but we do not go in that direction in this paper; one can find such results in [13]).

As an application of the Jacquet module interpretation of invariants, we give (an expected) description of partially defined function of an irreducible square integrable representation when one Jordan block of the representation is deformed. This is one of two important constructions of square integrable representations from [15]. The other construction is adding two neighbor Jordan blocks (the description of partially defined function for this case is obtained in [15]; see also 3.2 of this paper).

We explain very briefly the application that we mentioned above. Take an irreducible square integrable representation π of a classical group and take an irreducible selfdual cuspidal representation ρ of a general linear group . Let $a \in \mathrm{Jord}_\rho(\pi)$. Suppose that there exists an integer b of the same parity as a, greater than a, which satisfies $[a+1,b] \cap \mathrm{Jord}_\rho(\pi) = \emptyset$. Then the representation

$$\mathrm{Ind}(\delta([\nu^{(a+1)/2}\rho, \nu^{(b-1)/2}\rho]) \otimes \pi) \tag{1.7}$$

has an irreducible square integrable subrepresentation. Denote it by π'. Then π and π' have the same partial cuspidal supports, and we know how to get Jordan blocks of one of the representations from the other representation (see Proposition 3.1). In Theorem 8.2 we show how to get partially defined function of one of the representations from the partially defied function of the other representation.

We describe now the content of the paper. The notation that we use in this paper is introduced in the second section. The third section recalls some basic results that we use in the paper, in particular about Jordan blocks. We define basic irreducible tempered representations π_δ in the fourth section. The fifth section gives a description of all irreducible tempered representations by the basic ones. In the sixth section we present some simple observations on the action of the Bernstein center of a factor of a direct product of two reductive groups, on the representations of the direct product. We apply these simple observations in the seventh section and obtain the Jacquet module interpretation of invariants. The eighth section deals with connection of partially defined functions of square integrable representations of smaller and bigger classical group. For the convenience of reader, in the appendix we bring a proof of a result on irreducibility used in this paper, which follows from the paper [25] of G. Muić.

Let us note that at least some of the results of this paper were known to experts. Nevertheless, we present the proofs of the results for which we did not know written references. Further, description of tempered representations in some cases has been obtained and used already by some authors (when they where describing irreducible subquotients of parabolically induced representations; see for example [32], [11], [3] or [24]).

Some of the main topics of this paper (like irreducible tempered or square integrable representations of classical groups over F) show up as main local objects of the recent book [1] of J. Arthur. It would be very important to understand explicitly the relation between the fundamental classification in [1] and the problems studied in our paper. We do not go in that direction in this paper. We shall say only a few words regarding this.

Arthur's book contains Langlands classification of irreducible tempered representations of classical groups over F in characteristic zero (as far as we know, his classification is still conditional, but it is soon expected to be unconditional). Our description of irreducible tempered representations is modulo cuspidal data (i.e. irreducible cuspidal representations and cuspidal reducibilities in the generalized rank one cases). These are different aspects of the same problem (our description is only one of the steps in understanding the irreducible tempered representations of classical groups). For a number of problems of harmonic analysis of classical groups (and automorphic forms), it is important to understand irreducible tempered representations in terms of square integrable ones. As well, it is important to understand irreducible square integrable representations in terms of cuspidal representations. For example, the problem of unitarizability is the place where such information is crucial. Such understanding was also important in the case of general linear groups, where the Bernstein-Zelevinsky theory provides us with such understanding (clearly, the situation there is much simpler).

Arthur's classification of irreducible square integrable representations is very important for our approach. In that classification, cuspidal representations are very simply recognized among all the irreducible square integrable representations. This is done by C. Mœglin. The requirement on the admissible homomorphism corresponding to an irreducible cuspidal representation is a very simple condition, "without gaps", while the requirement on the character of the component group is also a simple condition, "being alternate" (for precise assumptions see [18], and Theorem 2.5.1 there). From the Langlands parameter of such a cuspidal representation, one sees directly all the "exceptional" cuspidal reducibilities, i.e. those ones which are ≥ 1 (at least for the odd orthogonal and symplectic groups). This is done by C. Mœglin and J.-L. Waldspurger (see the remark (ii) in Remarks 4.5.2 of [21]). The remaining cuspidal reducibilities are controlled already by the local Langlands correspondences for general linear groups. Therefore, the Arthur's results give us precisely the parameters that we are using. Let us recall that the case of unitary groups was completed earlier by C. Mœglin (in [17]).

Let us mention that C. Mœglin has obtained in [18] a parameterization of the packets determined by Arthur's parameters. Since special cases of these packets are tempered, there is also a parameterization of irreducible tempered representations there. We do not go in this paper into the relation between the parameterization in her paper and our paper.

After this paper has been completed (and submitted), we have learned for C. Jantzen's paper [13]. The main aims of both papers are very close. The results were independently obtained, and some of them are more or less the same (there is a slight difference in the choice of parameters of some tempered representations). The approaches and the methods of proofs in the papers are very different. Therefore, the papers are complementary and give different understanding of the problems studied in the papers. We are thankful to C. Jantzen for clearing the relation between parameterizations in the papers.

We are very thankful for particularly useful discussions on the topic of this paper to C. Mœglin, and for her explanations. Discussions with M. Hanzer, A. Moy and G. Muić were also very helpful. The referees gave some very useful suggestions. Parts of this paper were written while the author was the guest of the Hong Kong University of Science and Technology. We are thankful to the University for the hospitality.

2. Notation

In this section we shall briefly recall the notation that we use for general linear and classical groups in the paper. This notation we have already used in [19] and [36] (see also [31], [2] and [20]). More details regarding this notation can be found in those papers (see [40] for the case of general linear groups).

We have fixed a local non-archimedean field F. We consider in this paper symplectic, orthogonal and unitary groups. If we consider unitary groups, then F' denotes the separable quadratic extension of F which enters the definition of the unitary groups. For the other series of groups, we take $F' = F$. We denote by θ the non-trivial F-automorphism of F' if $F' \neq F$. In the case $F' = F$, θ denotes the identity mapping on F. The modulus character of F' is denoted by $|\ \ |_{F'}$, and the character $|\det|_{F'}$ of $GL(n, F')$ is denoted by ν.

For the group G of rational points of a reductive group defined over F, the Grothendieck group of the category $\mathrm{Alg}_{\text{f.l.}}(G)$ of the representations of G of finite length, is denoted by $\mathfrak{R}(G)$ (we consider only smooth representations in this paper). The Grothendieck group $\mathfrak{R}(G)$ carries a natural ordering $\leq$. The semi simplification of $\tau \in \mathrm{Alg}_{\text{f.l.}}(G)$ is denoted by s.s.(τ). For representations $\pi_1, \pi_2 \in \mathrm{Alg}_{\text{f.l.}}(G)$, the fact s.s.$(\pi_1) \leq$ s.s.(π_2) we write shortly as $\pi_1 \leq \pi_2$.

For $0 \leq k \leq n$, there exists a unique standard (with respect to the minimal parabolic subgroup consisting of the upper triangular matrices in $GL(n, F')$) parabolic subgroup $P_{(k,n-k)} = M_{(k,n-k)}N_{(k,n-k)}$ of $GL(n, F')$ whose Levi factor $M_{(k,n-k)}$ is naturally isomorphic to $GL(k, F') \times GL(n-k, F')$. For smooth representations π_1 of $GL(n_1, F')$ and π_2 of

$GL(n_i, F')$, $\pi_1 \times \pi_2$ denotes the smooth representation of $GL(n_1+n_2, F')$ parabolically induced by $\pi_1 \otimes \pi_2$ from $P_{(k,n-k)}$. Let

$$R = \oplus_{n\geq 0}\mathfrak{R}(GL(n, F')).$$

We lift $\times$ to a $\mathbb{Z}$-bilinear mapping $R \times R \to R$. We denote this operation again by $\times$. We can factor the mapping $\times : R \times R \to R$ through $R \otimes R$. The mapping $R \otimes R \to R$ which we obtain in this way is denoted by m.

For a smooth representation π of $GL(n, F')$, we denote by $r_{(k,n-k)}(\pi)$ the normalized Jacquet module with respect to the parabolic subgroup $P_{(k,n-k)}$. The comultiplication $m^* : R \to R \otimes R$ is an additive mapping which is given by $m^*(\pi) = \sum_{k=0}^{n} \text{s.s.}(r_{(k,n-k)}(\pi))$ on irreducible representations. With these two operations, the additive group R becomes a graded Hopf algebra.

Let ρ be an irreducible cuspidal representation of a general linear group (over F'), and $n \in \mathbb{Z}_{\geq 0}$. Define $[\rho, \nu^n\rho] := \{\rho, \nu\rho, \dots, \nu^n\rho\}$. The representation $\nu^n\rho \times \nu^{n-1}\rho \times \dots \times \nu\rho \times \rho$ has a unique irreducible subrepresentation. This is an essentially square integrable representations. It is denoted $\delta([\rho, \nu^n\rho])$. We have

$$m^*(\delta([\rho, \nu^n\rho])) = \sum_{i=-1}^{n} \delta([\rho^{i+1}, \nu^n\rho]) \otimes \delta([\rho, \nu^i\rho]) \tag{2.1}$$

(we take formally $\delta(\emptyset)$ to be the trivial representation of $GL(0, F) = \{1\}$, which is identity of algebra R). Denote

$$\delta(\rho, a) = \delta([\nu^{-\frac{a-1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]), \quad a \in \mathbb{Z}_{\geq 0}.$$

An irreducible essentially square integrable representation δ of $GL(n, F')$ can be written as $\delta = \nu^{e(\delta)}\delta^u$, where $e(\delta) \in \mathbb{R}$ and δ^u is an irreducible unitarizable square integrable representation. This defines $e(\delta)$ and δ^u.

In this paper, we shall fix either a series of symplectic, odd or even orthogonal, or unitary groups.

If we fix the symplectic series, we consider the Witt tower of symplectic spaces, and the space in this tower of dimension $2n$ is denoted by V_n (for symplectic groups we define Y_0 to be the trivial vector space $\{0\}$). Then S_n denotes the group of isomorphisms of V_n.

In the case of a series of odd orthogonal groups, we fix an anisotropic orthogonal vector space Y_0 over F of odd dimension (1 or 3) and consider the Witt tower based on Y_0. Take n such that $2n+1 \geq \dim Y_0$. Then we have exactly one space V_n in the tower of dimension $2n+1$. The special orthogonal group of this space is denoted by S_n.

For an even-orthogonal series of groups, we fix an anisotropic orthogonal space Y_0 over F of even dimension, and consider the Witt tower based on Y_0. Take n such that $2n \geq \dim_F(Y_0)$.

Then there is exactly one space V_n in the tower of dimension $2n$. The orthogonal group of V_n is denoted by S_n.

In the unitary case, we consider unitary groups $U(n, F'/F)$, where F' is the separable quadratic extension of F entering the definition of unitary groups. Here we also fix an anisotropic unitary space Y_0 over F', and consider the Witt tower of unitary spaces V_n based on Y_0. Similarly as in the case of orthogonal groups, we have here also odd and even cases. For $\dim_{F'}(Y_0)$ odd (i.e. 1) and for n satisfying $2n+1 \geq \dim_{F'}(Y_0)$, let V_n be the space in the tower of dimension $2n+1$. Denote the unitary group of this space by S_n. For $\dim_{F'}(Y_0)$ even (i.e. 0) and for n satisfying $2n \geq \dim_{F'}(Y_0)$, let V_n be the space in the tower of dimension $2n$. Denote its unitary group by S_n.

We fix a minimal parabolic subgroup in S_n (in this paper we shall consider only standard parabolic subgroups with respect to this minimal parabolic subgroup). One can see in [31] matrix realizations of the split connected classical groups (and a description of their standard parabolic subgroups).

In what follows, one series of groups $\{S_n\}_n$ as above will be fixed. Denote by n' the Witt index of V_n (i.e. $n' = n - \frac{1}{2}\dim_{F'}(Y_0)$ if V_n is symplectic or even-orthogonal or even-unitary group, and otherwise $n' = n - \frac{1}{2}(\dim_{F'}(Y_0) - 1)$). For an integer k which satisfies $0 \leq k \leq n'$, there is a standard parabolic subgroup $P_{(k)} = M_{(k)}N_{(k)}$ of S_n, whose Levi factor $M_{(k)}$ is naturally isomorphic to $GL(k, F') \times S_{n-k}$ (the group $P_{(k)}$ is the stabilizer of an isotropic space of dimension k). Let π and σ be smooth representations of $GL(k, F')$ and S_{n-k} respectively. We denote by

$$\pi \rtimes \sigma$$

the representation parabolically induced by $\pi \otimes \sigma$ (from $P_{(k)}$). A very useful property of operations $\times$ and $\rtimes$ is

$$\pi_1 \rtimes (\pi_2 \rtimes \sigma) \cong (\pi_1 \times \pi_2) \rtimes \sigma. \tag{2.2}$$

For a smooth representation π of $GL(k, F')$, denote by $\check{\pi}$ the representation $g \mapsto \tilde{\pi}(\theta(g))$ (here $\tilde{\pi}$ denotes the contragredient representation of π). The representation π is called F'/F-selfdual if $\pi \cong \check{\pi}$. In the case $F' = F$, we say also simply that π is selfdual. If π and σ are representations of finite length, then we have

$$\text{s.s.}(\pi \rtimes \sigma) = \text{s.s.}(\check{\pi} \rtimes \sigma). \tag{2.3}$$

Observe that in the case that $\pi \rtimes \sigma$ is irreducible, we have $\pi \rtimes \sigma \cong \check{\pi} \rtimes \sigma$. Denote

$$R(S) = \bigoplus_n \mathfrak{R}(S_n),$$

where the above sum runs over all integers $n \geq \frac{1}{2}(\dim_{F'}(Y_0) - 1)$ if we consider odd-orthogonal or odd-unitary groups, and otherwise over all $n \geq \frac{1}{2}\dim_{F'}(Y_0)$). Now $\rtimes$ lifts in a natural way to a bilinear mapping $R \times R(S) \to R(S)$, denoted again by $\rtimes$.

For a smooth representation τ of S_n, the normalized Jacquet module of τ with respect to $P_{(k)}$ is denoted by $s_{(k)}(\tau)$. If τ is irreducible, denote

$$\mu^*(\tau)=\sum_{k=0}^{n'} \text{s.s.}\left(s_{(k)}(\tau)\right),$$

where n' denotes the Witt index of V_n. Extend μ^* additively to $\mu^*: R(S)\to R\otimes R(S)$. Let

$$M^*=(m\otimes 1)\circ(\,\check{}\,\otimes m^*)\circ\kappa\circ m^*: R\to R\otimes R, \tag{2.4}$$

where $\check{}\,: R\to R$ is a group homomorphism determined by the requirement that $\pi\mapsto\check{\pi}$ for all irreducible representations π, and where $\kappa: R\times R\to R\times R$ maps $\sum x_i\otimes y_i$ to $\sum y_i\otimes x_i$. The action $\rtimes$ of $R\otimes R$ on $R\otimes R(S)$ is defined in a natural way. Then

$$\mu^*(\pi\rtimes\sigma)=M^*(\pi)\rtimes\mu^*(\sigma) \tag{2.5}$$

holds for $\pi\in R$ and $\sigma\in R(S)$.

Let ρ be an irreducible F'/F-selfdual cuspidal representation of a general linear group and $x,y\in\mathbb{R}$ which satisfy $y-x\in\mathbb{Z}_{\geq 0}$. Then (2.1) and (2.5) imply

$$M^*\big(\delta([\nu^x\rho,\nu^y\rho])\big)=\sum_{i=x-1}^{y}\sum_{j=i}^{y}\delta([\nu^{-i}\rho,\nu^{-x}\rho])\times\delta([\nu^{j+1}\rho,\nu^{y}\rho])\otimes\delta([\nu^{i+1}\rho,\nu^{j}\rho]), \tag{2.6}$$

where $y-i, y-j\in\mathbb{Z}_{\geq 0}$ in the above sums. The part of $M^*\big(\delta([\nu^x\rho,\nu^y\rho])\big)$ contained in $R\otimes R_0$ will be denoted by $M^*_{GL}\big(\delta([\nu^x\rho,\nu^y\rho])\big)\otimes 1$. Then

$$M^*_{GL}\big(\delta([\nu^x\rho,\nu^y\rho])\big)=\sum_{i=x-1}^{y}\delta([\nu^{-i}\rho,\nu^{-x}\rho])\times\delta([\nu^{i+1}\rho,\nu^{y}\rho]) \tag{2.7}$$

(again $y-i\in\mathbb{Z}_{\geq 0}$ in the above sum).

Now we shall recall the definition of Jordan blocks attached to an irreducible square integrable representation of a classical group. First we shall define representations R_k of the L-group of F-group $GL(k,F')$. Suppose that we consider the symplectic series of groups (resp. one of orthogonal series of groups), Let $k\in\mathbb{Z}_{>0}$. Then by R_k will be denote the representation of $GL(k,\mathbb{C})$ on $\wedge^2\mathbb{C}^k$ (resp., $\mathrm{Sym}^2\mathbb{C}^k$).

Now suppose that we consider a series of unitary groups. Then the L-group of F-group $GL(k,F')$ is isomorphic to a semidirect product $\big(GL(k,\mathbb{C})\times GL(k,\mathbb{C})\big)\leftthreetimes Gal(F'/F)$. The non-trivial element θ of $Gal(F'/F)$ acts on the normal subgroup $GL(k,\mathbb{C})\times GL(k,\mathbb{C})$ by $\theta(g_1,g_2,1)\theta^{-1}=({}^tg_2^{-1},{}^tg_1^{-1},1)$, where ${}^tg$ denotes the transposed matrix of g (see [15]). For $\eta\in\{\pm 1\}$ denote by $R_k^{(\eta)}$ the representation of the above L-group (of $GL(k,F')$) on $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^k)$, determined by $(g_1,g_2,1)\,u=g_1\,u\,{}^tg_2$ and $(1,1,\theta)\,u=\eta\,{}^tu$. Suppose that S_n is a series of (unitary) groups such that dimensions of V_n are even (resp. odd). Then by R_k will be denoted the representation $R_k^{(1)}$ (resp. $R_k^{(-1)}$).

Now we have the following definition of the Jordan blocks (from [15]; $L(\rho, R_{d_\rho}, s)$ denotes the L-function defined by F. Shahidi).

Definition 2.1. *The set $Jord(\pi)$ of Jordan blocks attached to an irreducible square integrable representation π of a classical group S_n is the set of all pairs (ρ, a) where ρ is an irreducible F'/F-selfdual cuspidal representation of $GL(d_\rho, F')$ and $a \in \mathbb{Z}_{>0}$, such that:*

(J1) *a is even if $L(\rho, R_{d_\rho}, s)$ has a pole at $s = 0$, and odd otherwise,*
(J2) *$\delta(\rho, a) \rtimes \pi$ is irreducible.*

Denote $Jord_\rho(\pi) = \{a; (\rho, a) \in Jord(\pi)\}$ (ρ is an irreducible F'/F-selfdual cuspidal representation of a general linear group).

In this paper we shall assume that the following (basic) assumption holds for any irreducible cuspidal representation π_c of any S_q, and for any irreducible F'/F-selfdual cuspidal representation ρ of any $GL(p, F')$:

(BA) If we denote by

$$a_{\pi_c,\max,\rho} = \begin{cases} \max Jord_\rho(\pi_c) & \text{if } Jord_\rho(\pi_c) \neq \emptyset, \\ 0 & \text{if } L(\rho, R_{d_\rho}, s) \text{ has a pole at } s = 0 \text{ and } Jord_\rho(\pi_c) = \emptyset, \\ -1 & \text{otherwise.} \end{cases}$$

then

$$\nu^{\pm(1+a_{\pi_c,\max,\rho})/2}\rho \rtimes \pi_c$$

reduces.

Remark 2.2. *The basic assumption is known to hold in some cases, while for (the most) of other cases, it is expected to be known fact very soon (when the facts on which relays* [1] *become available).*

For unitary groups, it follows from C. Mœglin's paper [17]*. Namely, by* [29] *we know that there is only one non-negative exponent x for which $\nu^x\rho \rtimes \pi_c$ is reducible. Propositions 3.1 of* [17] *gives that this exponent is integer or half-integer. If it is 0 or 1/2, then Proposition 13.2 of* [32] *and* [16] *imply that $Jord_\rho(\pi_c) = \emptyset$. Suppose that $\nu^x\rho \rtimes \pi_c$ reduces for some $x \geq 1$. Now Theorem 13.2 of* [32] *implies that $\delta(\rho, n) \rtimes \pi_c$ is irreducible for all positive integers of the parity same as the parity of $2x$, while for the other parity we have irreducibility for $n \leq 2x - 1$, and reducibility otherwise. Note that now the unique irreducible subrepresentation of $\nu^x\rho \rtimes \pi_c$ is square integrable. Denote it by π. Then the extended cuspidal support of π contains $\nu^x\rho$. Now Proposition 5.6 of* [17] *implies that $2x + 1$ has even parity if the L-function from the Definition 2.1 has a pole at $s = 0$, and odd otherwise. Therefore, the basic assumption holds for unitary groups.*

Observe that [17] *gives much more then the basic assumption. It gives the full classification of irreducible square integrable representations, and it gives also the full classification of the (subfamily) of cuspidal representations of these groups (moreover, from the parameters on the Galois side one can see what are the cuspidal reducibilities).*

For the odd orthogonal and symplectic groups, it follows from remark (ii) in Remarks 4.5.2 of the paper [21] *of C. Mœglin and J.-L. Waldspurger and the book* [1] *of J. Arthur (Theorem 1.5.1). We are very thankful to C. Mœglin for explaining this to us.*

In this section we have recalled the definition of invariants Jord(π) and π_{cusp}. The definition of the third invariant ϵ_π, convenient for our purposes, can be found in [36] (see also section 7 of this paper). Let us recall that ϵ_π is defined on a subset of $\mathrm{Jord}(\pi) \cup \mathrm{Jord}(\pi)\times \mathrm{Jord}(\pi)$, and that it takes values in $\{\pm 1\}$.

3. Two basic preliminary facts about Jordan blocks and partially defined functions

Here we shall recall two basic facts about Jordan blocks and partially defined functions which we shall use often in the paper.

First we shall recall a general fact regarding Jordan blocks (Proposition 2.1 of [19]):

Proposition 3.1. *Let π' be an irreducible square integrable representation of S_q and let $x, y \in (1/2)\mathbb{Z}$ such that $x - y \in \mathbb{Z}_{\geq 0}$. Let ρ be an F'/F-selfdual cuspidal (unitarizable) representation of $GL(d_\rho)$. We assume that x, $y \in \mathbb{Z}$ if and only if $L(\rho, R_{d_\rho}, s)$ has no pole at $s = 0$. Further, suppose that there is an irreducible square integrable representation π embedded in the induced representation*

$$\pi \hookrightarrow \delta([\nu^y\rho, \nu^x\rho]) \rtimes \pi'. \tag{3.1}$$

or more generally, into

$$\pi \hookrightarrow \nu^x\rho \times \cdots \times \nu^{x-i+1}\rho \times \cdots \times \nu^y\rho \rtimes \pi'. \tag{3.2}$$

Then holds:

(1) *If $y > 0$, then*
$$Jord(\pi) = (Jord(\pi') \setminus \{(\rho, 2y-1)\}) \cup \{(\rho, 2x+1)\}.$$
Further, $2y - 1 \in Jord_\rho(\pi')$ if $y \geq 1$.

(2) *If $y \leq 0$, then*
$$Jord(\pi) = Jord(\pi') \cup \{(\rho, 2x+1), (\rho, -2y+1)\}.$$
In particular, $2x + 1$ and $-2y + 1$ are not in $Jord_\rho(\pi')$.

The following theorem follows directly from Lemmas 5.4, 5.5 and (ii) of Proposition 6.1 (all) from [15]:

Theorem 3.2. *Let π' be an irreducible square integrable representation of S_q. Take $a, a_- \in \mathbb{Z}$ such that $a - a_- \in 2\mathbb{Z}_{>0}$. Let ρ be an F'/F-selfdual cuspidal representation of $GL(d_\rho)$. Suppose that a_-, $a \in 1 + 2\mathbb{Z}$ if and only if $L(\rho, R_{d_\rho}, s)$ has no pole at $s = 0$. Assume that*

$$\mathrm{Jord}_\rho(\pi') \cap [a_-, a] = \emptyset.$$

Let π be an irreducible subrepresentation of

$$\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'. \tag{3.3}$$

Then π is square integrable and

$$\mathrm{Jord}_\rho(\pi) = Jord_\rho(\pi') \cup \{(\rho, a_-), (\rho, a)\}.$$

Further, for $(\rho', b) \in Jord_\rho(\pi')$, $\epsilon_\pi(\rho', b)$ is defined if and only if $\epsilon_{\pi'}(\rho', b)$ is defined. If they are defined, then

$$\epsilon_\pi(\rho', b) = \epsilon_{\pi'}(\rho', b).$$

If there exists $(\rho', c) \in \mathrm{Jord}(\pi')$ with $c \neq b$, then

$$\epsilon_\pi(\rho', b)\epsilon_\pi(\rho', c)^{-1} = \epsilon_{\pi'}(\rho', b)\epsilon_{\pi'}(\rho', c)^{-1}.$$

Definition 3.3. *The square integrable representation π' in the above theorem is uniquely determined (up to an equivalence) by π, and we shall denote it by*

$$\pi' = \pi^{-\{(\rho, a_-), (\rho, a)\}}.$$

Now we shall recall Lemma 5.3 of [19]:

Lemma 3.4. *Let π be an irreducible square integrable representation and suppose that there exists $a \in Jord_\rho(\pi)$ such that $a + 2 \notin Jord_\rho(\pi)$. Then*

$$\nu^{(a+1)/2}\rho \rtimes \pi$$

reduces. Further, it contains a unique irreducible subrepresentation

We recall next the elementary Lemma 3.2 of [19] which shall be used often without mentioning.

Lemma 3.5. *Let π be an irreducible representation of a reductive p-adic group and let $P = MN$ be a parabolic subgroup of G. Suppose that M is a direct product of two reductive subgroups M_1 and M_2. Let τ_1 be an irreducible representation of M_1, and let τ_2 be a representation of M_2. Suppose*

$$\pi \hookrightarrow Ind_P^G(\tau_1 \otimes \tau_2).$$

Then there exists an irreducible representation τ_2' such that

$$\pi \hookrightarrow Ind_P^G(\tau_1 \otimes \tau_2').$$

We shall often use the following

Proposition 3.6. *Let π be an irreducible square integrable representation of S_q.*

(1) *Suppose that $\nu^x\rho \otimes \tau$ is an irreducible subquotient of a standard Jacquet module of π, where ρ is an irreducible F'/F-selfdual cuspidal representation of $GL(p, F')$, $x \in \mathbb{R}$, and τ is an irreducible representation of $S_{q'}$. Then*
$$(\rho, 2x + 1) \in Jord(\pi).$$
(2) *More generally, let $\sigma \otimes \tau$ be an irreducible subquotient of a standard Jacquet module of π, where σ is an irreducible representation of $GL(p, F')$, and τ is an irreducible representation of $S_{q'}$. Then there exists an irreducible cuspidal representation ρ' in the cuspidal support of σ such that if we write $\rho' = \nu^x\rho$ with $x \in \mathbb{R}$ and ρ unitarizable, then*
$$(\rho, 2x + 1) \in Jord(\pi).$$

One can find the definition of the cuspidal support in Remark 6.3 in what follows.

Proof. The claim (1) is just Lemma 3.6 of [19] (for a slightly different argument observe that (3) of Corollary 6.2 implies that $\pi \hookrightarrow \nu^x\rho \rtimes \sigma$ for some irreducible representation σ; now Remark 5.1.2 of [15] implies (1)).

For the second claim, observe that the the transitivity of Jacquet modules implies that there exists an irreducible cuspidal representation ρ' in the cuspidal support of σ and an irreducible representation μ of a classical group such that $\rho' \otimes \mu$ is a subquotient of the Jacquet module of π. Now (1) implies (2). □

Let (ρ, a) be a Jordan block of an irreducible square integrable representation π of a classical group. If there exists the element b in $\text{Jord}_\rho(\pi)$ such that $b < a$, and $\{x \in \text{Jord}_\rho(\pi); b < x < a\} = \emptyset$, then we shall denote b by
$$a_-.$$
In this case we shall say that $a \in \text{Jord}_\rho(\pi)$ has a_-.

4. First tempered reductions

We recall the definition of cuspidal support in the case of general linear groups. If an irreducible representation π of a general linear group is a subquotient of $\rho_1 \times \ldots \times \rho_k$ where ρ_i are irreducible and cusidal, then the multiset $(\rho_1, \ldots, \rho_k)$ is called the cuspidal support of π, and denoted by
$$\text{supp}(\pi).$$

Lemma 4.1. *Let π be an irreducible square integrable representation of a classical group, ρ an F'/F-selfdual irreducible cuspidal representation of a general linear group and $b \in \mathbb{Z}_{>0}$ such that*

$$\delta(\rho, b) \rtimes \pi$$

reduces (equivalently, ρ and b satisfy (J1), and $b \notin \mathrm{Jord}_\rho(\pi)$).

Suppose

$$\mathrm{Jord}_\rho(\pi) \cap [1, b] \neq \emptyset \tag{4.1}$$

and denote

$$a = \max(\mathrm{Jord}_\rho(\pi) \cap [1, b]). \tag{4.2}$$

Then:

(1) *There exists an irreducible representation $\theta \otimes \sigma$ satisfying*

$$\theta \otimes \sigma \leq \mu^*(\delta(\rho, b) \rtimes \pi)$$

and

$$supp(\theta) = [\nu^{(a-1)/2+1}\rho, \nu^{(b-1)/2}\rho] + [\nu^{(a-1)/2+1}\rho, \nu^{(b-1)/2}\rho]. \tag{4.3}$$

Further, such θ and σ are unique, and satisfy

$$\theta \otimes \sigma \cong \delta([\nu^{(a-1)/2+1}\rho, \nu^{(b-1)/2}\rho])^2 \otimes \delta(\rho, a) \rtimes \pi.$$

(2) *The multiplicity of*

$$\delta([\nu^{(a-1)/2+1}\rho, \nu^{(b-1)/2}\rho])^2 \otimes \delta(\rho, a) \rtimes \pi$$

in

$$\mu^*(\delta(\rho, b) \rtimes \pi)$$

is one.

(3) *$\delta([\nu^{(a-1)/2+1}\rho, \nu^{(b-1)/2}\rho])^2 \otimes \delta(\rho, a) \rtimes \pi$ is a direct summand in the corresponding Jacquet module of $\delta(\rho, b) \rtimes \pi$.*

Proof. Observe that $b \geq 3$. Suppose that $\theta \otimes \sigma$ is some irreducible representation satisfying the assumption (4.3). We shall now consider the formula for $\mu^*(\delta(\rho, b) \rtimes \pi)$, determine the terms of the sums where representation of type $\theta \otimes \sigma$ can be a subquotient, determine $\theta \otimes \sigma$ and determine the multiplicity of $\theta \otimes \sigma$.

From (2.5) and (2.6) follows that $\mu^*(\delta(\rho, b) \rtimes \pi)$ is

$$\left(\sum_{i=-(b-1)/2-1}^{(b-1)/2} \sum_{j=i}^{(b-1)/2} \delta([\nu^{-i}\rho, \nu^{(b-1)/2}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{(b-1)/2}\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]) \right) \rtimes \mu^*(\pi). \tag{4.4}$$

We shall now analyze in the above sums indexes i and j for which we have possibility to get a representation $\theta\otimes\sigma$ satisfying (4.3) for a subquotient. The condition on the support of θ implies that if $\theta\otimes\sigma$ as above is a subquotient, then the indexes must satisfy

$$(a-1)/2+1\leq -i \text{ and } (a-1)/2+1\leq j+1. \tag{4.5}$$

Suppose now that $\theta\otimes\sigma$ as above is a subquotient of some term corresponding to indexes i and j which satisfy

$$(a-1)/2+1< -i \text{ or } (a-1)/2+1< j+1. \tag{4.6}$$

Then there exists an irreducible subquotient $\gamma\otimes\lambda$ of $\mu^*(\pi)$ such that

$$\theta\otimes\sigma\leq \delta([\nu^{-i}\rho,\nu^{(b-1)/2}\rho])\times\delta([\nu^{j+1}\rho,\nu^{(b-1)/2}\rho])\times\gamma\otimes\delta([\nu^{i+1}\rho,\nu^{j}\rho])\rtimes\lambda.$$

The assumption on the cuspidal support of θ implies $\text{supp}(\gamma)=[\nu^{(a-1)/2+1}\rho,\nu^{-i-1}\rho]+[\nu^{(a-1)/2+1}\rho,\nu^{j}\rho]$. Recall that our assumption (4.6) implies $\text{supp}(\gamma)\neq\emptyset$. Now (2) of Proposition 3.6 implies the existence of

$$\nu^x\rho\in[\nu^{(a-1)/2+1}\rho,\nu^{-i-1}\rho]\cup[\nu^{(a-1)/2+1}\rho,\nu^{j}\rho]$$

such that $2x+1\in \text{Jord}_\rho(\pi)$. Observe that $(a-1)/2+1\leq x$ implies

$$a+2\leq 2x+1.$$

From the other side, $i\geq -(b-1)/2-1$ implies $x\leq -i-1\leq (b-1)/2$ which further implies $2x+1\leq b$, and $j\leq (b-1)/2$ implies again $2x+1\leq 2j+1\leq b$.

Therefore, we have obtained that (4.6) implies $[a+2,b]\cap\text{Jord}_\rho(\pi)\neq\emptyset$. This contradicts the assumption (4.2). This contradiction implies that $\theta\otimes\sigma$ as above can not be a subquotient of some term corresponding to indexes i and j which satisfy (4.6).

Thus, if $\theta\otimes\sigma$ as above is a subquotient of some term corresponding to indexes i and j, then these indexes must satisfy $(a-1)/2+1=-i$ and $(a-1)/2+1=j+1$.

Observe that if we take in (4.4) the term from the double sum corresponding to indexes $(a-1)/2+1=-i$ and $(a-1)/2+1=j+1$, and from $\mu^*(\pi)$ the term $1\otimes\pi$, then we get in (4.4) the term

$$\delta([\nu^{(a-1)/2+1}\rho,\nu^{(b-1)/2}\rho])^2\otimes\delta([\nu^{-(a-1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\pi.$$

Obsetve that this representation is irreducible, and satisfies (4.3). This shows that that there is at least one $\theta\otimes\sigma$ as above.

Suppose now that $\theta\otimes\sigma$ is any representation as above (i.e. satisfying (4.3)), which is a subquotient of $\mu^*(\delta(\rho,b)\rtimes\pi)$ (we have seen that there exists at least one such representation). Then

$$\theta\otimes\sigma\leq\delta([\nu^{(a-1)/2+1}\rho,\nu^{(b-1)/2}\rho])^2\times\gamma\otimes\delta([\nu^{-(a-1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\lambda$$

for some $\gamma\otimes\lambda\leq\mu^*(\pi)$. Now the assumption on the cuspidal support implies that $\gamma=1$. Therefore $1\otimes\lambda\leq\mu^*(\pi)$, which implies $1\otimes\lambda\cong 1\otimes\pi$. Thus

$$\theta\otimes\sigma\leq\delta([\nu^{(a-1)/2+1}\rho,\nu^{(b-1)/2}\rho])^2\otimes\delta([\nu^{-(a-1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\pi.$$

Since the representation on the right hand side is irreducible, we have the equality above. Further, since the multiplicity of $1\otimes\pi$ in $\mu^*(\pi)$ is one, we conclude that the multiplicity of $\theta\otimes\sigma$ in $\mu^*(\delta(\rho,b)\rtimes\pi)$ is one.

This completes the proof of (1) and (2) of the lemma. Further, using the properties of the Bernstein center (see section 6) (3) follows from (1). □

Corollary 4.2. *Let the assumptions be the same as in the previous lemma. Then the following requirements on an irreducible subquotient τ of $\delta(\rho,b)\rtimes\pi$ are equivalent:*

1. *τ embeds into $\delta([\nu^{(a-1)/2+1}\rho,\nu^{(b-1)/2}\rho])^2\times\delta([\nu^{-(a-1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\pi$.*
2. *τ embeds into $\delta([\nu^{(a-1)/2+1}\rho,\nu^{(b-1)/2}\rho])^2\rtimes\lambda$ for some irreducible representation λ.*
3. *τ embeds into $\theta\rtimes\lambda$ for some irreducible representations θ and λ, such that supp$(\theta)=2[\nu^{(a-1)/2+1}\rho,\nu^{(b-1)/2}\rho]$.*
4. *τ has $\delta([\nu^{(a-1)/2+1}\rho,\nu^{(b-1)/2}\rho])^2\otimes\delta([\nu^{-(a-1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\pi$ for a subquotient of the corresponding Jacquet module.*
5. *τ has $\delta([\nu^{(a-1)/2+1}\rho,\nu^{(b-1)/2}\rho])^2\otimes\lambda$ for a subquotient of the corresponding Jacquet module, for some irreducible representation λ.*
6. *τ has $\theta\otimes\lambda$ for a subquotient of the corresponding Jacquet module, for some irreducible representations θ and λ, such that supp$(\theta)=2[\nu^{(a-1)/2+1}\rho,\nu^{(b-1)/2}\rho]$.* □

Definition 4.3. *With the assumptions as in the previous lemma, the irreducible subquotient τ of $\delta(\rho,b)\rtimes\pi$ satisfying the equivalent requirements of the above corollary, will be denoted by*

$$\pi_{\delta(\rho,b)}.$$

The other irreducible subrepresentation of $\delta(\rho,b)\rtimes\pi$ will be denoted by

$$\pi_{-\delta(\rho,b)}.$$

Lemma 4.4. *Let π be an irreducible square integrable representation of a classical group, ρ an F'/F-selfdual irreducible cuspidal representation of a general linear group and*

$$b\in 2\mathbb{Z}_{>0}$$

such that

$$\delta(\rho,b)\rtimes\pi$$

reduces (i.e., ρ and b satisfy (J1), and $b\notin\mathrm{Jord}_\rho(\pi)$).

Suppose

$$\mathrm{Jord}_\rho(\pi)\cap[1,b]=\emptyset. \tag{4.7}$$

Then

(1) *There exists an irreducible representation* $\theta \otimes \sigma$ *satisfying* $\theta \otimes \sigma \leq \mu^*(\delta(\rho, b) \rtimes \pi)$ *and*
$$supp(\theta) = [\nu^{1/2}\rho, \nu^{(b-1)/2}\rho] + [\nu^{1/2}\rho, \nu^{(b-1)/2}\rho].$$
Such θ *and* σ *are unique, and holds*
$$\theta \otimes \sigma \cong \delta([\nu^{1/2}\rho, \nu^{(b-1)/2}\rho])^2 \otimes \pi.$$
(2) *The multiplicity of*
$$\delta([\nu^{1/2}\rho, \nu^{(b-1)/2}\rho])^2 \otimes \pi$$
in
$$\mu^*(\delta(\rho, b) \rtimes \pi)$$
is one.
(3) $\delta([\nu^{1/2}\rho, \nu^{(b-1)/2}\rho])^2 \otimes \pi$ *is a direct summand in the corresponding Jacquet module of* $\delta(\rho, b) \rtimes \pi$.

Proof. The proof proceeds in a very similar way as the proof of Lemma 4.1. We shall very briefly sketch the proof. To get any $\theta \otimes \sigma$ satisfying (1) from (4.4), we must take $1/2 \leq -i$ and $1/2 \leq j+1$. If at least one of these two inequalities is strict, we would get that some positive even integer $k \leq b$ is in $\mathrm{Jord}_\rho(\pi)$, which contradicts the assumption of the lemma. Now the proof proceeds in exactly the same way as the proof of Lemma 4.1. ☐

Corollary 4.5. *Let the assumptions be the same as in the previous lemma. Then the following requirements on an irreducible subquotient* τ *of* $\delta(\rho, b) \rtimes \pi$ *are equivalent:*

(1) τ *embeds into* $\delta([\nu^{1/2}\rho, \nu^{(b-1)/2}\rho])^2 \rtimes \pi$.
(2) τ *embeds into* $\delta([\nu^{1/2}\rho, \nu^{(b-1)/2}\rho])^2 \rtimes \lambda$ *for some irreducible representation* λ.
(3) τ *embeds into* $\theta \rtimes \lambda$ *for some irreducible representations* θ *and* λ, *such that* $supp(\theta) = 2[\nu^{1/2}\rho, \nu^{(b-1)/2}\rho]$.
(4) τ *has* $\delta([\nu^{1/2}\rho, \nu^{(b-1)/2}\rho])^2 \otimes \pi$ *for a subquotient of its Jacquet module.*
(5) τ *has* $\delta([\nu^{1/2}\rho, \nu^{(b-1)/2}\rho])^2 \otimes \lambda$ *for a subquotient of its Jacquet module, for some irreducible representation* λ.
(6) τ *has* $\theta \otimes \lambda$ *for a subquotient of its Jacquet module, for some irreducible representations* θ *and* λ, *such that* $supp(\theta) = 2[\nu^{1/2}\rho, \nu^{(b-1)/2}\rho]$. ☐

Definition 4.6. *With the assumptions as in the previous lemma, the irreducible subquotient* τ *of* $\delta(\rho, b) \rtimes \pi$ *satisfying the equivalent requirements of the above corollary, will be denoted by*
$$\pi_{\delta(\rho,b)}.$$
The other irreducible subrepresentation of $\delta(\rho, b) \rtimes \pi$ *will be denoted by* $\pi_{-\delta(\rho,b)}$.

Lemma 4.7. *Let* π *be an irreducible square integrable representation of a classical group,* ρ *an* F'/F*-selfdual irreducible cuspidal representation of a general linear group and*
$$b \in 1 + 2\mathbb{Z}_{\geq 0}$$

such that

$$\delta(\rho, b) \rtimes \pi$$

reduces (i.e., ρ and b satisfy (J1), and $b \notin \mathrm{Jord}_\rho(\pi)$).

Suppose

$$\mathrm{Jord}_\rho(\pi) \cap [1, b] = \emptyset \tag{4.8}$$

and

$$\mathrm{Jord}_\rho(\pi) \neq \emptyset.$$

Denote

$$a := \min(\mathrm{Jord}_\rho(\pi)).$$

Then:

(1) *There exists an irreducible representation $\theta \otimes \sigma$ satisfying $\theta \otimes \sigma \leq \mu^*(\delta(\rho, b) \rtimes \pi)$ and*

$$supp(\theta) = [\nu\rho, \nu^{(a-1)/2}\rho] + [\nu\rho, \nu^{(b-1)/2}\rho] + [\nu\rho, \nu^{(b-1)/2}\rho].$$

Such θ and σ are unique. More precisely, we have

$$\theta \cong \delta([\nu\rho, \nu^{(a-1)/2}\rho]) \times \delta([\nu\rho, \nu^{(b-1)/2}\rho])^2.$$

Further, (ii) of Lemma 5.4.2 in [15] *implies that $\pi \hookrightarrow \delta([\nu\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'$ for some irreducible square integrable representation π' of a classical group. Then $\rho \rtimes \pi'$ is irreducible and*

$$\sigma \cong \rho \rtimes \pi'.$$

(2) *The multiplicity of $\delta([\nu\rho, \nu^{(a-1)/2}\rho]) \times \delta([\nu\rho, \nu^{(b-1)/2}\rho])^2 \otimes \rho \rtimes \pi'$ in $\mu^*(\delta(\rho, b) \rtimes \pi)$ is one.*

(3) *$\delta([\nu\rho, \nu^{(a-1)/2}\rho]) \times \delta([\nu\rho, \nu^{(b-1)/2}\rho])^2 \otimes \rho \rtimes \pi'$ is a direct summand in the corresponding Jacquet module of $\delta(\rho, b) \rtimes \pi$.*

Remark 4.8. *The representation π' in (2) of the above lemma has explicit description by the admissible triple (see section 8).*

Proof. (1) Clearly, $a \geq 3$. Observe that from $\pi \hookrightarrow \delta([\nu\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'$ and (1) of Proposition 3.1 follow

$$Jord(\pi) = \big(Jord(\pi') \backslash \{(\rho, 1)\}\big) \cup \{(\rho, a)\},$$

i.e.

$$Jord(\pi') = \big(Jord(\pi) \backslash \{(\rho, a)\}\big) \cup \{(\rho, 1)\}. \tag{4.9}$$

We have obviously

$$\mu^*(\delta(\rho, b) \rtimes \pi) \leq \mu^*(\delta(\rho, b) \times \delta([\nu\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi') =$$
$$M^*(\delta(\rho, b)) \times M^*(\delta([\nu\rho, \nu^{(a-1)/2}\rho])) \rtimes \mu^*(\pi').$$

Now we shall analyze when we can get an irreducible term $\theta \otimes \sigma$ such that

$$\text{supp}(\theta) = [\nu\rho, \nu^{(a-1)/2}\rho] + [\nu\rho, \nu^{(b-1)/2}\rho] + [\nu\rho, \nu^{(b-1)/2}\rho]$$

in the right hand side of the above inequality. Considering the cuspidal support of θ and Jordan blocks of π' (see (4.9)), we see that to get $\theta \otimes \sigma$ as a subquotient, we must take from $M^*(\delta(\rho, b))$ the term $\delta([\nu\rho, \nu^{(b-1)/2}\rho])^2 \otimes \rho$, from $M^*(\delta([\nu\rho, \nu^{(a-1)/2}\rho])$ the term $\delta([\nu\rho, \nu^{(a-1)/2}\rho]) \otimes 1$ and from $\mu^*(\pi')$ the term $1 \otimes \pi'$. Thus

$$\theta \otimes \sigma \cong \delta([\nu\rho, \nu^{(a-1)/2}\rho]) \times \delta([\nu\rho, \nu^{(b-1)/2}\rho])^2 \otimes \rho \rtimes \pi'. \tag{4.10}$$

This proves multiplicity one in $\mu^*(\delta(\rho, b) \times \delta([\nu\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi')$. Note that the irreducibility of $\rho \rtimes \pi'$ follows from (4.9). Uniqueness of σ will follow from section 8.

From the other side $\pi \hookrightarrow \delta([\nu\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'$ implies that $\delta([\nu\rho, \nu^{(a-1)/2}\rho]) \otimes \pi'$ is in the Jacquet module of π. Observe that $\delta([\nu\rho, \nu^{(b-1)/2}\rho])^2 \otimes \rho \leq M^*(\delta(\rho, b))$. From this we get that the term $\delta([\nu\rho, \nu^{(a-1)/2}\rho]) \times \delta([\nu\rho, \nu^{(b-1)/2}\rho])^2 \otimes \rho \rtimes \pi'$ must show up in $\mu^*(\delta(\rho, b) \rtimes \pi)$. The condition on the cuspidal support of θ implies that this term is a direct summand in the corresponding Jacquet module. □

Corollary 4.9. *Let the assumptions be the same as in the previous lemma. Then the following requirements on an irreducible subquotient τ of $\delta(\rho, b) \rtimes \pi$ are equivalent:*

1. *τ embeds into $\delta([\nu\rho, \nu^{(b-1)/2}\rho])^2 \times \delta([\nu\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'$.*
2. *τ embeds into $\delta([\nu\rho, \nu^{(b-1)/2}\rho])^2 \times \delta([\nu\rho, \nu^{(a-1)/2}\rho]) \rtimes \lambda$ for some irreducible representation λ.*
3. *τ embeds into $\theta \rtimes \lambda$ for some irreducible representations θ and λ, such that supp$(\theta) = 2[\nu\rho, \nu^{(b-1)/2}\rho] + [\nu\rho, \nu^{(a-1)/2}\rho]$.*
4. *τ has $\delta([\nu\rho, \nu^{(b-1)/2}\rho])^2 \times \delta([\nu\rho, \nu^{(a-1)/2}\rho]) \otimes \pi'$ for a subquotient of its Jacquet module.*
5. *τ has $\delta([\nu\rho, \nu^{(b-1)/2}\rho])^2 \times \delta([\nu\rho, \nu^{(a-1)/2}\rho]) \otimes \lambda$ for a subquotient of its Jacquet module, for some irreducible representation λ.*
6. *τ has $\theta \otimes \lambda$ for a subquotient of its Jacquet module, for some irreducible representations θ and λ, such that supp$(\theta) = 2[\nu\rho, \nu^{(b-1)/2}\rho] + [\nu\rho, \nu^{(a-1)/2}\rho]$.* □

Definition 4.10. *With the assumptions as in the previous lemma, the irreducible subquotient τ of $\delta(\rho, b) \rtimes \pi$ satisfying the equivalent requirements of the above corollary, will be denoted by*

$$\pi_{\delta(\rho,b)}.$$

The other irreducible subrepresentation of $\delta(\rho, b) \rtimes \pi$ will be denoted by $\pi_{-\delta(\rho,b)}$.

Lemma 4.11. *Let π be an irreducible square integrable representation of a classical group, ρ an F'/F-selfdual irreducible cuspidal representation of a general linear group and*

$$b \in 1 + 2\mathbb{Z}_{\geq 0}$$

such that

$$\delta(\rho, b) \rtimes \pi$$

reduces (i.e., ρ and b satisfy (J1), and $b \notin \mathrm{Jord}_\rho(\pi)$).

Suppose

$$\mathrm{Jord}_\rho(\pi) = \emptyset.$$

Then $\rho \rtimes \pi_{cusp}$ reduces. Decompose

$$\rho \rtimes \pi_{cusp} = \tau_1 \oplus \tau_{-1}.$$

Then for an irreducible subrepresentation T of

$$\delta(\rho, b) \rtimes \pi$$

there is a unique $i \in \{\pm 1\}$ such that there exists an irreducible representation θ of a general linear group such that

$$T \hookrightarrow \theta \rtimes \tau_i.$$

We shall denote this subrepresentation T by

$$\pi_{i\delta(\rho,b)}$$

(we consider $i\delta(\rho, b)$ as an element of the Hopf algebra R in a natural way). We can also characterize above T by the requirement that a term of the form $\theta \otimes \tau_i$ is in the Jacquet module of T.

Proof. Observe

$$T \hookrightarrow \delta(\rho, b) \rtimes \pi \hookrightarrow \delta([\rho, \nu^{(b-1)/2}\rho]) \times \delta([\nu^{-(b-1)/2}\rho, \nu^{-1}\rho]) \rtimes \pi.$$

We have $\delta([\nu^{-(b-1)/2}\rho, \nu^{-1}\rho]) \rtimes \pi \cong \delta([\nu\rho, \nu^{(b-1)/2}\rho]) \rtimes \pi$ by (2.3), Lemma 9.2 and Proposition 8.1. Thus

$$\begin{aligned} T &\hookrightarrow \delta([\rho, \nu^{(b-1)/2}\rho]) \times \delta([\nu\rho, \nu^{(b-1)/2}\rho]) \rtimes \pi \\ &\cong \delta([\nu\rho, \nu^{(b-1)/2}\rho]) \times \delta([\rho, \nu^{(b-1)/2}\rho]) \rtimes \pi \\ &\hookrightarrow \delta([\nu\rho, \nu^{(b-1)/2}\rho]) \times \delta([\nu\rho, \nu^{(b-1)/2}\rho]) \times \rho \rtimes \pi. \end{aligned}$$

We know that $\pi \hookrightarrow \theta' \rtimes \pi_{cusp}$ for some irreducible representation θ' of a general linear group (this is the definition of π_{cusp}). Note that the condition $\mathrm{Jord}_\rho(\pi) = \emptyset$ implies $\rho \times \theta' \cong \theta' \times \rho$. Then

$$\begin{aligned} T &\hookrightarrow \delta([\nu\rho, \nu^{(b-1)/2}\rho]) \times \delta([\nu\rho, \nu^{(b-1)/2}\rho]) \times \rho \times \theta' \rtimes \pi_{cusp} \\ &\cong \delta([\nu\rho, \nu^{(b-1)/2}\rho]) \times \delta([\nu\rho, \nu^{(b-1)/2}\rho]) \times \theta' \times \rho \rtimes \pi_{cusp}. \end{aligned}$$

Obviously there exists $i \in \{\pm 1\}$ such that

$$T \hookrightarrow \delta([\nu\rho, \nu^{(b-1)/2}\rho]) \times \delta([\nu\rho, \nu^{(b-1)/2}\rho]) \times \theta' \rtimes \tau_i.$$

Now Lemma 3.4 implies that there exists an irreducible representation θ of a general linear group such that

$$T \hookrightarrow \theta \rtimes \tau_i.$$

Note that ρ is not in $\mathrm{supp}(\theta)$.

Denote by T' the irreducible subrepresentation of $\delta(\rho, b) \rtimes \pi$ different from T. Suppose that $T' \hookrightarrow \theta' \rtimes \tau_i$ for some irreducible θ'. Again, ρ is not in $\text{supp}(\theta')$. Then

$$\delta(\rho, b) \rtimes \pi = T \oplus T' \leq (\theta \oplus \theta') \rtimes \tau_i \tag{4.11}$$

(ρ is neither in the cuspidal support of θ, nor of θ').

The formula for μ^* implies that the term of form $* \otimes \tau_{-i}$ cannot be in the Jacquet module $(\theta \oplus \theta') \rtimes \tau_i$ (use that $\mu^*(\tau_i) = 1 \otimes \tau_i + \rho \otimes \pi_{cusp}$, the fact that ρ is not in the cuspidal supports of θ and θ', and use (2.4) and (2.5)).

From the other side, take some irreducible $\theta'' \otimes \pi_{cusp} \leq \mu^*(\pi)$. Then (2.5) and (2.6) imply that

$$\delta([\nu\rho, \nu^{(b-1)/2}\rho])^2 \times \theta'' \otimes \rho \rtimes \pi_{cusp} \tag{4.12}$$

is in the Jacquet module of $\delta(\rho, b) \rtimes \pi$. Observe that (4.11) has a subquotient of the form $* \otimes \tau_{-i}$. Now (4.11) implies that $(\theta \oplus \theta') \rtimes \tau_i$ has a subquotient of the form $* \otimes \tau_{-i}$. This contradiction ends the proof. □

Observe that in the above lemma we have again defined

$$\pi_{i\delta(\rho,b)}, \quad i \in \{\pm 1\}$$

in this case.

5. Tempered representations

We denote by

$$D$$

the set of all equivalence classes of irreducible essentially square integrable representations of $GL(n, F')$, for all $n \geq 1$. The subset of all unitarizable classes in D is denoted by

$$D^u.$$

For an irreducible square integrable representation π of a classical group denote

$$D^u_{\pi,\text{red}} = \{\delta \in D^u; \delta \rtimes \pi \text{ reduces}\}.$$

Recall that $\delta = \delta(\rho, b) \in D^u$ is in $D^u_{\pi,\text{red}}$ if and only if ρ is F'/F-selfdual, ρ and b satisfy, (J1) and $(\rho, b) \notin \text{Jord}(\pi)$. Let

$$D^u_{\pi,\text{irr}} = D^u \setminus D^u_{\pi,\text{red}}.$$

Proposition 5.1. *Let π be an irreducible square integrable representation of a classical group, $\delta_1, \dots, \delta_n$ different (i.e. nonequivalent) representations in $D^u_{\pi,red}$ and $j_1, \dots, j_n \in \{\pm 1\}$. Then there exists a unique irreducible subrepresentation T of $\delta_1 \times \dots \times \delta_n \rtimes \pi$ such that*

$$T \hookrightarrow \left(\underset{k\in\{1,\dots,n\}\setminus\{i\}}{\times} \delta_k \right) \rtimes \pi_{j_i\delta_i}$$

for all $i \in \{1, \dots, n\}$. We shall denote T by

$$\pi_{j_1\delta_1,\dots,j_n\delta_n}.$$

Further, if we have some other representation $\pi'_{j'_1\delta'_1,\dots,j'_{n'}\delta'_{n'}}$ defined in the above way, then

$$\pi_{j_1\delta_1,\dots,j_n\delta_n}. \cong \pi'_{j'_1\delta'_1,\dots,j'_{n'}\delta'_{n'}}$$

if and only if $\pi \cong \pi'$ and $\{j_1\delta_1, \dots, j_n\delta_n\} = \{j'_1\delta'_1, \dots, j'_{n'}\delta'_{n'}\}$.

Proof. We know that $\delta_1 \times \dots \times \delta_n \rtimes \pi$ is a multiplicity one representation of length 2^n (see Theorem 13.1 of [19], or [8]).

Now we shall prove by induction that the description of irreducible subrepresentations of $\delta_1 \times \dots \times \delta_n \rtimes \pi$ is well-defined. For $n = 1$ there is nothing to prove. Suppose $n \geq 2$ and that the description is well-defined for $n - 1$. Write $\delta_i = \delta(\rho_i, a_i)$. Renumerate (δ_i, j_i)'s in a way that $a_1 \geq a_i$ for all $i > 1$. Denote $\tau = \pi_{j_2\delta_2,\dots,j_n\delta_n}$.

Observe that

$$\mu^*(\delta_1 \times \dots \times \delta_n \rtimes \pi) = \Bigg(\prod_{k=1}^{n} \Bigg(\sum_{i_k=-(a_k-1)/2-1}^{(a_k-1)/2} \sum_{j_k=i_k}^{(a_k-1)/2} \delta([\nu^{-i_k}\rho, \nu^{(a_k-1)/2}\rho]) \times \delta([\nu^{j_k+1}\rho, \nu^{(a_k-1)/2}\rho]) \otimes \delta([\nu^{i_k+1}\rho, \nu^{j_k}\rho]) \Bigg) \Bigg) \rtimes \mu^*(\pi). \tag{5.1}$$

From this easily follows that the multiplicity of $\delta_1 \otimes \tau$ in (5.1) is two (consider the term $\nu^{-(a_1-1)/2}\rho$).

One gets also directly that the multiplicity of $\delta_1 \otimes \tau$ in $\mu^*(\delta_1 \rtimes \tau)$ is at least two. The above fact about (5.1) implies that the multiplicity is 2.

From this (using Frobenius reciprocity) follows that $\delta_1 \rtimes \tau$ reduces into two non-equivalent irreducible representations. Denote them by T_1 and T_2. For description proposed in the proposition, it is enough to prove that both T_1 and T_2 cannot be in the same time subrepresentations of $\delta_2 \times \dots \times \delta_n \rtimes \pi_{j\delta_1}$ for both $j \in \{\pm 1\}$. Suppose that they are. Then the multiplicity of $\delta_1 \otimes \tau$ in $\mu^*(\delta_2 \times \dots \times \delta_n \rtimes \pi_{j\delta_1})$ must be 2.

Observe $1 \otimes \delta_i \leq M^*(\delta_i)$ for $i = 2, \dots, n$, and $\delta_1 \otimes \pi \leq \mu^*(\pi_{\pm\delta_1})$ (by Frobenius reciprocity). From this follows that $\delta_1 \otimes \tau \leq \mu^*(\delta_2 \times \dots \times \delta_n \rtimes \pi_{\pm\delta_1})$. From this and the fact that the multiplicity of $\delta_1 \otimes \tau$ in (5.1) is two, follow that the multiplicity of $\delta_1 \otimes \tau$ in $\mu^*(\delta_2 \times \ldots \times \delta_n \rtimes \pi_{\pm\delta_1})$ is one. This contradiction completes the proof that representations $\pi_{j_1\delta_1,\ldots,j_n\delta_n}$ are well defined.

The rest of proposition directly follows from Proposition III.4.1 of [39]. □

Definition 5.2. *If an irreducible (tempered) representation π is equivalent to some representation $\pi_{j_1\delta_1,\ldots,j_n\delta_n}$ as in the above proposition, we shall say that π is e-tempered representation.*

Note that from [10] follows that in the case of symplectic and split odd-orthogonal groups, the notion of an irreducible e-tempered representation coincides coincides with the notion of elliptic tempered representation.

Now we have:

Theorem 5.3. (1) *Let $\pi_{j_1\delta_1,\ldots,j_n\delta_n}$ be an irreducible e-tempered representation (like in the above proposition) and $\gamma_1,\ldots,\gamma_m \in D^u_{\pi,irr} \cup \{\delta_1,\ldots,\delta_n\}$. Then*

$$\gamma_1 \times \ldots \times \gamma_m \rtimes \pi_{j_1\delta_1,\ldots,j_n\delta_n}$$

is an irreducible tempered representation.

(2) *If we have additionally an irreducible e-tempered representation $\pi'_{j'_1\delta'_1,\ldots,j'_{n'}\delta'_{n'}}$ and $\gamma'_1,\ldots,\gamma'_{m'} \in D^u_{\pi',irr} \cup \{\delta'_1,\ldots,\delta'_{n'}\}$. Then*

$$\gamma_1 \times \ldots \times \gamma_m \rtimes \pi_{j_1\delta_1,\ldots,j_n\delta_n} \cong \gamma'_1 \times \ldots \times \gamma'_{m'} \rtimes \pi_{j'_1\delta'_1,\ldots,j'_{n'}\delta'_{n'}}$$

if and only if we have equality of multisets

$$(\gamma_1,\ldots,\gamma_m,\check{\gamma}_1,\ldots,\check{\gamma}_m) = (\gamma'_1,\ldots,\gamma'_{m'},\check{\gamma}'_1,\ldots,\check{\gamma}'_{m'}),$$

and $\pi_{j_1\delta_1,\ldots,j_n\delta_n} \cong \pi_{j'_1\delta'_1,\ldots,j'_n\delta'_{n'}}$ (see the previous proposition for the description of this equivalence).

(3) *Each irreducible tempered representation of a classical group S_q is equivalent to some representation from (1).* □

Now we shall define in a natural way Jordan blocks of irreducible tempered representations of classical groups. We shall use here rather irreducible square integrable representations, than the pairs that parameterize them. Therefore, we define

$$Jord(\pi)_{\text{d.s.}} = \{\delta(\rho,n); (\rho,n) \in Jord(\pi)\}.$$

We can consider a set in an obvious way as a multiset (we shall do this below).

We now extend the definition of Jordan blocks to the case of irreducible tempered representations. This is very simple and natural extension, which seems to be present in the literature for a long time (at least implicitly).

Definition 5.4. *Let $\pi_{j_1\delta_1,\dots,j_n\delta_n}$ be an irreducible e-tempered representation defined in Proposition 5.1, and let*

$$\gamma_1, \dots, \gamma_m \in D^u_{\pi,irr} \cup \{\delta_1, \dots, \delta_n\}.$$

Then the Jordan blocks

$$Jord(\gamma_1 \times \dots \times \gamma_m \rtimes \pi_{j_1\delta_1,\dots,j_n\delta_n})$$

attached to the irreducible tempered representation $\gamma_1 \times \dots \times \gamma_m \rtimes \pi_{j_1\delta_1,\dots,j_n\delta_n}$ are defined to be the multiset

$$(\gamma_1, \dots, \gamma_m, \check{\gamma}_1, \dots, \check{\gamma}_m) + 2(\delta_1, \dots, \delta_n) + Jord(\pi)_{d.s.}.$$

We attach to $Jord(\gamma_1 \times \dots \times \gamma_m \rtimes \pi_{j_1\delta_1,\dots,j_n\delta_n})$ the corresponding set

$$\{\gamma_1, \dots, \gamma_m, \check{\gamma}_1, \dots, \check{\gamma}_m\} \cup \{\delta_1, \dots, \delta_n\} \cup Jord(\pi)_{\text{d.s.}},$$

which will be denoted by

$$|Jord(\gamma_1 \times \dots \times \gamma_m \rtimes \pi_{j_1\delta_1,\dots,j_n\delta_n})|.$$

We can now extend the definition of partially defined function to the tempered case:

Definition 5.5. *Let $\pi_{j_1\delta_1,\dots,j_n\delta_n}$ be an irreducible e-tempered representation defined in Proposition 5.1, and let $\gamma_1, \dots, \gamma_m \in D^u_{\pi,irr} \cup \{\delta_1, \dots, \delta_n\}$. Denote by τ the irreducible tempered representation*

$$\tau = \gamma_1 \times \dots \times \gamma_m \rtimes \pi_{j_1\delta_1,\dots,j_n\delta_n}.$$

Then the partially defined function ϵ_τ attached to τ is a function defined on a subset of $|Jord(\tau)| \cup |Jord(\tau)| \times |Jord(\tau)|$ (with values in $\{\pm 1\}$), which satisfies the following requirements:

1. *If $\delta \in |Jord(\tau)|$ is not F'/F-selfdual, then $\epsilon_\tau(\delta)$ is not defined.*
2. *If $\delta \in |Jord(\tau)|$ is F'/F-selfdual, if it has even multiplicity in $Jord(\tau)$ and if $\delta \notin \{\delta_1, \dots, \delta_n\}$*[6]*, then $\epsilon_\tau(\delta)$ is not defined.*
3. *If $\delta \in |Jord(\tau)|$ is F'/F-selfdual, if it has even multiplicity in $Jord(\tau)$ and if $\delta \in \{\delta_1, \dots, \delta_n\}$, then $\epsilon_\tau(\delta)$ is defined, and*
$$\epsilon_\tau(\delta_i) = j_i.$$
4. *If $\delta = \delta(\rho, a) \in |Jord(\tau)|$ has odd multiplicity in $Jord(\tau)$, then $\delta \in Jord(\pi)_{d.s.}$, and further $\epsilon_\tau(\delta)$ is defined if and only if $\epsilon_\pi((\rho, a))$ is defined. If it is defined, then*
$$\epsilon_\tau(\delta) = \epsilon_\pi((\rho, a)).$$
5. *Let $\delta_1 = \delta(\rho_1, a_1), \delta_2 = \delta(\rho_2, a_2) \in |Jord(\tau)|$. If $\epsilon_\tau((\delta_1, \delta_2))$ is defined, then $\delta_1, \delta_2 \in Jord(\pi)_{d.s.}$. In that case $\epsilon_\tau((\delta_1, \delta_2))$ is defined if and only if $\epsilon_\pi((\rho_1, a_1))e_\pi((\rho_2, a_2))^{-1}$ is defined, and then*
$$\epsilon_\tau((\delta_1, \delta_2)) = \epsilon_\pi((\rho_1, a_1))e_\pi((\rho_2, a_2))^{-1}.$$

[6] In this case, if we write δ as $\delta(\rho, a)$, then (ρ, a) does not satisfy (J1)

Now we shall give very brief description of the irreducible tempered representations in terms of Jordan blocks, partially defined functions and partial cuspidal supports (as it was done for irreducible square integrable representations in Theorem 6.1 of [19]).

Definition 5.6. *A tempered triple*

$$(Jord, \sigma, \epsilon)$$

is a triple for which holds:

(1) *Jord is a finite multiset in D^u which satisfies:*
   (a) *Jord is F'/F-selfdual (i.e. if $Jord = (\delta_1, \dots, \delta_k)$, then $Jord = (\check{\delta}_1, \dots, \check{\delta}_k)$).*
   (b) *If $\delta = \delta(\rho, a)$ from Jord is F'/F-selfdual and (ρ, a) does not satisfy (J1), then the multiplicity of δ in Jord is even.*

(2) *σ is an irreducible cuspidal representation of a classical group.*

(3) *ϵ is a function which takes values in $\{\pm 1\}$. It is defined on a subset of the set $|Jord| \cup |Jord| \times |Jord|$. Each δ_1 or (δ_2, δ_3) from the domain of ϵ satisfies:*

   If $\delta = \delta(\rho, a) = \delta_1$, δ_2 or δ_3, then δ is F'/F-selfdual and (ρ, a) satisfies (J1).

   The domain of ϵ is described as follows:

   (a) *If the multiplicity of δ in Jord is even, then ϵ is defined on δ.*
   (b) *If the multiplicity of δ in Jord is odd, then ϵ is defined on $\delta = \delta(\rho, a)$ if and only if a is even or a is odd and $Jord_\rho(\sigma) = \emptyset$.*
   (c) *ϵ is defined on $(\delta(\rho_1, a_1), \delta(\rho_2, a_2))$ if and only if the multipicities of both $\delta(\rho_1, a_1)$ and $\delta(\rho_2, a_2)$ in Jord are odd, and if $\rho_1 = \rho_2$.*

   The partially defined function ϵ needs to satisfy:

   (a) *If ϵ is defined on δ_1, δ_2 and (δ_1, d_2), then*

   $$\epsilon(\delta_1, \delta_2) = \epsilon(\delta_1)\epsilon(\delta_2)^{-1}.$$

   (b) *If ϵ is define on (δ_1, δ_2) and (δ_2, δ_3), then*

   $$\epsilon(\delta_1, \delta_3) = \epsilon(\delta_1, \delta_2)\epsilon(\delta_2, \delta_3).$$

(4) *Denote by $Jord^{(J1),odd}$ the multiset that we get from Jord deleting all $\delta = \delta(\rho, a)$'s which are not F'/F-selfdual, or which are, but have even multiplicity in Jord (all what remains satisfy (J1)). Let ϵ' be the natural restriction of ϵ to $|Jord^{(J1),odd}| \cup |Jord^{(J1),odd}| \times |Jord^{(J1),odd}|$. Denote $\{(\rho, a); \delta(\rho, a) \in |Jord^{(J1),odd}|\}$ by $|Jord^{(J1),odd}|'$, and denote by ϵ'' the function that we get on $|Jord^{(J1),odd}|$ transferring ϵ' in obvious way. Then*

   $$(|Jord^{(J1),odd}|', \sigma, \epsilon'')$$

   needs to be an admissible triple (as it is defined in [15]*; see also* [19]*).*

Now we can express the parameterization of the irreducibe tempered representations that we have obtained in the following way. The map

$$\tau \to (Jord(\tau), \tau_{cusp}, \epsilon_\tau) \tag{5.2}$$

defines a bijection from the set of equivalence classes of irreducible tempered representations of groups S_n onto the set of all tempered triples.

Remark 5.7. *We consider quasi-split classical groups and their generic irreducible tempered representations in this remark. For these groups, we shall follow the conventions of* [9] *regarding non-degenerate characters of the maximal unipotent subgroups.*

The definition of the partially defined function attached to an irreducible square integrable representation of a classical group depends (only) on the choice of indexing of irreducible constituents of

$$\rho \rtimes \pi_{cusp} = \tau_1 \oplus \tau_{-1},$$

when (π_{cusp}, ρ) *runs over all pairs of irreducible* F'/F*-selfdual cuspidal representations* ρ *of general linear groups and irreducible cuspidal representations* π_{cusp} *of classical groups, such that* $\rho \rtimes \pi_{cusp}$ *reduces (see* [36] *for more details). In this remark, we shall assume that in the case that* π_{cusp} *is generic, we have chosen always* τ_1 *to be generic. Then from* [9] *and* [23] *follows directly that an irreducible square integrable representation* π *of a quasi-split classical group is generic (for the fixed non-degenerate character of the maximal unipotent subgroup) if and only if the partial cuspidal support* π_{cusp} *of* π *is generic and if the partially defined function* ϵ_π *attached to* π *takes value one on elements and pairs from* $Jord(\pi)$*, whenever it is defined on them (see also* [22]*). This was also proved by C. Mœglin in an unpublished manuscript. We shall briefly comment this result.*

If π *is a generic irreducible square integrable representation, then obviously (by Theorem 2 of* [26]*) the partial cuspidal support* π_{cusp} *of* π *is generic. Further, in Proposition 3.1 of* [9] *is shown that the partially defined function* ϵ_π *attached to* π *takes value one on pairs from* $Jord(\pi)$*, whenever it is defined on them (see also* [22]*). Moreover, by the remark after Proposition 3.1 of* [9]*, the partially defined function takes value one on elements from* $Jord(\pi)$*, whenever it is defined on them (one can consult also* [22] *again). Let* $(\rho, a) \in Jord(\pi)$*. We shall additionally comment here the case of odd* a*. Then* $\rho \rtimes \pi_{cusp}$ *reduces. Now Proposition 4.1 of* [36] *and* [26] *(and also Proposition 3.1 of* [9]*) imply that the partially defined function on the element* (ρ, a) *takes the value one.*

Let π *be an irreducible square integrable representation of a classical group such that the partial cuspidal support* π_{cusp} *of* π *is generic, and such that the partially defined function* ϵ_π *attached to* π *takes value one on elements and pairs from* $Jord(\pi)$*, whenever it is defined on them. Now first Proposition 1.1 of* [23] *implies that the generic subquotient of the representation parabolically induced by corresponding irreducible cuspidal representation, where* π *is a subquotient, must be square integrable. Now the above discussion implies that this generic square integrable subquotient is* π*. Thus,* π *is generic.*

Let π be an irreducible generic square integrable representation of a classical group and let δ be an irreducible F'/F-selfdual square integrable representation of a general linear group, such that $\delta \rtimes \pi$ reduces. Then precisely one of the irreducible constituents of $\delta \rtimes \pi$ is generic ([26]). Denote it by τ_{gen}. Let $\delta = \delta(\rho, b)$, where ρ is irreducible F'/F-selfdual cuspidal representation of a general linear group and $b \geq 1$.

Suppose that we are in the setting of Lemma 4.1. The representation

$$\delta([\nu^{(a-1)/2+1}\rho, \nu^{(b-1)/2}\rho])^2 \times \delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi$$

has a unique generic irreducible subquotient, and it must be equivalent to τ_{gen} ([26]). By the generalized injectivity conjecture proved for classical groups in [9], the generic subquotient must be a subrepresentation. Thus,

$$\tau_{gen} \hookrightarrow \delta([\nu^{(a-1)/2+1}\rho, \nu^{(b-1)/2}\rho])^2 \times \delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi.$$

Now Definition 4.3 (see also Corollary 4.2) implies $\pi_{\delta(\rho,b)} \cong \tau_{gen}$. Thus, $\pi_{\delta(\rho,b)}$ is generic (and $\pi_{-\delta(\rho,b)}$ is not). In the same way one sees that also in the settings of Lemma 4.4 and Lemma 4.7, $\pi_{\delta(\rho,b)}$ is also generic and $\pi_{-\delta(\rho,b)}$ is not. In the setting of 4.11, [26] implies that $\pi_{\delta(\rho,b)}$ is generic.

This implies that $\pi_{\epsilon\delta}$ is generic if and only if π is generic and $\epsilon = 1$.

From this and directly from [26] follows that an irreducible e-tempered representation

$$\pi_{\epsilon_1\delta_1,\dots,\epsilon_n\delta_n}$$

is generic if and only if π is generic and $\epsilon_1 = \cdots = \epsilon_n = 1$.

At the end we conclude (from above discussion and [26]) that the irreducible tempered representation in (1) of Theorem 5.3 is generic, if and only if the e-tempered representation there is generic.

6. Some elementary facts on Bernstein center of direct products

First we shall recall the definition of the Bernstein center (see [4] and [5] for more details).

Let G be the group of F-rational points of a connected reductive group defined over F (we have fixed a local non-archimedean field F). The set of equivalence classes of irreducible smooth representations of G is called the non-unitary dual of G and denoted by $\tilde{G}$. It carries a natural topology of uniform convergence of matrix coefficients over compact subsets (see [30] for more details). Denote the Hausdorffization of $\tilde{G}$ by $\Theta(G)$. Then $\Theta(G)$ is the set of conjugacy classes of all pairs (M, ρ), where M is a Levi subgroup of G and ρ is an equivalence class of irreducible cuspidal representations of M (the conjugacy class of (M, ρ) will be denoted by $[(M, \rho)]$). The fibers of the Hausdorffization map $\tilde{G} \to \Theta(G)$ are finite. The structure of the complex algebraic variety on the unramified characters of Levi subgroups defines in a natural way a structure of algebraic variety on $\Theta(G)$. The algebra of all the regular functions on $\Theta(G)$ will be denoted by $\mathfrak{Z}(G)$. The subalgebra of all regular functions supported only on finitely many connected components is denoted by $\mathfrak{Z}(G)_0$.

The Bernstein center of G is the algebra $\mathcal{Z}(G)$ of all endomorphisms of the category of all smooth representations of G. For such an endomorphism z, denote by $\tilde{z} : \tilde{G} \to \mathbb{C}$ the mapping which attaches to $\pi \in \mathbb{C}$ the scalar $\chi_\pi(z)$ by which z acts in the representation space of π. Now $\tilde{z}$ factors through the Hausdorffization $\tilde{G} \to \Theta(G)$, and it is a regular function on $\Theta(G)$, i.e. an element of $\mathfrak{Z}(G)$. In this way one gets an isomorphism of the Bernstein center $\mathcal{Z}(G)$ onto the algebra $\mathfrak{Z}(G)$. In what follows, we shall identify the Bernstein center $\mathcal{Z}(G)$ with $\mathfrak{Z}(G)$.

Let χ be a character of $\mathfrak{Z}(G)$ which does not vanish on $\mathfrak{Z}(G)_0$. In this paper we, shall consider only such characters of $\mathfrak{Z}(G)$, and call them infinitesimal characters of G. For such a character, there exists $[(M, \rho)] \in \Theta(G)$ such that χ is the evaluation in that point.

Let (π, V) be a smooth representation of G. Then π is called a χ-representation, or a representation with infinitesimal character χ, if each $z \in \mathfrak{Z}(G)$ acts in the representation space of π as the multiplication by the scalar $\chi(z) \in \mathbb{C}$. Further, π will be called a representation of type χ if each irreducible subquotient of π is a χ-representation.

For a smooth representation (π, V) of G we denote by

$$(\pi_{[\chi]}, V_{[\chi]})$$

the biggest subrepresentation of (π, V) which is of type χ (one easily shows that such subrepresentation exists). Note that the possibility that this subrepresentation is $\{0\}$ is not excluded (actually, in the cases that we shall consider, this will be almost always the case). Denote

$$V_\chi = \{v \in V; z.v = \chi(z)v, \forall z \in \mathfrak{Z}(G)\}.$$

This is a subrepresentation of π, denoted by π_χ. Clearly, this is a χ-representation.

Let H and L be the groups of F-rational points of a connected reductive group defined over F. Denote

$$M = H \times L.$$

Then

$$(\sigma, \tau) \mapsto \sigma \otimes \tau$$

gives identification of $\tilde{H} \times \tilde{L}$ with $\tilde{M}$. In what follows, we shall always assume this identification.

In the rest of this section we shall consider representations of M, and the action of the Bernstein center of H on the representations of M.

Lemma 6.1. (1) *Let χ be an infinitesimal character of H, let σ be a smooth χ-representation of H and let τ be a smooth representation of L. Then the representation $\sigma \otimes \tau$ of M is a χ-representation, when we look at it as a representation of H (by restriction).*

(2) *Let (μ, V) be a smooth representation of M of finite length. Then there exists finitely many infinitesimal characters $\chi_1, \dots, \chi_k$ of H such that*

$$V = V_{[\chi_1]} \oplus \cdots \oplus V_{[\chi_k]}. \tag{6.1}$$

Moreover, each $V_{[\chi_i]}$ and each V_{χ_i} is an M-subrepresentation of V.

Proof. The claim (1) is evident.

For (2), recall that by the definition, $V_{[\chi_i]}$ is invariant for the action of H. From the other side, the action of L commutes with the action of H (i.e. the operators corresponding to the action of L can be regarded as an H-intertwinings). This implies that the action of $\mathfrak{Z}(H)$ commutes with the action of M. Therefore, if we act by M on $V_{[\chi_i]}$, we get again representation of type χ_i. The maximality of $V_{[\chi_i]}$ implies M-invariance of it. Further, V_{χ_i} is an M-subrepresentation since the action of L commutes with the action of H.

Let $\sigma_1 \otimes \tau_1$, $\sigma_2 \otimes \tau_2$, ..., $\sigma_n \otimes \tau_n$ be the set of all equivalence classes of irreducible subquotients of μ. Write $\{\chi_{\sigma_1}, \dots, \chi_{\sigma_n}\} = \{\chi_1, \dots, \chi_k\}$, where we assume $\chi_i \neq \chi_j$ for $i \neq j$. Then each χ_i is evaluation at some point $\theta_i \in \Theta(M)$. Now we shall prove that (6.1) holds (the proof is standard, but nevertheless we give details below). We prove by induction on k that the sum $V_{[\chi_1]} + \cdots + V_{[\chi_k]}$ is direct. Suppose $k = 2$ and chose $z \in \mathfrak{Z}(G)$ such that $z(\theta_1) = 0$ and $z(\theta_2) = 1$. Let $v \in V_{[\chi_1]} \cap V_{[\chi_2]}$. Suppose $v \neq 0$. Now since $v \in V_{[\chi_1]}$, we can chose integer $e \geq 0$ such that $z^e.v \neq 0$ and $z^{e+1}.v = 0$ (for $e = 0$ we take formally $z^0.v = v$). Denote $v' = z^e.v$. Then $v' \neq 0$ and $z.v' = 0$. But condition $v' \in V_{[\chi_2]}$ and $v' \neq 0$ easily implies $z.v' \neq 0$ (using $z(\theta_2) = 1$). This contradiction shows that the sum $V_{[\chi_1]} + V_{[\chi_2]}$ is direct.

Now we shall present the inductive step. Let $k \geq 3$. Fix $i_0 \in \{1, \dots, k\}$. Take $z_0 \in \mathfrak{Z}(G)$ such that $z_0(\theta_{i_0}) = 1$ and $z_0(\theta_i) = 0$ for $i \neq i_0$. Take

$$v \in V_{[\chi_{i_0}]} \cap (\oplus_{i \neq i_0} V_{[\chi_i]}).$$

Suppose $v \neq 0$. Now since $v \in V_{[\chi_{i_0}]}$, for any integer $\ell \geq 1$ the fact that $z_0^\ell(\theta_{i_0}) = 1$ easily implies

$$z_0^\ell.v \neq 0 \tag{6.2}$$

(look at the subrepresentation generated by v, and some its maximal subrepresentation).

We can write $v' = \sum_{i \neq i_0} v_i'$ where $v_i' \in V_{[\chi_i]}$. Now this formula and the fact that $z_0(\theta_i) = 0$ for $i \neq i_0$ imply that

$$z_0^m.v = 0$$

for some (big enough) integer $m \geq 1$. This contradicts (6.2).

Suppose $V_{[\chi_1]} + \cdots + V_{[\chi_k]} \neq V$. Chose an M-subrepresentation V' of V containing $V_{[\chi_1]} + \cdots + V_{[\chi_k]}$, such that the quotient representation $V'/(V_{[\chi_1]} + \cdots + V_{[\chi_k]})$ is irreducible. Then the infinitesimal character of this quotient must be some χ_{i_0}. Take again some $z_0 \in \mathfrak{Z}(G)$ such that $z_0(\theta_{i_0}) = 1$ and $z_0(\theta_i) = 0$ for $i \neq i_0$. Now we can chose integer $l \geq 1$ such that holds

$$z_0^l.w \in V_{[\chi_{i_0}]}$$

for any $w \in V_{[\chi_1]} + \cdots + V_{[\chi_k]}$. Let $v \in V' \backslash (V_{[\chi_1]} + \cdots + V_{[\chi_k]})$. Then

$$z_0^m.v - v \in V_{[\chi_1]} + \cdots + V_{[\chi_k]}$$

for any integer $m \geq 1$, which implies $z_0^m.v \in V' \backslash (V_{[\chi_1]} + \cdots + V_{[\chi_k]})$ since $z_0(\theta_{i_0}) = 1$. Now for any $z \in \mathfrak{Z}(H)$ holds

$$z.(v + (V_{[\chi_1]} + \cdots + V_{[\chi_k]})) = \chi_{i_0}(z)v + (V_{[\chi_1]} + \cdots + V_{[\chi_k]}),$$

which implies

$$z.v - \chi_{i_0}(z)v \in V_{[\chi_1]} + \cdots + V_{[\chi_k]},$$

and further

$$z.(z_0^l.v) - \chi_{i_0}(z)(z_0^l.v) \in V_{[\chi_{i_0}]}. \tag{6.3}$$

Recall $z_0^l.v \notin V_{[\chi_{i_0}]}$ since $z_0^m.v - v \in V_{[\chi_1]} + \cdots + V_{[\chi_k]}$ for any integer $m \geq 1$, and $v \notin V_{[\chi_1]} + \cdots + V_{[\chi_k]}$. Now (6.3) implies that $(V'/V_{[\chi_{i_0}]})_\chi$ is non-trivial. Consider the projection $p : V' \to V'/V_{[\chi_{i_0}]}$. Then $p^{-1}((V'/V_{[\chi_{i_0}]})_\chi)$ is a representation of type χ_{i_0}, and it strictly contains $V_{[\chi_{i_0}]}$. This contradiction completes the proof of the lemma. □

Corollary 6.2. *Let G be the group of F-rational points of a connected reductive group defined over F and let $P = MN$ be its Levi subgroup. Suppose that M is (isomorphic to) a direct product of reductive groups H and L. Take an irreducible smooth representation (π, V) of G such that the normalized Jacquet module $r_M^G(\pi)$ of π with respect to P is non-trivial (i.e. $\neq \{0\}$). Let $(\sigma \otimes \tau, U)$ be an irreducible subquotient of $r_M^G(\pi)$. Then*

(1) *There exists an irreducible smooth representation $\sigma' \otimes \tau'$ of M such that*

$$\chi_{\sigma'} = \chi_\sigma$$

and

$$\pi \hookrightarrow \mathrm{Ind}_P^G(\sigma' \otimes \tau').$$

(2) *If for any irreducible subquotient μ of $(r_M^G(\pi))_{[\chi_\sigma]}$ there exists an irreducible representation τ' of L such that $\mu \cong \sigma \otimes \tau'$, then*

$$\pi \hookrightarrow \mathrm{Ind}_P^G(\sigma \otimes \tau'')$$

for some irreducible representation τ'' of L.

(3) *If σ is cuspidal, then there exists an irreducible smooth representation τ' of L such that*

$$\pi \hookrightarrow \mathrm{Ind}_P^G(\sigma \otimes \tau').$$

(4) *If* $(r_M^G(\pi))_{[\chi_\sigma]}$ *is an irreducible* M*-representation, then*

$$\pi \hookrightarrow \mathrm{Ind}_P^G(\sigma \otimes \tau).$$

More generally, if all irreducible subquotients of $(r_M^G(\pi))_{[\chi_\sigma]}$ *are isomorphic to* $\sigma\otimes\tau$*, then again* $\pi \hookrightarrow \mathrm{Ind}_P^G(\sigma \otimes \tau)$.

Proof. (1) We know by (1) of Lemma 6.1 that $(r_M^G(\pi))_{[\chi_\sigma]} \neq \{0\}$. By (2) of the same lemma, $(r_M^G(\pi))_{[\chi_\sigma]}$ is M-invariant. Since it is of finite length, it has an irreducible quotient. Denote it by $\sigma' \otimes \tau'$. Frobenius reciprocity implies $\pi \hookrightarrow \mathrm{Ind}_P^G(\sigma' \otimes \tau')$. Observe that $\sigma' \otimes \tau'$ is a $\chi_{\sigma'}$-representations (as H-representation). But irreducible quotient of $(r_M^G(\pi))_{[\chi_\sigma]}$ has infinitesimal character χ_σ as H-representation. Therefore, $\chi_{\sigma'} = \chi_\sigma$. The proof of the (1) is now complete.

(2) is immediate consequence of (1).

(3) If σ is cuspidal and σ' is irreducible such that $\chi_\sigma = \chi_{\sigma'}$, then $\sigma \cong \sigma'$. Now from (2) immediately follows (3).

(4) follows also directly from (1). □

Remark 6.3. *Two irreducible representations of general linear groups have the same infinitesimal character if and only if they have the same cuspidal support.*

7. Interpretation of invariants of square integrable representations in terms of Jacquet modules

Recall that an irreducible cuspidal representation σ of a classical group (from the same series) is called the partial cuspidal support of an irreducible square integrable representation π of a classical group, if there exists a smooth representations θ of a general linear group such that $\pi \hookrightarrow \theta \rtimes \sigma$. Now Frobenius reciprocity implies that $\theta \otimes \sigma$ is a quotient of the appropriate Jacquet module of π. In particular, $\theta \otimes \sigma$ is a subquotient, i.e.

$$\theta \otimes \sigma \leq \mu^*(\pi).$$

The following well-known proposition (which directly follows from (3) of Corollary 6.2) implies that the converse holds:

Proposition 7.1. *Let* π *be an irreducible square integrable representation of a classical group. Suppose that there exits an irreducible cuspidal representation* σ *of a classical group and an irreducible representations* θ *of a general linear group such that*

$$\theta \otimes \sigma \leq \mu^*(\pi).$$

Then σ *is the partial cuspidal support of* π*.* □

Let π be an irreducible square integrable representation of a classical group S_q. Suppose that we have $a \in \mathrm{Jord}_\rho(\pi)$ which has $a_- \in \mathrm{Jord}_\rho(\pi)$. Suppose that

$$\epsilon_\pi((\rho, a))\epsilon_\pi((\rho, a_-))^{-1} = 1.$$

This is the case if and only if there exists an irreducible representation σ of classical group such that

$$\pi \hookrightarrow \delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a-1)/2}\rho] \rtimes \sigma.$$

This and Frobenius reciprocity imply that $\delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \otimes \sigma$ is a quotient of the Jacquet module of π. In particular,

$$\delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \otimes \sigma \leq \mu^*(\pi).$$

We have the converse:

Proposition 7.2. *Let π be an irreducible square integrable representation of a classical group S_q. Suppose that there exists $a \in \mathrm{Jord}_\rho(\pi)$ which has $a_- \in \mathrm{Jord}_\rho(\pi)$, such that*

$$\delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \otimes \sigma \leq \mu^*(\pi)$$

for some irreducible representation σ. Then

$$\epsilon_\pi((\rho, a))\epsilon_\pi((\rho, a_-))^{-1} = 1.$$

Proof. The condition on the Jacquet module in the proposition and transitivity of Jacquet modules imply that

$$\nu^{(a-1)/2}\rho \otimes \nu^{(a-1)/2-1}\rho \otimes \ldots \otimes \nu^{(a_--1)/2+1}\rho \otimes \sigma$$

is a subquotient of the Jacquet module of π. Now (3) of Corollary 6.2 implies that

$$\pi \hookrightarrow \nu^{(a-1)/2}\rho \times \nu^{(a-1)/2-1}\rho \times \ldots \times \nu^{(a_--1)/2+1}\rho \rtimes \sigma'$$

for some irreducible representation σ'. Further, (1) of Corollary 6.2 and Remark 6.3 imply that

$$\pi \hookrightarrow \theta \rtimes \sigma''$$

for some irreducible subquotient θ of $\nu^{(a-1)/2}\rho \times \nu^{(a-1)/2-1}\rho \times \ldots \times \nu^{(a_--1)/2+1}$ and some irreducible representation σ''. Denote

$$k_i = (a-1)/2 - i + 1, \quad i = 1, \ldots, (a - a_-)/2.$$

Then

$$\theta \hookrightarrow \nu^{k_{\alpha(1)}}\rho \times \cdots \times \nu^{k_{\alpha((a-a_-)/2)}}\rho \tag{7.1}$$

for some permutation α of $\{1, \ldots, (a - a_-)/2\}$. This and $\pi \hookrightarrow \theta \rtimes \sigma''$ imply

$$\pi \hookrightarrow \nu^{k_{\alpha(1)}}\rho \times \cdots \times \nu^{k_{\alpha((a-a_-)/2)}}\rho \rtimes \sigma''.$$

Let

$$j = \max\{1 \leq i \leq (a - a_-)/2 + 1; \alpha(s) = s \text{ for } s = 1, \ldots, i - 1\}.$$

If $j=(a-a_-)/2+1$, then (7.1) implies $\theta\cong\delta([\nu^{(a_--1)/2+1}\rho,\nu^{(a-1)/2}\rho])$ and now $\pi\hookrightarrow\theta\rtimes\sigma''$ implies

$$\epsilon_\pi((\rho,a_-))\epsilon_\pi((\rho,a))^{-1}=1.$$

Suppose

$$j\leq (a-a_-)/2.$$

Then $j<\alpha(j)$ and in (7.1) we can bring $\nu^{k_{\alpha(j)}}\rho$ at the "beginning", i.e. $\theta\hookrightarrow\nu^{k_{\alpha(j)}}\rho\times\dots$. Now Frobenius reciprocity and (1) of Proposition 3.6 imply

$$2k_{\alpha(j)}+1\in \mathrm{Jord}_\rho(\pi).$$

Note that

$$2k_{\alpha(j)}+1=2((a-1)/2-\alpha(j)+1)+1=a-2\alpha(j)+2.$$

From $-(a-a_-)/2\leq -\alpha(j)<-j$ we get

$$a_-+2=a-(a-a_-)+2\leq a-2\alpha(j)+2(=2k_{\alpha(j)}+1)<a-2j-2\leq a.$$

Thus $a_-<2k_{\alpha(j)}+1<a$. Since $2k_{\alpha(j)}+1\in\mathrm{Jord}_\rho(\pi)$, this contradicts to the assumption that a and a_- are the neighbors in $\mathrm{Jord}_\rho(\pi)$. □

Remark 7.3. *We shall consider the situation as in previous proposition, i.e. π is an irreducible square integrable representation of a classical group S_q, $a\in\mathrm{Jord}_\rho(\pi)$ which has $a_-\in\mathrm{Jord}_\rho(\pi)$, such that*

$$\epsilon_\pi((\rho,a))=\epsilon_\pi((\rho,a_-)).$$

Now Lemma 5.1 in [19] *(of C. Mœglin) implies*

$$\pi\hookrightarrow\delta([\nu^{-(a_--1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\pi'$$

for an irreducible square integrable representation π'. By Proposition 3.1, we know

$$Jord(\pi')=\mathrm{Jord}_\rho(\pi)\backslash\{(\rho,a),\rho,a_-)\}.$$

This implies that

$$\delta([\nu^{-(a_--1)/2}\rho,\nu^{(a_--1)/2}\rho])\rtimes\pi'$$

reduces (into a sum of two nonequivalent irreducible tempered representations).

Suppose that $\theta\otimes\sigma$ is a subquotient of the Jacquet module of π such that

$$\chi_\theta=\chi_{\delta([\nu^{(a_--1)/2+1}\rho,\nu^{(a-1)/2}\rho])}.$$

Then in the proof of the above proposition, we have shown that

$$\theta\cong\delta([\nu^{(a_--1)/2+1}\rho,\nu^{(a-1)/2}\rho]).$$

From $\pi\hookrightarrow\delta([\nu^{-(a_--1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\pi'$ follows

$$\pi\hookrightarrow\delta([\nu^{(a_--1)/2+1}\rho,\nu^{(a-1)/2}\rho])\times\delta([\nu^{-(a_--1)/2}\rho,\nu^{(a_--1)/2}\rho])\rtimes\pi'$$

and further

$$\theta\otimes\sigma\leq M^*(\delta([\nu^{(a_--1)/2+1}\rho,\nu^{(a-1)/2}\rho]))\times M^*(\delta([\nu^{-(a_--1)/2}\rho,\nu^{(a_--1)/2}\rho]))\rtimes\mu^*(\pi').$$

To get $\theta \otimes \sigma$ *from the right hand side of the above inequality, one needs to take from* $M^*(\delta([\nu^{(a_- -1)/2+1}\rho, \nu^{(a-1)/2}\rho]))$ *the term* $\theta \otimes 1$ *(use the formula for Jord*(π')*). Therefore, we need to take from*

$$M^*(\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a_- -1)/2}\rho])) \rtimes \mu^*(\pi')$$

the term $1 \otimes \sigma$*. It can come only from*

$$(1 \otimes \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a_- -1)/2}\rho])) \rtimes (1 \otimes \pi') = 1 \otimes \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a_- -1)/2}\rho]) \rtimes \pi'.$$

Therefore, σ *is equivalent to the precisely one irreducible subrepresentation of*

$$\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a_- -1)/2}\rho]) \rtimes \pi'.$$

This implies that $\theta \otimes \sigma$ *has the multiplicity one in the Jacquet module of* π*.*

We consider again an irreducible square integrable representation π of a classical group S_q. Let ρ be an irreducible F'/F-selfdual cuspidal representation of a general linear group such that $\mathrm{Jord}_\rho(\pi) \cap 2\mathbb{Z} \neq \emptyset$. Then $\epsilon_\pi((\rho, a))$ is defined for $a \in \mathrm{Jord}_\rho(\pi)$. Denote

$$a = \min(\mathrm{Jord}_\rho(\pi)).$$

Assume

$$\epsilon_\pi((\rho, a)) = 1.$$

This is equivalent to the fact that

$$\pi \hookrightarrow \delta([\nu^{1/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma$$

for some irreducible representation σ. This implies that $\delta([\nu^{1/2}\rho, \nu^{(a-1)/2}\rho]) \otimes \sigma$ is a quotient of the Jacquet module of π. In particular,

$$\delta([\nu^{1/2}\rho, \nu^{(a-1)/2}\rho]) \otimes \sigma \leq \mu^*(\pi).$$

The following proposition tells that the converse holds:

Proposition 7.4. *Let* π *be an irreducible square integrable representation of a classical group* S_q*. Suppose* $\mathrm{Jord}_\rho(\pi) \cap 2\mathbb{Z} \neq \emptyset$*. Denote*

$$a = \min(\mathrm{Jord}_\rho(\pi)).$$

Suppose that

$$\delta([\nu^{1/2}\rho, \nu^{(a-1)/2}\rho]) \otimes \sigma \leq \mu^*(\pi)$$

for some irreducible representation σ*. Then*

$$\epsilon_\pi((\rho, a)) = 1.$$

Proof. The condition in the proposition on the Jacquet module and the transitivity of Jacquet modules imply that

$$\nu^{(a-1)/2}\rho \otimes \nu^{(a-1)/2-1}\rho \otimes \ldots \otimes \nu^{1/2}\rho \otimes \sigma$$

is a subquotient in the Jacquet module of π. Now (3) of Corollary 6.2 implies

$$\pi \hookrightarrow \nu^{(a-1)/2}\rho \times \nu^{(a-1)/2-1}\rho \times \ldots \times \nu^{1/2}\rho \rtimes \sigma'$$

for some irreducible representation σ'. Now (1) of Corollary 6.2 and Remark 6.3 imply that

$$\pi \hookrightarrow \theta \rtimes \sigma''$$

for some irreducible subquotient θ of $\nu^{(a-1)/2}\rho\times\nu^{(a-1)/2-1}\rho\times\ldots\times\nu^{1/2}\rho$ and some irreducible representation σ''. Denote

$$k_i = (a-1)/2 - i + 1, \quad i = 1, \ldots, a/2.$$

Then

$$\theta \hookrightarrow \nu^{k_{\alpha(1)}}\rho \times \cdots \times \nu^{k_{\alpha(a/2)}}\rho \tag{7.2}$$

for some permutation α of $\{1, \ldots, a/2\}$. This and $\pi \hookrightarrow \theta \rtimes \sigma''$ imply

$$\pi \hookrightarrow \nu^{k_{\alpha(1)}} \times \cdots \times \nu^{k_{\alpha(a/2)}} \rtimes \sigma''.$$

Let

$$j = \max\{1 \leq i \leq a/2 + 1; \alpha(s) = s \text{ for } s = 1, \ldots, i-1\}.$$

If $j = a/2 + 1$, then(7.2) implies $\theta = \delta([\nu^{1/2}\rho, \nu^{(a-1)/2}\rho])$ and now $\pi \hookrightarrow \theta \rtimes \sigma''$ implies

$$\epsilon_\pi((\rho, a)) = 1.$$

Suppose

$$j \leq a/2.$$

Then $j < \alpha(j)$ and in (7.2) we can bring $\nu^{k_{\alpha(j)}}\rho$ at the "beginning" (as in the previous proof). By (1) of Proposition 3.6 we know

$$2k_{\alpha(j)} + 1 \in \mathrm{Jord}_\rho(\pi).$$

Note

$$2k_{\alpha(j)} + 1 = 2((a-1)/2 - \alpha(j) + 1) + 1 = a - 2\alpha(j) + 2.$$

From $-a/2 \leq -\alpha(j) < -j$ we get

$$2 = a - a + 2 \leq a - 2\alpha(j) + 2 (= 2k_{\alpha(j)} + 1) < a - 2j + 2 \leq a.$$

This contradicts to the minimality of a in $\mathrm{Jord}_\rho(\pi)$. □

Let σ be an irreducible cuspidal representation of a classical group and let ρ be an irreducible F'/F-selfdual representation of a general linear group (over F'). Suppose that $\rho \rtimes \sigma$ reduces. Write

$$\rho \rtimes \sigma = \tau_1 \oplus \tau_{-1}. \tag{7.3}$$

Then for any $k \in \mathbb{Z}_{>0}$ the representation $\delta([\nu\rho, \nu^k\rho]) \rtimes \tau_i$, $i \in \{\pm 1\}$, has the unique irreducible subrepresentation. Denoted it by

$$\delta([\nu\rho, \nu^k\rho]_{\tau_i}; \sigma).$$

For $k = 0$ we take formally $\delta(\emptyset_{\tau_i}; \sigma) = \sigma$.

Take an irreducible square integrable representation π of a classical group S_q such that

$$\pi_{cusp} = \sigma.$$

Then $Jord_\rho(\pi_{cusp}) = \emptyset$. Suppose $\mathrm{Jord}_\rho(\pi) \neq \emptyset$. Then $\epsilon_\pi((\rho, a))$ is defined for $a \in \mathrm{Jord}_\rho(\pi)$. Denote

$$a = \max(\mathrm{Jord}_\rho(\pi)).$$

Suppose that

$$\epsilon_\pi((\rho, a)) = i.$$

By [36], this is equivalent to the fact that

$$\pi \hookrightarrow \theta \rtimes \delta([\nu\rho, \nu^{(a-1)/2}\rho]_{\tau_i}; \pi_{cusp})$$

for some irreducible representation θ (of a general linear group). This implies that $\theta \otimes \delta([\nu\rho, \nu^{(a-1)/2}\rho]_{\tau_i}; \pi_{cusp})$ is a quotient of the Jacquet module of π. In particular.

$$\theta \otimes \delta([\nu\rho, \nu^{(a-1)/2}\rho]_{\tau_i}; \pi_{cusp}) \leq \mu^*(\pi).$$

The following proposition implies that the converse holds:

Proposition 7.5. *Let π be an irreducible square integrable representation of a classical group S_q. Suppose* $\mathrm{Jord}_\rho(\pi) \neq \emptyset$ *and $Jord_\rho(\pi_{cusp}) = \emptyset$. Denote*

$$a = \max(\mathrm{Jord}_\rho(\pi)).$$

Chose a decomposition into a sum of irreducible (tempered) subrepresentations:

$$\rho \rtimes \pi_{cusp} = \tau_1 \oplus \tau_{-1}.$$

Suppose that

$$\theta \otimes \delta([\nu\rho, \nu^{(a-1)/2}\rho]_{\tau_i}; \pi_{cusp})$$

in a subquotient of the corresponding Jacquet module of π for some irreducible representation θ. Then

$$\epsilon_\pi((\rho, a)) = i.$$

Proof. Suppose that $\theta \otimes \delta([\nu\rho, \nu^{(a-1)/2}\rho]_{\tau_i}; \pi_{cusp})$ is is a subquotient of a Jacquet module of π and suppose $\epsilon_\pi((\rho, a)) = -i$. At the end of proof of Proposition 4.1 of [36] we have proved that then

$$\theta \otimes \delta([\nu\rho, \nu^{(a-1)/2}\rho]_{\tau_{-(-i)}}; \pi_{cusp}) = \theta \otimes \delta([\nu\rho, \nu^{(a-1)/2}\rho]_{\tau_i}; \pi_{cusp})$$

is not a subquotient of the Jacquet module of π. This contradiction completes the proof of the lemma. □

Remark 7.6. *(1) Let ρ be an F'/F-selfdual irreducible cuspidal representation of a general linear group and let σ be an irreducible cuspidal representation of a classical group such that both representations*

$$\rho \rtimes \sigma \text{ and } \nu\rho \rtimes \sigma$$

are irreducible. Denote by $L((\rho, \nu\rho))$ the unique irreducible quotient of $\nu\rho \times \rho$. Then from (6.1) of [32] *we directly see*

$$L((\rho, \nu\rho)) \otimes \sigma \leq \mu^*(\delta([\rho, \nu\rho]) \rtimes \sigma) \tag{7.4}$$

and

$$L(\widetilde{(\rho,\nu\rho)})\otimes\sigma\not\leq\mu^*(\delta([\rho,\nu\rho])\rtimes\sigma). \tag{7.5}$$

From Proposition 6.3 of [32] *we know that the representation* $\delta([\rho,\nu\rho])\rtimes\sigma$ *is irreducible.*

Suppose that we have an embedding

$$\delta([\rho,\nu\rho])\rtimes\sigma\hookrightarrow L((\rho,\nu\rho))\rtimes\sigma. \tag{7.6}$$

Then Corollary 6.4 of [32] *implies that we have also embedding*

$$\delta([\rho,\nu\rho])\rtimes\sigma\hookrightarrow L(\widetilde{(\rho,\nu\rho)})\rtimes\sigma. \tag{7.7}$$

Now Frobenius reciprocity implies that $L(\widetilde{(\rho,\nu\rho)})\otimes\sigma$ *is a quotient of the corresponding Jacquet module of* $\delta([\rho,\nu\rho])\rtimes\sigma$*. In particular* $L(\widetilde{(\rho,\nu\rho)})\otimes\sigma\leq\mu^*(\delta([\rho,\nu\rho])\rtimes\sigma)$*. This contradicts* (7.5)*. Therefore,* (7.6) *can not hold.*

The simplest example of the above situation is when we take $\rho=\mathbf{1}_{F^\times}$ *and* $\sigma=\mathbf{1}_{SO(1,F)}$ *(*$\mathbf{1}_G$ *denotes the trivial one-dimensional representation, while* St_G *denotes the Steinberg representation of a reductive p-adic group* G*). Then* $\nu^{1/2}St_{GL(2,F)}\rtimes\mathbf{1}_{SO(1,F)}$ *is an irreducible representation of* $SO(5,F)$*. This representation has* $\nu^{1/2}\mathbf{1}_{GL(2,F)}\otimes\mathbf{1}_{SO(1,F)}$ *for a subquotient in the corresponding Jacquet module, but it does not embed into* $\nu^{1/2}\mathbf{1}_{GL(2,F)}\rtimes\mathbf{1}_{SO(1,F)}$.

(2) Let ρ *be an irreducible unitarizable cuspidal representation of* $GL(p,F)$ *and let* σ *be an irreducible cuspidal representation of a classical group. Suppose that* $\beta>1/2$ *is in* $(1/2)\,\mathbb{Z}$ *and that* $\nu^\beta\rho\rtimes\sigma$ *reduces. Then Proposition 5.1 of* [32] *says that representations* $\nu^\beta\rho\rtimes\delta(\nu^\beta\rho,\sigma)$ *and* $\nu^\beta\rho\rtimes L(\nu^\beta\rho,\sigma)$ *are irreducible. Compute*

$$\begin{aligned}\mu^*(\nu^\beta\rho\rtimes\delta(\nu^\beta\rho,\sigma))&=(1\otimes\nu^\beta\rho+\nu^\beta\rho\otimes1+\nu\rho^{-\beta}\otimes1)\rtimes(1\otimes\delta(\nu^\beta\rho,\sigma)+\nu^\beta\rho\otimes\sigma)\\&=1\otimes\nu^\beta\rho\rtimes\delta(\nu^\beta\rho,\sigma)+[\nu^\beta\rho\otimes\delta(\nu^\beta\rho,\sigma)+\nu^\beta\rho\otimes\nu^\beta\rho\rtimes\sigma+\nu^{-\beta}\rho\otimes\delta(\nu^\beta\rho,\sigma)]\\&\quad+[\nu^\beta\rho\times\nu^\beta\rho\otimes\sigma+\nu^\beta\rho\times\nu^{-\beta}\rho\otimes\sigma].\end{aligned}$$

Observe that $\nu^\beta\rho\otimes L(\nu^\beta\rho;\sigma)\leq\mu^*(\nu^\beta\rho\rtimes\delta(\nu^\beta\rho,\sigma))$ *but* $\nu^{-\beta}\rho\otimes L(\nu^\beta\rho;\sigma)\not\leq\mu^*(\nu^\beta\rho\rtimes\delta(\nu^\beta\rho,\sigma))$.

Suppose

$$\nu^\beta\rho\rtimes\delta(\nu^\beta\rho,\sigma)\hookrightarrow\nu^\beta\rho\rtimes L(\nu^\beta\rho;\sigma).$$

Then $\nu^\beta\rho\rtimes\delta(\nu^\beta\rho,\sigma)\hookrightarrow\nu^{-\beta}\rho\rtimes L(\nu^\beta\rho;\sigma)$*, which implies that* $\nu^{-\beta}\rho\otimes L(\nu^\beta\rho;\sigma)$ *is a subquotient of the corresponding Jacquet module of* $\nu^\beta\rho\rtimes\delta(\nu^\beta\rho,\sigma)$*. This contradicts* $\nu^{-\beta}\rho\otimes L(\nu^\beta\rho;\sigma)\not\leq\mu^*(\nu^\beta\rho\rtimes\delta(\nu^\beta\rho,\sigma))$*. Therefore* $\nu^\beta\rho\rtimes\delta(\nu^\beta\rho,\sigma)$ *does not embed into* $\nu^\beta\rho\rtimes L(\nu^\beta\rho;\sigma)$*.*

The simplest example of the above situation is when we take $\rho=\mathbf{1}_{F^\times}$ *and* $\sigma=\mathbf{1}_{Sp(0,F)}$*. Then* $\nu\mathbf{1}_{F^\times}\rtimes St_{Sp(2,F)}$ *is an irreducible representation of* $Sp(4,F)$*. This representation has* $\nu\mathbf{1}_{F^\times}\otimes\mathbf{1}_{Sp(2,F)}$ *for a subquotient in its corresponding Jacquet module, but it does not embed into* $\nu\mathbf{1}_{F^\times}\rtimes\mathbf{1}_{Sp(2,F)}$.

8. Behavior of partially defined function for deforming Jordan blocks

In Proposition 3.1, square integrable representations of a smaller and bigger classical groups are related. There are two types of relation (see (1) and (2) of Proposition 3.1). The proposition describes the behavior of the Jordan blocks in both cases. Note that the partial cuspidal supports are preserved. Theorem 3.2 of C. Mœglin describes the behavior of the partially defined function for the relation of type (2) of that proposition. In this section we shall describe the behavior of the partially defined function for the relation of type (1) of that proposition.

Lemma 8.1. *Let π be an irreducible square integrable representation of a classical group S_q. Suppose that we have $a \in \mathrm{Jord}_\rho(\pi)$, $a \geq 3$, which satisfies $a-2 \notin \mathrm{Jord}_\rho(\pi)$. Then there exists an irreducible square integrable representation π' such that*

$$\pi \hookrightarrow \nu^{(a-1)/2}\rho \rtimes \pi'. \tag{8.1}$$

Let π' be any irreducible square integrable representation satisfying (8.1)*. Then:*

(1) *π is the unique irreducible subrepresentation of $\nu^{(a-1)/2}\rho \rtimes \pi'$.*

(2)
$$\pi'_{cusp} = \pi_{cusp}.$$

(3)
$$Jord(\pi') = (Jord(\pi) \setminus \{(\rho, a)\}) \cup \{(\rho, a-2)\}.$$

(4) *Let*
$$(\rho', b), (\rho', c) \in Jord(\pi)$$
(the possibility $b = c$ is not excluded). Observe that this implies $(\rho, a-2) \notin \{(\rho', b), (\rho', c)\}$.

Suppose $\rho' \not\cong \rho$, or $\rho' \cong \rho$ but $a \notin \{b, c\}$. If $b \neq c$, then

$$\epsilon_{\pi'}((\rho', b))\epsilon_{\pi'}((\rho', c))^{-1} = \epsilon_\pi((\rho', b))\epsilon_\pi((\rho', c))^{-1}. \tag{8.2}$$

Further, $\epsilon_{\pi'}((\rho', b))$ is defined if and only if $\epsilon_\pi((\rho', b))$ is defined. If it is defined, then

$$\epsilon_{\pi'}((\rho', b)) = \epsilon_\pi((\rho', b)). \tag{8.3}$$

Suppose $\rho' \cong \rho$. If $b \neq a$, then

$$\epsilon_{\pi'}((\rho, b))\epsilon_{\pi'}((\rho, a-2))^{-1} = \epsilon_\pi((\rho, b))\epsilon_\pi((\rho, a))^{-1}. \tag{8.4}$$

Further, $\epsilon_{\pi'}((\rho, a-2))$ is defined if and only if $\epsilon_\pi((\rho, a))$ is defined. If it is defined, then

$$\epsilon_{\pi'}((\rho, a-2)) = \epsilon_\pi((\rho, a)). \tag{8.5}$$

(5) *If σ is an irreducible representation of a classical group such that*

$$\pi \hookrightarrow \nu^{(a-1)/2}\rho \rtimes \sigma,$$

then $\sigma \cong \pi'$. In particular, σ is uniquely determined by π (and it is square integrable).

Note that (4) tells us that $\epsilon_{\pi'}$ is completely determined by ϵ_π.

Proof. The existence of an irreducible square integrable representation π' satisfying (8.1) is just Lemma in 10.2.2 of [19] (see there also Remark after that lemma). It also implies that claim (3) holds. Now Lemma 5.3 of [19] applied to π' gives (1). Claim (2) follows directly from the definition of the cuspidal support: $\pi \hookrightarrow \nu^{(a-1)/2}\rho \times \theta \rtimes \pi'_{cusp}$ for some irreducible representation θ of a general linear group. It remains to prove (4) and (5).

First we shall prove (5). Suppose $\pi \hookrightarrow \nu^{(a-1)/2}\rho \rtimes \sigma$. This and (8.1) imply

$$\nu^{(a-1)/2}\rho \otimes \sigma \leq \mu^*(\pi) \leq \mu^*(\nu^{(a-1)/2}\rho \rtimes \pi'). \tag{8.6}$$

We have

$$\mu^*(\nu^{(a-1)/2}\rho \rtimes \pi') = (1 \otimes \nu^{(a-1)/2}\rho + \nu^{(a-1)/2}\rho \otimes 1 + \nu^{-(a-1)/2}\rho \otimes 1) \rtimes \mu^*(\pi'). \tag{8.7}$$

Now (8.6) and the above formula imply

$$\nu^{(a-1)/2}\rho \otimes \sigma \leq (1 \otimes \nu^{(a-1)/2}\rho) \rtimes \mu^*(\pi') + (\nu^{(a-1)/2}\rho \otimes 1) \rtimes \mu^*(\pi'). \tag{8.8}$$

Suppose that

$$\nu^{(a-1)/2}\rho \otimes \sigma \leq (\nu^{(a-1)/2}\rho \otimes 1) \rtimes \mu^*(\pi'). \tag{8.9}$$

Then the formula for $\mu^*(\pi')$ implies $\sigma \cong \pi'$, so (5) holds. Suppose that (8.9) does not hold. In that case

$$\nu^{(a-1)/2}\rho \otimes \sigma \leq (1 \otimes \nu^{(a-1)/2}\rho) \rtimes \mu^*(\pi').$$

This implies that that there exits an irreducible representation $\tau \otimes \tau' \leq s_{(p)}(\sigma)$ such that

$$\nu^{(a-1)/2}\rho \otimes \sigma \leq \tau \otimes \nu^{(a-1)/2}\rho \rtimes \tau'.$$

This implies $\tau \cong \nu^{(a-1)/2}\rho$. Now (1) of Proposition 3.6 implies that $(\rho, a) \in Jord(\pi')$, which contradicts (3). This completes the proof of (5).

Now we shall prove (4). The proof proceeds in a number of steps.

(A) Suppose

$$b = \min(Jord_{\rho'}(\pi')).$$

Recall $(\rho', b) \neq (\rho, a-2)$ (since $(\rho', b) \in \mathrm{Jord}(\pi)$ and $(\rho, a-2) \notin \mathrm{Jord}(\pi)$). Therefore

$$b = \min(Jord_{\rho'}(\pi)).$$

Clearly, $(\rho', b) \neq (\rho, a)$. Assume additionally

$$b \in 2\mathbb{Z}.$$

(1) Let $\epsilon_{\pi'}((\rho', b)) = 1$. Then

$$\pi' \hookrightarrow \delta([\nu^{1/2}\rho', \nu^{(b-1)/2}\rho']) \rtimes \sigma$$

for some irreducible representation σ. Now

$$\pi \hookrightarrow \nu^{(a-1)/2}\rho \times \pi' \hookrightarrow \nu^{(a-1)/2}\rho \times \delta([\nu^{1/2}\rho', \nu^{(b-1)/2}\rho']) \rtimes \sigma.$$

The condition $(\rho', b) \neq (\rho, a-2)$ implies

$$\pi \hookrightarrow \delta([\nu^{1/2}\rho', \nu^{(b-1)/2}\rho']) \times \nu^{(a-1)/2}\rho \rtimes \sigma.$$

Thus $\epsilon_{\pi}((\rho', b)) = 1$.

(2) Let now $\epsilon_{\pi}((\rho', b)) = 1$. Then

$$\pi \hookrightarrow \delta([\nu^{1/2}\rho', \nu^{(b-1)/2}\rho']) \rtimes \sigma$$

for some irreducible representation σ. This implies

$$\begin{aligned}\delta([\nu^{1/2}\rho', \nu^{(b-1)/2}\rho']) \otimes \sigma &\leq \mu^*(\nu^{(a-1)/2}\rho \rtimes \pi') \\ &= (1 \otimes \nu^{(a-1)/2}\rho + \nu^{(a-1)/2}\rho \otimes 1 + \nu^{-(a-1)/2}\rho \otimes 1) \rtimes \mu^*(\pi').\end{aligned}$$

Now the above formula and $(\rho', b) \neq (\rho, a)$, $(\rho', b) \neq (\rho, a-2)$ directly imply that

$$\delta([\nu^{1/2}\rho', \nu^{(b-1)/2}\rho']) \otimes \sigma \leq (1 \otimes \nu^{(a-1)/2}\rho) \rtimes \mu^*(\pi').$$

From this follows that $\delta([\nu^{1/2}\rho', \nu^{(b-1)/2}\rho']) \otimes \sigma' \leq \varphi \otimes \nu^{(a-1)/2}\rho \rtimes \psi$ for some irreducible subquotient $\varphi \otimes \psi \leq \mu^*(\pi')$. Clearly, $\delta([\nu^{1/2}\rho', \nu^{(b-1)/2}\rho']) \cong \varphi$. Now from Proposition 7.4 we get $\epsilon_{\pi'}((\rho', b)) = 1$.

(B) Suppose

$$a - 2 = \min(Jord_{\rho}(\pi')).$$

Then also $a = \min(Jord_{\rho}(\pi))$ (and conversely). Assume additionally

$$a \in 2\mathbb{Z}.$$

(1) Suppose $\epsilon_{\pi'}((\rho, a-2)) = 1$. Then

$$\pi' \hookrightarrow \delta([\nu^{1/2}\rho, \nu^{(a-3)/2}\rho]) \rtimes \sigma$$

for some irreducible representation σ. Now

$$\pi \hookrightarrow \nu^{(a-1)/2}\rho \times \pi' \hookrightarrow \nu^{(a-1)/2}\rho \times \delta([\nu^{1/2}\rho, \nu^{(a-3)/2}\rho]) \rtimes \sigma.$$

Now $\nu^{(a-1)/2}\rho \otimes \delta([\nu^{1/2}\rho, \nu^{(a-3)/2}\rho]) \otimes \sigma$ is in the Jacquet module of π. Transitivity of Jacquet modules implies that also $\delta([\nu^{1/2}\rho, \nu^{(a-1)/2}\rho]) \otimes \sigma$ must be a subquotient of the Jacquet module of π. Now Proposition 7.4 implies $\epsilon_{\pi}((\rho, a)) = 1$.

(2) Let now $\epsilon_\pi((\rho,a))=1$. Then

$$\pi \hookrightarrow \delta([\nu^{1/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\sigma$$

for some irreducible representation σ. This implies

$$\delta([\nu^{1/2}\rho,\nu^{(a-1)/2}\rho])\otimes\sigma \le \mu^*(\nu^{(a-1)/2}\rho\rtimes\pi')$$
$$= (1\otimes\nu^{(a-1)/2}\rho+\nu^{(a-1)/2}\rho\otimes 1+\nu^{-(a-1)/2}\rho\otimes 1)\rtimes\mu^*(\pi').$$

From this follows that

$$\delta([\nu^{1/2}\rho,\nu^{(a-3)/2}\rho])\otimes\sigma' \le \mu^*(\pi') \tag{8.10}$$

or

$$\delta([\nu^{1/2}\rho,\nu^{(a-1)/2}\rho])\otimes\sigma'' \le \mu^*(\pi')$$

for some irreducible representations σ' and σ''. The last inequality implies $(\rho,a)\in Jord(\pi')$, which is impossible. Therefore, (8.10) holds. Now Proposition 7.4 implies $\epsilon_{\pi'}((\rho,a-2))=1$.

(C) Suppose

$$\rho'\not\cong\rho, \text{ or } \rho'\cong\rho \text{ but } a\notin\{b,c\}.$$

Let $b\neq c$. Consider the case $b=c_-$.

(1) First suppose that $\epsilon_{\pi'}((\rho',c_-))\epsilon_{\pi'}((\rho',c))^{-1}=1$. Then by Remark 5.1.3 of [15] (or Lemma 5.1 of [19]) we have

$$\pi'\hookrightarrow\delta([\nu^{-(c_--1)/2}\rho',\nu^{(c-1)/2}\rho'])\rtimes\pi''$$

for some irreducible square integrable representation π''. Now

$$\pi\hookrightarrow\nu^{(a-1)/2}\rho\rtimes\pi'\hookrightarrow\nu^{(a-1)/2}\rho\times\delta([\nu^{-(c_--1)/2}\rho',\nu^{(c-1)/2}\rho'])\rtimes\pi''.$$

Suppose that the segments $\{\nu^{(a-1)/2}\rho\}$ and $\delta([\nu^{-(c_--1)/2}\rho',\nu^{(c-1)/2}\rho'])$ are linked. In this case $\rho'\cong\rho$ and we have two possibilities. Then first is $(c-1)/2+1=(a-1)/2$, i.e. $c=a-2$, which is not possible since $c\in Jord_\rho(\pi)$ and $a-2\notin Jord_\rho(\pi)$. The second possibility is $(a-1)/2+1=-(c_--1)/2$. This is obviously impossible. Therefore, the segments $\{\nu^{(a-1)/2}\rho\}$ and $\delta([\nu^{-(c_--1)/2}\rho',\nu^{(c-1)/2}\rho'])$ are not linked. This implies

$$\nu^{(a-1)/2}\rho\times\delta([\nu^{-(c_--1)/2}\rho',\nu^{(c-1)/2}\rho'])\cong\delta([\nu^{-(c_--1)/2}\rho',\nu^{(c-1)/2}\rho'])\times\nu^{(a-1)/2}\rho.$$

Therefore

$$\pi\hookrightarrow\delta([\nu^{-(c_--1)/2}\rho',\nu^{(c-1)/2}\rho'])\times\nu^{(a-1)/2}\rho\rtimes\pi''\hookrightarrow$$
$$\delta([\nu^{(c_-+1)/2}\rho',\nu^{(c-1)/2}\rho'])\times\delta([\nu^{-(c_-1)/2}\rho',\nu^{(c_--1)/2}\rho'])\times\nu^{(a-1)/2}\rho\rtimes\pi''.$$

Now the definition of the partially defined function implies $\epsilon_\pi((\rho',c_-))\epsilon_\pi((\rho,c))^{-1}=1$. Therefore in this case we have $\epsilon_{\pi'}((\rho',c_-))\epsilon_{\pi'}((\rho',c))^{-1}=\epsilon_\pi((\rho',c_-))\epsilon_\pi((\rho',c))^{-1}$.

(2) Suppose now $\epsilon_\pi((\rho',c_-))\epsilon_\pi((\rho',c))^{-1}=1$. Then

$$\pi \hookrightarrow \delta([\nu^{-(c_--1)/2}\rho',\nu^{(c-1)/2}\rho'])\rtimes\pi'' \tag{8.11}$$

for some irreducible square integrable representation π''. Now (8.11) and $\pi\hookrightarrow\nu^{(a-1)/2}\rho\rtimes\pi'$ imply

$$\begin{aligned}\delta([\nu^{-(c_--1)/2}\rho',\nu^{(c-1)/2}\rho'])\otimes\pi'' &\le \mu^*(\nu^{(a-1)/2}\rho\rtimes\pi')\\ &= (1\otimes\nu^{(a-1)/2}\rho+\nu^{(a-1)/2}\rho\otimes 1+\nu^{-(a-1)/2}\rho\otimes 1)\rtimes\mu^*(\pi').\end{aligned}$$

Suppose

$$\delta([\nu^{-(c_--1)/2}\rho',\nu^{(c-1)/2}\rho'])\otimes\pi''\le(\nu^{\pm(a-1)/2}\rho\otimes 1)\rtimes\mu^*(\pi').$$

Then $\rho'\cong\rho$ and $a\le c$, which implies $a<c$ since we consider the case $a\ne c$. Therefore, $a<c_-$ also (since $a\ne b$). In this case we have two possibilities (corresponding to the choice of sign in the term $\nu^{\pm(a-1)/2}\rho\otimes 1$). The first possibility implies

$$\delta([\nu^{-(c_--1)/2}\rho',\nu^{(a-1)/2-1}\rho'])\times\delta([\nu^{(a-1)/2+1}\rho',\nu^{(c-1)/2}\rho'])\otimes\sigma\le\mu^*(\pi')$$

for some irreducible representation σ. This would imply that π' is not square integrable. The second possibility implies in the same way that π' is not square integrable.

Thus

$$\delta([\nu^{-(c_--1)/2}\rho',\nu^{(c-1)/2}\rho'])\otimes\pi''\le(1\otimes\nu^{(a-1)/2}\rho)\rtimes\mu^*(\pi').$$

This implies that

$$\delta([\nu^{-(c_--1)/2}\rho',\nu^{(c-1)/2}\rho'])\otimes\sigma'\le\mu^*(\pi')$$

for some irreducible representation σ'. Now Proposition 3.6 easily implies that $\epsilon_{\pi'}((\rho',c_-))\epsilon_{\pi'}((\rho,c))^{-1}=1$. Thus $\epsilon_{\pi'}((\rho',c_-))\epsilon_{\pi'}((\rho',c))^{-1}=\epsilon_\pi((\rho',c_-))\epsilon_\pi((\rho',c))^{-1}$ holds also in this case.

(D) Suppose that a_- is defined. Then $(a-2)_-$ is defined, and conversely. In that case

$$(a-2)_-=a_-.$$

(1) Suppose $\epsilon_{\pi'}((\rho,a_-))\epsilon_{\pi'}((\rho,a-2))^{-1}=1$. Then

$$\pi'\hookrightarrow\delta([\nu^{-(a_--1)/2}\rho,\nu^{(a-3)/2}\rho])\rtimes\pi''$$

for some irreducible square integrable representation π''. Now

$$\pi\hookrightarrow\nu^{(a-1)/2}\rho\rtimes\pi'\hookrightarrow\nu^{(a-1)/2}\rho\times\delta([\nu^{-(a_--1)/2}\rho,\nu^{(a-3)/2}\rho])\rtimes\pi''.$$

In a standard way we get that $\delta([\nu^{(a_--1)/2+1}\rho,\nu^{(a-1)/2}\rho])\otimes\pi''$ is a subquotient of the Jacquet module of π, and now (1) of Proposition 7.2 implies $\epsilon_\pi((\rho,a_-))\epsilon_\pi((\rho,a))^{-1}=1$. Therefore in this case holds $\epsilon_{\pi'}((\rho,a_-))\epsilon_{\pi'}((\rho,a-2))^{-1}=\epsilon_\pi((\rho,a_-))\epsilon_\pi((\rho,a))^{-1}$.

(2) Suppose now $\epsilon_\pi((\rho,a_-))\epsilon_\pi((\rho,a))^{-1}=1$. Then

$$\pi\hookrightarrow\delta([\nu^{-(a_--1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\pi'' \tag{8.12}$$

for some irreducible square integrable representation π''. Now (8.12) and $\pi\hookrightarrow\nu^{(a-1)/2}\rho\rtimes\pi'$ imply

$$\begin{aligned}\delta([\nu^{-(a_--1)/2}\rho,\nu^{(a-1)/2}\rho])\otimes\pi''&\leq\mu^*(\nu^{(a-1)/2}\rho\rtimes\pi')\\&=(1\otimes\nu^{(a-1)/2}\rho+\nu^{(a-1)/2}\rho\otimes1+\nu^{-(a-1)/2}\rho\otimes1)\rtimes\mu^*(\pi').\end{aligned}$$

Suppose

$$\delta([\nu^{-(a_--1)/2}\rho,\nu^{(a-1)/2}\rho])\otimes\pi''\leq(\nu^{\pm(a-1)/2}\rho\otimes1)\rtimes\mu^*(\pi').$$

In this case we must have

$$\delta([\nu^{-(a_--1)/2}\rho,\nu^{(a-1)/2-1}\rho])\otimes\sigma\leq\mu^*(\pi')$$

for some irreducible representation σ. Now Proposition 7.2 implies that we have $\epsilon_{\pi'}((\rho,a_-))\epsilon_{\pi'}((\rho,a-2))^{-1}=1$. It remains to consider the case

$$\delta([\nu^{-(a_--1)/2}\rho,\nu^{(a-1)/2}\rho])\otimes\pi''\leq(1\otimes\nu^{(a-1)/2}\rho)\rtimes\mu^*(\pi').$$

Now one directly gets that

$$\delta([\nu^{-(a_--1)/2}\rho,\nu^{(a-1)/2}\rho])\otimes\sigma'\leq\mu^*(\pi')$$

for some irreducible representation σ'. From (1) of Proposition 3.6 now follows $a\in Jord_\rho(\pi')$, which is a contradiction.

Therefore, we have proved that also in this case we have $\epsilon_{\pi'}((\rho,a-2))\epsilon_{\pi'}((\rho,a_-))^{-1}=\epsilon_\pi((\rho,a))\epsilon_\pi((\rho,a_-))^{-1}$.

(E) Suppose that

$$\rho'\cong\rho \text{ and } b_-=a \text{ is defined in } \mathrm{Jord}_\rho(\pi).$$

Then b_- is $a-2$ in $Jord_\rho(\pi')$ (the converse also holds). We denote

$$a_+=b.$$

(1) Suppose $\epsilon_{\pi'}((\rho,a_+))\epsilon_{\pi'}((\rho,a-2))^{-1}=1$. Then

$$\pi'\hookrightarrow\delta([\nu^{-(a-3)/2}\rho,\nu^{(a_+-1)/2}\rho])\rtimes\pi''$$

for some irreducible square integrable representation π''. Now

$$\pi\hookrightarrow\nu^{(a-1)/2}\rho\rtimes\pi'\hookrightarrow\nu^{(a-1)/2}\rho\times\delta([\nu^{-(a-3)/2}\rho,\nu^{(a_+-1)/2}\rho])\rtimes\pi''.$$

In standard way we get that $\delta([\nu^{(a-1)/2+1}\rho,\nu^{(a_+-1)/2}\rho])\otimes\pi''$ is a subquotient of the Jacquet module of π. Now Proposition 7.2 implies $\epsilon_\pi((\rho,a_+))\epsilon_\pi((\rho,a))^{-1}=1$. Therefore we have $\epsilon_{\pi'}((\rho,a_+))\epsilon_{\pi'}((\rho,a-2))^{-1}=\epsilon_\pi((\rho,a_+))\epsilon_\pi((\rho,a))^{-1}$.

(2) Suppose now $\epsilon_\pi((\rho,a_+))\epsilon_\pi((\rho,a))^{-1}=1$. Then

$$\pi \hookrightarrow \delta([\nu^{-(a-1)/2}\rho,\nu^{(a_+-1)/2}\rho])\rtimes\pi'' \tag{8.13}$$

for some irreducible square integrable representation π''. Now (8.13) and $\pi\hookrightarrow \nu^{(a-1)/2}\rho\rtimes\pi'$ imply

$$\delta([\nu^{-(a-1)/2}\rho,\nu^{(a_+-1)/2}\rho])\otimes\pi''\le\mu^*(\nu^{(a-1)/2}\rho\rtimes\pi')$$
$$=(1\otimes\nu^{(a-1)/2}\rho+\nu^{(a-1)/2}\rho\otimes 1+\nu^{-(a-1)/2}\rho\otimes 1)\rtimes\mu^*(\pi').$$

Suppose

$$\delta([\nu^{-(a-1)/2}\rho,\nu^{(a_+-1)/2}\rho])\otimes\pi''\le(\nu^{\pm(a-1)/2}\rho\otimes 1)\rtimes\mu^*(\pi').$$

In this case we must have

$$\delta([\nu^{-(a-1)/2}\rho,\nu^{(a-1)/2-1}\rho])\times\delta([\nu^{(a-1)/2+1}\rho,\nu^{(a_+-1)/2}\rho])\otimes\sigma\le\mu^*(\pi')$$

or

$$\delta([\nu^{-(a-1)/2+1}\rho,\nu^{(a_+-1)/2}\rho])\otimes\sigma'\le\mu^*(\pi')$$

for some irreducible representations σ and σ'. The first possibility would imply that π' is not square integrable, which is contradiction. The second possibility and Proposition 7.2 easily imply that $\epsilon_{\pi'}((\rho,a-2))\epsilon_{\pi'}((\rho,a_+))^{-1}=1$.

It remains to consider the case

$$\delta([\nu^{-(a-1)/2}\rho,\nu^{(a_+-1)/2}\rho])\otimes\pi''\le(1\otimes\nu^{(a-1)/2}\rho)\rtimes\mu^*(\pi').$$

Now one directly gets that

$$\delta([\nu^{(a-1)/2+1}\rho,\nu^{(a_+-1)/2}\rho])\otimes\sigma''\le\mu^*(\pi')$$

for some irreducible representation σ''. Proposition 7.2 now implies $\epsilon_{\pi'}((\rho,a-2))\epsilon_{\pi'}((\rho,a_+))^{-1}=1$. Therefore in this case also holds $\epsilon_{\pi'}((\rho,a-2))\epsilon_{\pi'}((\rho,a_+))^{-1}=\epsilon_\pi((\rho,a))\epsilon_\pi((\rho,a_+))^{-1}$.

(F) Suppose

$$b=\max(Jord_{\rho'}(\pi')).$$

Then $(\rho',b)\ne(\rho,a)$ since $(\rho,a)\notin Jord(\pi')$, and therefore $b=\max(Jord_{\rho'}(\pi))$ (and conversely). Suppose additionally

$$Jord_{\rho'}(\pi_{cusp})=\emptyset.$$

Write

$$\rho'\rtimes\pi_{cusp}=\tau'_1\oplus\tau'_{-1}. \tag{8.14}$$

Let

$$\epsilon_{\pi'}((\rho',b))=i.$$

Then

$$\pi'\hookrightarrow\theta\rtimes\delta([\nu\rho',\nu^{(b-1)/2}\rho']_{\tau'_i};\pi_{cusp})$$

for some irreducible representation θ. Now

$$\pi\hookrightarrow\nu^{(a-1)/2}\rho\rtimes\pi'\hookrightarrow\nu^{(a-1)/2}\rho\times\theta\rtimes\delta([\nu\rho',\nu^{(b-1)/2}\rho']_{\tau'_i};\pi_{cusp}).$$

Then $\epsilon_\pi((\rho', b)) = i$, and therefore $\epsilon_{\pi'}((\rho', b)) = \epsilon_\pi((\rho', b))$.

(G) Suppose

$$a - 2 = \max(Jord_\rho(\pi')).$$

Then $a = \max(Jord_\rho(\pi))$ (and conversely). Suppose

$$Jord_\rho(\pi_{cusp}) = \emptyset$$

and assume the decomposition $\rho' \rtimes \pi_{cusp} = \tau'_1 \oplus \tau'_{-1}$ from (8.14) to hold.

Let

$$\epsilon_\pi((\rho, a)) = i.$$

Then

$$\pi \hookrightarrow \theta \rtimes \delta([\nu\rho, \nu^{(a-1)/2}\rho]_{\tau_i}; \pi_{cusp})$$

for some irreducible representation θ. This implies

$$\theta \otimes \delta([\nu\rho, \nu^{(a-1)/2}\rho]_{\tau_i}, \pi_{cusp}) \leq \mu^*(\pi') \tag{8.15}$$

for some (irreducible) representation θ. This implies

$$\theta \otimes \delta([\nu\rho, \nu^{(a-1)/2}\rho]_{\tau_i}; \pi_{cusp}) \leq \mu^*(\nu^{(a-1)/2}\rho \rtimes \pi')$$
$$= (1 \otimes \nu^{(a-1)/2}\rho + \nu^{(a-1)/2}\rho \otimes 1 + \nu^{-(a-1)/2}\rho \otimes 1) \rtimes \mu^*(\pi').$$

From (ii) of Proposition 5.2 from [33] we know that $\nu^{(a-1)/2}\rho \otimes \delta([\nu\rho, \nu^{(a-3)/2}\rho]_{\tau_i}; \pi_{cusp}) \leq \mu^*(\delta([\nu\rho, \nu^{(a-1)/2}\rho]_{\tau_i}; \pi_{cusp}))$. From this and (8.15) follow that

$$\theta' \otimes \delta([\nu\rho, \nu^{(a-3)/2}\rho]_{\tau_i}; \pi_{cusp}) \leq \mu^*(\pi')$$

for some irreducible representation θ'. Now Proposition 7.4 implies $\epsilon_{\pi'}((\rho, a-2)) = i$, which gives $\epsilon_{\pi'}((\rho, a-2)) = \epsilon_\pi((\rho, a))$.

The proof of lemma is now complete. □

Theorem 8.2. *Let π be an irreducible square integrable representation of a classical group S_q, let ρ be an irreducible cuspidal F'/F-selfdual representation of $GL(p, F')$ and let $a \in$ $\mathrm{Jord}_\rho(\pi)$, $a \geq 3$. Suppose that there exists $k \in \mathbb{Z}_{>0}$ such that*

$$[a - 2k, a - 2] \cap \mathrm{Jord}_\rho(\pi) = \emptyset. \tag{8.16}$$

Then there exists an irreducible square integrable representation π' of S_{q-kp} such that π embeds into

$$\delta([\nu^{(a-2(k-1)-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'. \tag{8.17}$$

Then π embeds also into

$$\nu^{(a-1)/2}\rho \times \nu^{(a-3)/2}\rho \times \ldots \times \nu^{(a-2(k-1)-1)/2}\rho \rtimes \pi'. \tag{8.18}$$

Let π' be any irreducible square integrable representation such that π embeds into (8.18). Then:

(1) *π is the unique irreducible subrepresentation of*

$$\nu^{(a-1)/2}\rho \times \nu^{(a-3)/2}\rho \dots \nu^{(a-2(k-1)-1)/2}\rho \rtimes \pi'.$$

(2)

$$\pi'_{cusp} = \pi_{cusp}.$$

(3)

$$Jord(\pi') = (Jord(\pi) \setminus \{(\rho, a)\}) \cup \{(\rho, a-2k)\}. \tag{8.19}$$

(4) *Let $(\rho', b), (\rho', c) \in Jord(\pi)$ (the possibility $b = c$ is not excluded). Suppose $\rho' \not\cong \rho$, or $\rho' \cong \rho$ but $a \notin \{b, c\}$. If $b \neq c$, then*

$$\epsilon_{\pi'}((\rho', b))\epsilon_{\pi'}((\rho', c))^{-1} = \epsilon_{\pi}((\rho', b))\epsilon_{\pi}((\rho', c))^{-1}. \tag{8.20}$$

Further, $\epsilon_{\pi'}((\rho', b))$ is defined if and only if $\epsilon_{\pi}((\rho', b))$ is defined. If it is defined, then

$$\epsilon_{\pi'}((\rho', b)) = \epsilon_{\pi}((\rho', b)). \tag{8.21}$$

Suppose $\rho' \cong \rho$. If $b \neq a$, then

$$\epsilon_{\pi'}((\rho, b))\epsilon_{\pi'}((\rho, a-2k))^{-1} = \epsilon_{\pi}((\rho, b))\epsilon_{\pi}((\rho, a))^{-1}. \tag{8.22}$$

Further, $\epsilon_{\pi'}((\rho, a-2k))$ is defined if and only if $\epsilon_{\pi}((\rho, a))$ is defined. If it is defined, then

$$\epsilon_{\pi'}((\rho, a-2k)) = \epsilon_{\pi}((\rho, a)). \tag{8.23}$$

(5) *If σ is an irreducible representation of a classical group such that*

$$\pi \hookrightarrow \nu^{(a-1)/2}\rho \times \nu^{(a-3)/2}\rho \times \dots \times \nu^{(a-2(k-1)-1)/2}\rho \rtimes \sigma, \tag{8.24}$$

then $\sigma \cong \pi'$. In particular, σ is uniquely determined by π (and it is square integrable).

Proof. First we shall prove by induction that π can be embedded into representation of type (8.17). For $k = 1$ this follows Lemma 3.4. Suppose that $[a-2(k+1), a-2] \cap \mathrm{Jord}_\rho(\pi) = \emptyset$, and that we have an embedding

$$\pi \hookrightarrow \delta([\nu^{(a-2(k-1)-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'.$$

Now Lemma 3.4, (8.19) and assumption $[a-2(k+1), a-2] \cap \mathrm{Jord}_\rho(\pi) = \emptyset$ implies that

$$\pi' \hookrightarrow \nu^{(a-2(k+1)-1)/2}\rho \rtimes \pi''$$

for some square integrable representation π' (use (1) of Proposition 3.1). Further, (1) of Proposition 3.1 implies

$$Jord(\pi'') = (Jord(\pi) \setminus \{(\rho, a)\}) \cup \{(\rho, a-2(k+1))\}.$$

Observe that

$$\pi \hookrightarrow \delta([\nu^{(a-2(k-1)-1)/2}\rho, \nu^{(a-1)/2}\rho]) \times \nu^{(a-2(k+1)-1)/2}\rho \rtimes \pi''.$$

We know

$\delta([\nu^{(a-2(k+1)-1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\pi''\hookrightarrow\delta([\nu^{(a-2(k-1)-1)/2}\rho,\nu^{(a-1)/2}\rho])\times\nu^{(a-2(k+1)-1)/2}\rho\rtimes\pi''$.

Suppose

$$\pi\not\hookrightarrow\delta([\nu^{(a-2(k+1)-1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\pi''. \tag{8.25}$$

Then we can easily get

$$\pi\hookrightarrow\nu^{(a-2(k+1)-1)/2}\rho\times\delta([\nu^{(a-2(k-1)-1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\pi''.$$

Now (1) of Proposition 3.6 implies $a-2(k+1)\in\mathrm{Jord}_\rho(\pi)$, which is a contradiction. This tells that (8.25) can not happen. Therefore, we have proved the existence of an embedding of type (8.17).

Observe that (2) follows directly from Proposition 7.1. Further, (3) follows from Proposition 3.6.

Let us now prove (1). Denote representation (8.18) by Π. To prove (1), it is enough to prove that the multiplicity of

$$\nu^{(a-1)/2}\rho\otimes\nu^{(a-3)/2}\rho\otimes\dots\otimes\nu^{(a-2(k-1)-1)/2}\rho\otimes\pi' \tag{8.26}$$

in the Jacquet module of Π is 1. For this, it is enough to prove that the multiplicity of $\delta([\nu^{(a-2(k-1)-1)/2}\rho,\nu^{(a-1)/2}\rho])\otimes\pi'$ in $\mu^*(\Pi)$ is one. We shall now prove that if an irreducible representation

$$\omega=\delta([\nu^{(a-2(k-1)-1)/2}\rho,\nu^{(a-1)/2}\rho])\otimes\sigma$$

is in $\mu^*(\Pi)$, then $\sigma\cong\pi'$ and it has multiplicity one in $\mu^*(\Pi)$.

We have

$$\mu^*(\Pi)=\underset{i=(a-2(k-1)-1)/2}{\overset{(a-1)/2}{\times}}(1\otimes\nu^i\rho+\nu^i\rho\otimes 1+\nu^{-i}\rho\otimes 1)\rtimes\mu^*(\pi'). \tag{8.27}$$

Suppose that for each index i in the above formula we have picked up the term $\nu^i\rho\otimes\rho$. To get ω for a subquotient, we must take from $\mu^*(\pi')$ the term $1\otimes\pi'$. In the corresponding product, if ω is a subquotient, then $\sigma\cong\pi'$ and it has multiplicity one.

Suppose that for some index i in (8.27) we have picked up one of two terms different from $\nu^i\rho\otimes 1$ and suppose that ω is a subquotient of the corresponding product. Obviously, the term $\nu^{-i}\rho\otimes 1$ can not give ω (we see this from the cuspidal supports). Thus, the only remaining possibility is $1\otimes\nu^i\rho$. Now to get ω for a subquotient, we must take from $\mu^*(\pi')$ a non-zero term of form $\tau'\otimes\sigma$, where τ' is an irreducible representation of some $GL(l,F')$ with $l\geq 1$ such that the cuspidal support of τ' is contained in $\{\nu^{\frac{a-2(k-1)-1}{2}}\rho,\dots,\nu^{\frac{a-3}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\}$. This would imply that $\ell\in Jord_\rho(\pi')$ for some $\ell\in\{a-2(k-1),\dots,a-2,a\}$. This contradict to (8.16) and (3). This contradiction ends the proof of the claim.

We get (4) applying Lemma 8.1 several times.

Suppose that we have an embedding (8.24). Then ω which we have defined above must be a quotient of the Jacquet module of π, and therefore in $\mu^*(\Pi)$. We have seen that then $\sigma \cong \pi'$. This ends the proof of (5).

The proof of the theorem is now complete. $\square$

Definition 8.3. *The representation π' from the above theorem will be denoted by*

$$\pi^{(\rho,a\downarrow a-2k)}.$$

Further, the representation π from the above theorem will be denoted by

$$(\pi')^{(\rho,a-2k\uparrow a)}.$$

9. Appendix: an irreducibility result

The following simple, but often useful result is proved in [32] (Theorem 13.2). Let Δ be a segment in irreducible cuspidal representations of general linear groups and let σ be an irreducible cuspidal representation of a classical group. Then, assuming that (BA) holds, we have

$$\delta(\Delta) \rtimes \sigma \text{ reduces } \iff \rho \rtimes \sigma \text{ reduces for some } \rho \in \Delta,$$

or equivalently

$$\rho \rtimes \sigma \text{ is irreducible for all } \rho \in \Delta \iff \delta(\Delta) \rtimes \sigma \text{ is irreducible.}$$

If we take instead of cuspidal σ an irreducible square integrable representation π, the above equivalence does not hold in general (we can have $\delta(\Delta) \rtimes \pi$ irreducible despite the fact that $\rho \rtimes \pi$ reduces for some $\rho \in \Delta$). Instead of the equivalence, for the square integrable representation π one implication still holds. This follows from G. Muić's paper [25], where he has described completely (besides others) reducibility points of representations $\delta(\Delta) \rtimes \pi$. Proof of this implication is elementary in comparison with his results. For the convenience of the reader, we present here a proof of this implication (which we have used in this paper). Before we give the proof, we shall recall (general) Proposition 6.1 from [35], which we shall use several times in the proof below (note that in (vii) of Proposition 6.1 in [35], the condition weather $(\rho, 2)$ satisfy or not satisfy (J1) was forgotten).

Proposition 9.1. *Let ρ be an irreducible F'/F-self dual cuspidal representation of a general linear group, and let π be an irreducible square integrable representation of S_q. Suppose that (BA) holds. Let a be a positive integer. Then:*

(i) *For $\alpha \in \mathbb{R}$, $\nu^{\alpha}\rho \rtimes \pi$ reduces if and only if $\nu^{-\alpha}\rho \rtimes \pi$ reduces.*
(ii) *If $\alpha \in \mathbb{R}\backslash(1/2)\mathbb{Z}$, then $\nu^{\alpha}\rho \rtimes \pi$ is irreducible.*
(iii) *$\rho \rtimes \pi$ reduces if and only if ρ has odd parity with respect to π and $1 \notin Jord_\rho(\pi)$.*
(iv) *If $a \notin Jord_\rho(\pi)$, then $\nu^{(a+1)/2}\rho \rtimes \pi$ is irreducible.*
(v) *If $a \in \mathrm{Jord}_\rho(\pi)$ and $a+2 \notin \mathrm{Jord}_\rho(\pi)$, then $\nu^{(a+1)/2}\rho \rtimes \pi$ reduces.*

(vi) *Suppose that* a *and* $a+2$ *are in* $\mathrm{Jord}_\rho(\pi)$. *Then* $\nu^{(a+1)/2}\rho \rtimes \pi$ *reduces if and only if* $\epsilon((\rho,a))\epsilon((\rho,a+2)) = 1$.

(vii) $\nu^{1/2}\rho \rtimes \pi$ *reduces if and only if* $(\rho,2)$ *satisfies* (J1) *and* $2 \notin \mathrm{Jord}_\rho(\pi)$, *or* $2 \in \mathrm{Jord}_\rho(\pi)$ *and* $\epsilon((\rho,2)) = 1$.
In other words, $\nu^{1/2}\rho \rtimes \pi$ *is irreducible if and only if* $(\rho,2)$ *does not satisfy* (J1), *or* $2 \in \mathrm{Jord}_\rho(\pi)$ *and* $\epsilon((\rho,2)) = -1$.

Recall, if ρ is a not F'/F-self dual irreducible cuspidal representation of a general linear group and $\alpha \in \mathbb{R}$, then $\nu^\alpha\rho \rtimes \pi$ is irreducible (π is an irreducible square integrable representation of S_q).

Lemma 9.2. *Let* Δ *be a segment in irreducible cuspidal representations of general linear groups. Suppose that* $\tau \rtimes \pi$ *is irreducible for all* $\tau \in \Delta$. *Then*

$$\delta(\Delta) \rtimes \pi$$

is irreducible.

Not that for the above result, we assume also that the basic assumption (BA) from section 2 holds.

Proof. We shall prove the lemma by induction with respect to $\mathrm{Card}(\Delta)$. For $\mathrm{Card}(\Delta) = 1$ there is nothing to prove. Therefore we shall assume in what follows that $\mathrm{Card}(\Delta) \geq 2$.

First consider the case when $\delta(\Delta)$ is unitarizable. Write $\delta(\Delta) = \delta(\rho,a)$. Suppose that $\delta(\Delta) \rtimes \pi$ is reducible. Then (ρ,a) satisfies (J1) and $a \notin \mathrm{Jord}_\rho(\pi)$. Further suppose that $\tau \rtimes \pi$ is irreducible for all $\tau \in \Delta$. Assume that a is odd. Now (v) of Proposition 9.1 (and the fact that $\nu^{(a-1)/2}\rho \rtimes \pi, \nu^{(a-1)/2-1}\rho \rtimes \pi, \dots, \nu\rho \rtimes \pi, \nu\rho \rtimes \pi$ are all irreducible by our assumptions) implies that $a-2, a-4, \dots, 3, 1$ are not in $\mathrm{Jord}_\rho(\pi)$. But $1 \notin \mathrm{Jord}_\rho(\pi)$ implies that $\rho \rtimes \pi$ reduces. Since $\rho \in \Delta$, we have obtained a contradiction. It remains to consider the case of a even. Now in the same way as before (using (v) of Proposition 9.1), we get that $a-2, a-4, \dots, 2$ are not in $\mathrm{Jord}_\rho(\pi)$. Since $(\rho,2)$ satisfies (J1), (vii) of the same proposition implies that $\nu^{1/2}\rho \rtimes \pi$ reduces. This is again a contradiction. This completes the proof of the lemma in the case of $\delta(\Delta)$ unitarizable.

Now we go to the case when $\delta(\Delta)$ is not unitarizable. We shall first consider a delicate case, the case of $\Delta = \{\rho, \nu\rho\}$, where ρ is an irreducible F'/F-self dual cuspidal representation of a general linear group such that $\rho \rtimes \pi$ and $\nu\rho \rtimes \pi$ are both irreducible. Note that a representation of the form $\nu^\alpha\rho \otimes \tau$, $\alpha \leq 0$, cannot be a subquotient of the Jacquet module of π, since π is square integrable (this follows directly from the square integrability criterion of Casselman from [7]). We consider now two possibilities. Suppose first that the parity of ρ is odd (i.e. that $(\rho,1)$ satisfies (J1)). Then first $1 \in \mathrm{Jord}_\rho(\pi)$ (because $\rho \rtimes \pi$ is irreducible). Now (v) of Proposition 9.1 implies $3 \in \mathrm{Jord}_\rho(\pi)$ (because $\nu\rho \rtimes \pi$ is irreducible). Further, (vi) of the same proposition and the irreducibility of $\nu\rho \rtimes \pi$ imply

$\epsilon_\pi((\rho,1))\epsilon_\pi((\rho,3))^{-1}=1$. This and the definition of ϵ_π (using Proposition 7.2) imply that a representation of the form $\nu\rho\otimes\tau$ can not be a subquotient of the Jacquet module of π. If the parity of ρ is even, then a representation of the form $\nu\rho\otimes\tau$ can not again be a subquotient of of the Jacquet module of π by (1) of Proposition 3.6 (since 3 is odd).

By the proof of the lemma for $\delta(\Delta)$ unitarizable, our assumptions on ρ and π imply that $\delta([\nu^{-1}\rho,\nu\rho])\rtimes\pi$ is irreducible. Now [8] (or [19]) implies that $\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\pi$ is irreducible. Recall

$$\mu^*(\rho\times\rho\times\nu\rho\times\nu\rho\rtimes\pi)=(1\otimes\rho+2\rho\otimes 1)^2\times(1\otimes\nu\rho+\nu\rho\otimes 1+\nu^{-1}\rho\otimes 1)^2\rtimes\mu^*(\pi).$$

From this formula and from the above observations about Jacquet modules of π, one gets directly that the multiplicity of $\delta([\rho,\nu\rho])\times\delta([\rho,\nu\rho])\otimes\pi$ in $\mu^*(\rho\times\rho\times\nu\rho\times\nu\rho\rtimes\pi)$ is four. Further, one gets easily that the multiplicity of $\delta([\rho,\nu\rho])\times\delta([\rho,\nu\rho])\otimes\pi$ in $\mu^*(\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\pi)$ is also four.

Let θ_s be an irreducible subrepresentation of $\delta([\rho,\nu\rho])\rtimes\pi$. Note that $\delta([\rho,\nu\rho])\otimes\pi\leq\mu^*(\theta_s)$ (by Frobenius reciprocity). This implies

$$\delta([\rho,\nu\rho])\times\delta([\rho,\nu\rho])\otimes\pi\leq\mu^*(\delta([\rho,\nu\rho])\rtimes\theta_s). \tag{9.1}$$

Since the multiplicity of $\delta([\rho,\nu\rho])\times\delta([\rho,\nu\rho])\otimes\pi$ in the Jacquet modules of $\rho\times\rho\times\nu\rho\times\nu\rho\rtimes\pi$ and $\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\pi$ is four in both cases, and the last representation is irreducible, we see that

$$\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\pi\leq\delta([\rho,\nu\rho])\rtimes\theta_s. \tag{9.2}$$

From the Langlands classification follows that the representation $\delta([\rho,\nu\rho])\rtimes\pi$ has a unique irreducible quotient. Denote it by θ_q. Then $\theta_q\hookrightarrow\delta([\nu^{-1}\rho,\rho])\rtimes\pi$, which implies $\delta([\nu^{-1}\rho,\rho])\otimes\pi\leq\mu^*(\theta_q)$.

Suppose that $\delta([\rho,\nu\rho])\rtimes\pi$ reduces. Then $\theta_s\not\cong\theta_q$. Now starting with (9.2), we get

$$\begin{aligned}\mu^*(\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\pi)&\leq\mu^*(\delta([\rho,\nu\rho])\rtimes\theta_s)\\&\leq M^*(\delta([\rho,\nu\rho]))\rtimes\Big(M^*(\delta([\rho,\nu\rho]))\rtimes\mu^*(\pi)-\delta([\nu^{-1}\rho,\rho])\otimes\pi\Big).\end{aligned}$$

One directly gets that the multiplicity of $\rho\times\delta([\nu^{-1}\rho,\nu\rho])\otimes\pi$ in $\mu^*(\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\pi)$ is four (for the proof we need only that it is at least four). This and the above remarks about Jacquet modules of π (regarding the terms of the form $\nu\rho\otimes\tau$ and $\nu^\alpha\rho\otimes\tau$ for $\alpha\leq 0$) imply

$$4\rho\times\delta([\nu^{-1}\rho,\nu\rho])\otimes\pi\leq\big(\delta([\rho,\nu\rho])+\rho\times\nu\rho+\delta([\nu^{-1}\rho,\rho])\big)\times\big(\delta([\rho,\nu\rho])+\rho\times\nu\rho\big)\otimes\pi.$$

This implies

$$4\rho\times\delta([\nu^{-1}\rho,\nu\rho])\otimes\pi\leq\delta([\nu^{-1}\rho,\rho])\times\big(\delta([\rho,\nu\rho])+\rho\times\nu\rho\big)\otimes\pi.$$

Obviously, this cannot happen. Therefore, we have proved the irreducibility of $\delta([\rho,\nu\rho])\rtimes\pi$.

We consider now the case $\Delta = [\nu^\alpha\rho, \nu^\beta\rho]$, where $\alpha, \beta \in \mathbb{R}$, $\beta - \alpha \in \mathbb{Z}_{>0}$ and ρ is an F'/F-selfdual irreducible cuspidal representation of a general linear group. Suppose that Δ satisfies the condition of the lemma (with respect to π). Since $\delta([\nu^\alpha\rho, \nu^\beta\rho]) \rtimes \pi$ and $\delta([\nu^{-\beta}\rho, \nu^{-\alpha}\rho]) \rtimes \pi$ have the same composition series, it is enough to prove the irreducibility of $\delta(\Delta) \rtimes \pi$ in the case $\alpha+\beta > 0$. By the previous part of the proof, it is enough to consider the case $\Delta \neq \{\rho, \nu\rho\}$ (therefore if $\alpha + 1 = \beta$, then $\alpha \neq 0$), which we shall assume in what follows.

Let θ_s be an irreducible subrepresentation of $\delta(\Delta) \rtimes \pi$. Note that $\delta(\Delta) \rtimes \pi$ has a unique irreducible quotient. Now a well-known embedding of $\delta(\Delta)$ and the inductive assumption imply

$$\theta_s \hookrightarrow \delta(\Delta) \rtimes \pi \hookrightarrow \nu^\beta\rho \times \delta([\nu^\alpha\rho, \nu^{\beta-1}\rho]) \rtimes \pi \cong \nu^\beta\rho \times \delta([\nu^{-\beta+1}\rho, \nu^{-\alpha}\rho]) \rtimes \pi.$$

Observe $\beta + 1 \neq -\beta + 1$ (since $\beta > 0$).

Suppose $-\alpha + 1 \neq \beta$, i.e. $\alpha \neq -\beta + 1$. Then $\nu^\beta\rho \times \delta([\nu^{-\beta+1}\rho, \nu^{-\alpha}\rho])$ is irreducible (recall $\beta + 1 \neq -\beta + 1$), which implies

$$\theta_s \hookrightarrow \delta([\nu^{-\beta+1}\rho, \nu^{-\alpha}\rho]) \times \nu^\beta\rho \rtimes \pi \cong \delta([\nu^{-\beta+1}\rho, \nu^{-\alpha}\rho]) \times \nu^{-\beta}\rho \rtimes \pi.$$

Now Frobenius reciprocity implies that $\delta([\nu^{-\beta+1}\rho, \nu^{-\alpha}\rho]) \otimes \nu^{-\beta}\rho \otimes \pi$ is a quotient of the Jacquet module of θ_s. This and the transitivity of Jacquet modules imply that $\delta([\nu^{-\beta}\rho, \nu^{-\alpha}\rho]) \otimes \pi$ must be also a subquotient of the Jacquet module of θ_s. But then θ_s must be the unique irreducible quotient of $\delta(\Delta) \rtimes \pi$ (see Lemma 4.4 of [19]). This implies the irreducibility of $\delta(\Delta) \rtimes \pi$.

It remains to consider the case $\alpha = -\beta+1$. Note that $\beta-\alpha \in \mathbb{Z}_{>0}$ implies $\beta \geq 1$, and further $\Delta \neq \{\rho, \nu\rho\}$ implies $b > 1$. For an irreducible subrepresentation θ_s of $\delta([\nu^{-\beta+1}\rho, \nu^\beta\rho]) \rtimes \pi$, we proceed similarly as above:

$$\theta_s \hookrightarrow \delta([\nu^{-\beta+1}\rho, \nu^\beta\rho]) \rtimes \pi \hookrightarrow \delta([\nu^{-\beta+2}\rho, \nu^\beta\rho]) \times \nu^{-\beta+1}\rho \rtimes \pi \cong \delta([\nu^{-\beta+2}\rho, \nu^\beta\rho]) \times \nu^{\beta-1}\rho \rtimes \pi.$$

Clearly, $\beta + 1 \neq \beta - 1$. Further, $\beta - 1 + 1 = -\beta + 2$ implies $\beta = 1$, which contradicts to $b > 1$. Therefore, $\delta([\nu^{-\beta+2}\rho, \nu^\beta\rho]) \times \nu^{\beta-1}\rho$ is irreducible. Using the inductive assumption, we continue similarly as in the previous case:

$$\theta_s \hookrightarrow \nu^{\beta-1}\rho \times \delta([\nu^{-\beta+2}\rho, \nu^\beta\rho]) \rtimes \pi \cong \nu^{\beta-1}\rho \times \delta([\nu^{-\beta}\rho, \nu^{\beta-2}\rho]) \rtimes \pi.$$

This implies that $\nu^{\beta-1}\rho \otimes \nu^{\beta-2}\rho \otimes \ldots \otimes \nu^{-\beta}\rho \otimes \pi$ is in the Jacquet module of σ. The transitivity of Jacquet modules imply that $\delta([\nu^{-\beta}\rho, \nu^{\beta-1}\rho]) \otimes \pi$ is also in the Jacquet module of θ_s. One concludes the irreducibility of $\delta(\Delta) \rtimes \pi$ in the same way as above, using Lemma 4.4 of [19].

We end with the case $\Delta = [\nu^\alpha\rho, \nu^\beta\rho]$, where $\alpha, \beta \in \mathbb{R}$, $\beta - \alpha \in \mathbb{Z}_{>0}$ and ρ is a unitarizable irreducible cuspidal representation of a general linear group which is not F'/F-selfdual (then Δ satisfies the condition of the lemma). It is enough to prove the irreducibility of

$\delta(\Delta) \rtimes \pi$ in the case $\alpha + \beta > 0$. Let θ_s be an irreducible subrepresentation of $\delta(\Delta) \rtimes \pi$. Then

$$\theta_s \hookrightarrow \delta(\Delta) \rtimes \pi \hookrightarrow \nu^{\beta}\rho \times \delta([\nu^{\alpha}\rho, \nu^{\beta-1}\rho]) \rtimes \pi \cong \nu^{\beta}\rho \times \delta([\nu^{-\beta+1}\check{\rho}, \nu^{-\alpha}\check{\rho}]) \rtimes \pi$$
$$\cong \delta([\nu^{-\beta+1}\check{\rho}, \nu^{-\alpha}\check{\rho}]) \times \nu^{\beta}\rho \rtimes \pi \cong \delta([\nu^{-\beta+1}\check{\rho}, \nu^{-\alpha}\check{\rho}]) \times \nu^{-\beta}\check{\rho} \rtimes \pi.$$

We conclude now the irreducibility of $\delta(\Delta)$ in the same way as in the previous two cases.

The proof of the lemma is now complete. □

References

[1] Arthur, J., *The Endoscopic Classification of Representations: Orthogonal and Symplectic Groups*, preprint (http://www.claymath.org/cw/arthur/pdf/Book.pdf).

[2] Ban, D., *Parabolic induction and Jacquet modules of representations of $O(2n, F)$*, Glasnik Mat. 34 (1999), pp. 147-185.

[3] Ban, D. and Jantzen, C., *Degenerate principal series for even-orthogonal groups*, Represent. Theory 7 (2003), pp. 440480.

[4] Bernstein, J. (written by K. Rumelhart), *Draft of: Representations of p-adic groups*, preprint.

[5] Bernstein, J.,rédigé par Deligne, P., *Le "centre" de Bernstein*, "Représentations des Groupes Réductifs sur un Corps Local" written by J.-N. Bernstein, P. Deligne, D. Kazhdan and M.-F. Vignéras, Hermann, Paris, 1984.

[6] Bernstein, J. and Zelevinsky, A.V., *Induced representations of reductive p-adic groups I* , Ann. Sci. École Norm Sup. 10 (1977), pp. 441-472.

[7] Casselman, W., *Introduction to the theory of admissible representations of p-adic reductive groups*, preprint (http://www.math.ubc.ca/~cass/research/pdf/p-adic-book.pdf).

[8] Goldberg, D., *Reducibility of induced representations for $Sp(2n)$ and $SO(n)$*, Amer. J. Math. 116 (1994), pp. 1101-1151.

[9] Hanzer, M., *The generalized injectivity conjecture for the classical p-adic groups*, Intern. Math. Research Notices 2010 (2010), pp. 195-237.

[10] Herb, R., *Elliptic representations for $Sp(2n)$ and $SO(n)$*, Pacific J. Math. 161(1993), pp. 347-358.

[11] Jantzen, C., *Degenerate principal series for symplectic and odd-orthogonal groups*, Mem. Amer. Math. Society 590 (1996), pp. 1-100.

[12] Jantzen, C., *On supports of induced representations for symplectic and odd-orthogonal groups*, Amer. J. Math. 119 (1997), pp. 1213-1262.

[13] Jantzen, C., *Tempered representations for classical p-adic groups*, preprint.

[14] Jiang, D. and Soudry, D., *Generic representations and local Langlands reciprocity law for p-adic $SO(2n+1)$* in "Contributions to automorphic forms, geometry, and number theory", Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 457-519.

[15] Mœglin, C., *Sur la classification des séries discrètes des groupes classiques p-adiques: paramètres de Langlands et exhaustivité*, J. Eur. Math. Soc. 4 (2002), pp. 143-200.

[16] Mœglin, C., *Points de réductibilité pour les induites de cuspidales*, J. Algebra 268 (2003), pp. 81-117.

[17] Mœglin, C., *Classification et Changement de base pour les séries discrètes des groupes unitaires p-adiques*, Pacific J. Math. 233 (2007), pp. 159-204.

[18] Mœglin, C., *Multiplicité 1 dans les paquets d'Arthur aux places p-adiques*, in "On certain L-functions", Clay Math. Proc. 13, Amer. Math. Soc., Providence, RI, 2011, pp. 333-374.

[19] Mœglin, C. and Tadić, M., *Construction of discrete series for classical p-adic groups*, J. Amer. Math. Soc. 15 (2002), pp. 715-786.

[20] Mœglin, C.; Vignéras, M.-F. and Waldspurger, J.-L., *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Math. 1291, Springer-Verlag, Berlin, 1987.

[21] Mœglin, C. and Waldspurger, J.-L., *Sur le transfert des traces tordues d'un group linéaire à un groupe classique p-adique*, Selecta mathematica 12 (2006), pp. 433-516.

[22] Muić, G., *On generic irreducible representations of $Sp(n, F)$ and $SO(2n+1, F)$*, Glasnik Mat. 33(53) (1998), pp. 19-31.

[23] Muić, G., *A proof of Casselman-Shahidi's conjecture for quasi-split classical groups*, Canad. Math. Bull. 44 (2001), pp. 298-312.

[24] Muić, G., *Composition Series of Generalized Principal Series; The Case of Strongly Positive Discrete Series*, Israel J. Math 149 (2004), pp. 157-202.

[25] Muić, G., *Reducibility of Generalized Principal Series*, Canad. J. Math. 57 (2005), pp. 616-647.

[26] Rodier, F., *Whittaker models for admissible representations*, Proc. Sympos. Pure Math. AMS 26 (1983), pp. 425-430.

[27] Shahidi, F., *A proof of Langlands conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. 132 (1990), pp. 273-330.

[28] Shahidi, F., *Twisted endoscopy and reducibility of induced representations for p-adic groups*, Duke Math. J. 66 (1992), pp. 1-41.

[29] Silberger, A., *Special representations of reductive p-adic groups are not integrable*, Ann. of Math. 111 (1980), pp. 571-587.

[30] Tadić, M., *Geometry of dual spaces of reductive groups (non-archimedean case)*, J. Analyse Math. 51 (1988), pp. 139-181.

[31] Tadić, M., *Structure arising from induction and Jacquet modules of representations of classical p-adic groups*, J. of Algebra 177 (1995), pp. 1-33.

[32] Tadić, M., *On reducibility of parabolic induction*, Israel J. Math. 107 (1998), pp. 29-91.

[33] Tadić, M., *Square integrable representations of classical p-adic groups corresponding to segments*, Represent. Theory 3 (1999), pp. 58-89.

[34] Tadić, M., *On classification of some classes of irreducible representations of classical groups*, in book Representations of real and p-adic groups, Singapore University Press and World Scientific, Singapore, 2004, 95-162.

[35] Tadić, M., *On reducibility and unitarizability for classical p-adic groups, some general results*, Canad. J. Math. 61 (2009), pp. 427-450.

[36] Tadić, M., *On invariants of discrete series representations of classical p-adic groups*, Manuscripta Math. 136 (2011), pp.. 417-435.

[37] Tadić, M., *Reducibility and discrete series in the case of classical p-adic groups; an approach based on examples*, in "Geometry and Analysis of Automorphic Forms of Several Variables", World Scientific, Singapore, 2012, pp. 254-333.

[38] Tadić, M., *On interactions between harmonic analysis and the theory of automorphic forms*, preprint (`http://www.hazu.hr/~tadic/50-harmonic-auto.pdf`).

[39] Waldspurger, J.-L., *La formule de Plancherel pour les groupes p-adiques*, J. Inst. Math. Jussieu 2 (2003), pp. 235–333.

[40] Zelevinsky, A. V., *Induced representations of reductive p-adic groups II. On irreducible representations of GL(n)*, Ann. Sci. École Norm. Sup. 13 (1980), pp. 165-210.

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