A method of proving non-unitarity of representations of p-adic groups
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Abstract

In this paper we exhibit a new method of proving non-unitarity of representations, based on the semi simplicity of unitarizable representations. The non-unitarity is proved for (a little bit more then) half of the irreducible representations of the classical p-adic groups whose infinitesimal character is the same as the infinitesimal character of a generalized Steinberg representation (as defined in [T6]), but different from generalized Steinberg representations and its Aubert dual. In this way we partially generalize a result of W. Casselman to the case of classical groups. Our argument is completely different from Casselman's argument (which is hard to extend to this case). It requires a very limited knowledge of the inducing cuspidal representation.

1 Introduction

For a connected reductive group G over a p-adic field F, W. Casselman has shown in [C2] that if an irreducible unitary representation π of G has a non-trivial cohomology $H^*(\pi) \neq 0$, then π is the trivial representation or the Steinberg representation of G. The most delicate problem in proving this result was to prove:

(NU) All the irreducible representations having the same infinitesimal character (in the sense of Bernstein - [BD]) as the Steinberg representation, other then Steinberg representation and its Aubert dual (which is the trivial representation in this case) are not unitary.

In the case of the general linear groups, irreducible square integrable representations were classified by J. Bernstein and A.V. Zelevinsky modulo cuspidal representations (see [Z1]). Their construction and properties resemble very much of the Steinberg representation, and therefore they are often called generalized Steinberg representations (for general linear groups). From [T1] follows that (NU) holds if one in (NU) put these generalized Steinberg representations (instead of the usual Steinberg representations).

The construction and properties of irreducible square integrable representations for the classical p-adic groups are very different from the case of general linear groups (already for $Sp(4, F)$; see [SaT]). Very few irreducible square integrable representations there behave like the Steinberg representation (one of the main characteristics of the Steinberg representation, is that all the Jacquet modules are irreducible essentially square integrable representations).

The Generalized Steinberg representations were defined for classical p-adic groups in [T6] (basic property which they satisfy is that all their Jacquet modules are irreducible essentially square integrable representations). There is a considerable number of generalized Steinberg representations. The aim of this paper is to prove (NU) for a half of all irreducible representations that one gets when one put these generalized Steinberg representations of classical p-adic groups in (NU) instead of usual Steinberg representations (classical groups that we consider here are symplectic, odd-orthogonal and unitary groups). The first author has proved in [Ha1], under a natural assumption, that Aubert duals of generalized Steinberg representations are unitary.

The Casselman's proof of (NU) is based on the fact that in the case of the Steinberg representation, we are dealing with representations which we understanding well (they have a non-trivial Iwahori fixed vector). We cannot apply his approach to the case that we consider. A different proof of (NU) can be obtained using asymptotic of matrix coefficients (developed in [C1]) and a nice result of R. Howe and C.C. Moore, that matrix coefficients of an infinite dimensional irreducible unitary representation of a semi simple group vanish at infinity. Since the matrix coefficients of representations for which we want to prove the non-unitarity usually vanish at infinity, we can not use Howe-Moore's result (note that (NU) for generalized Steinberg representations of general linear groups follows from compete solution of the unitarizability problem [T1] in this case).

The problem that we are facing here, is that for the definition of generalized Steinberg representation of classical group besides the cuspidal reducibility point, we know nothing else about the cuspidal inducing representation. But we have complete control of Jacquet modules in terms of inducing cuspidal representation (about which we do not know much).

There are no many direct methods to prove the non–unitarity of a representation; to show a part of (NU) in our case, we apply a very simple strategy, that unitarity implies semi simplicity. Let us explain very roughly the idea. Let π be an irreducible unitary representation of reductive p-adic group G. Suppose that G_1 is a reductive p-adic group such that the direct product $G \times G_1$ is a Levi subgroup of a parabolic subgroup P_1 in a reductive p-adic group $\mathcal{G}_1$. Let σ_1 be an irreducible unitary representation of G_1. Then $\mathrm{Ind}_{P_1}^{\mathcal{G}_1}(\pi \otimes \sigma_1)$ is a semi simple representation. Therefore, if τ_1 is any irreducible subquotient of $\mathrm{Ind}_{P_1}^{\mathcal{G}_1}(\pi \otimes \sigma_1)$, then it is a subrepresentation. Applying now the Frobenius reciprocity we get that $\pi \otimes \sigma_1$ is in corresponding Jacquet module of τ_1. So, if we do not get this, this means that π could not be unitary. [1]

Classical groups are particularly convenient for applying the above strategy, since proper Levi subgroups are direct products of a smaller classic group and the general linear groups

[1] We can try to continue the procedure: if π was unitary, so should be also τ_1. So for τ_1, we can try to find G_2, σ_2, $\mathcal{G}_2$, τ_2 etc. Again, if we do not get $\tau_1 \otimes \sigma_2$ in the Jacquet module of τ_2, π could not be unitary.

(and for latter we understand irreducible unitary representations).

Regarding the above strategy of proving the non-unitary of π, the main problem is to find appropriate σ_1, τ_1 etc., necessary to perform the above strategy, and to show appropriate properties (a problem is that very often the irreducible representations are given as irreducible quotients of some reducible representations, and then we need additional work to understand the size of Jacquet modules).

We will now describe our result more explicitly. Let $|\ \ |_F$ be the normalized absolute value on F. Fix a series of symplectic or odd-orthogonal groups over F (see the paper for details)[2]. Denote by S_n the group of split rank n from the series. Let ρ be an irreducible unitary cuspidal representation of some $GL(p, F)$ and σ an irreducible cuspidal representation of some S_q.

Suppose that

$$\mathrm{Ind}^{S_{p+q}}(|\det|_F^{\alpha}\rho \otimes \sigma)$$

reduces for some $\alpha \in (1/2)\mathbb{Z}_{>0}$.[3] The aim of this and the sequel of this paper is to prove that each irreducible subquotient of

$$\mathrm{Ind}^{S_{(n+1)p+q}}(|\det|_F^{\alpha+n}\rho \otimes |\det|_F^{\alpha+n-1}\rho \otimes \cdots \otimes |\det|_F^{\alpha}\rho \otimes \sigma), \quad n \geq 0,$$

different from the irreducible subrepresentation and and the irreducible quotient, is not unitary.[4] This irreducible subrepresentation we shall denote by

$$\delta([|\det|_F^{\alpha}\rho, |\det|_F^{\alpha+n}\rho], \sigma).$$

In this paper we prove non-unitarity for roughly half of these representations. More precisely, we prove the following result.

Theorem 1.1. *Let $n \geq 1$. Then each irreducible subquotient of*

$$Ind^{S_{(n+1)p+q}}(|det|_F^{\alpha+n}\rho \otimes |det|_F^{\alpha+n-1}\rho \otimes \cdots \otimes |det|_F^{\alpha+1}\rho \otimes \delta(|det|_F^{\alpha}\rho, \sigma)),$$

which is not a subquotient of

$$Ind^{S_{(n+1)p+q}}(|det|_F^{\alpha+n}\rho \otimes |det|_F^{\alpha+n-1}\rho \otimes \cdots \otimes |det|_F^{\alpha+2}\rho \otimes \delta([|det|_F^{\alpha}\rho|, det|_F^{\alpha+1}\rho], \sigma)),$$

is not unitarizable, neither is its Aubert involution unitarizable.

[2]The case of unitary groups is similar, but requires slightly different notation.

[3]If we have reduction for some $\alpha \in \mathbb{R}$, then ρ is selfdual, i.e. we have for the contragredient $\tilde{\rho}$ that holds $\tilde{\rho} \cong \rho$. Form the other side, if ρ is selfdual, then we have always reducibility for unique $\alpha \geq 0$. It is expected that such α is always in $(1/2)\mathbb{Z}$ (this is proved by F. Shahidi if σ is generic).

[4]The irreducible subrepresentation is a generalized Steinberg representation, which is square integrable, so it is unitary. The quotient is an Aubert involution of the irreducible subrepresentation, and the first author has proved that it is unitary (assuming a natural conjecture). Both of them, the irreducible subrepresentation and the irreducible quotient, are unique.

In this way we show the non-unitarizability of 2^n representations.

It remains to consider remaining $2^n - 2$ representations. We have done most of the job regarding this (but not all parts are written with all details). This case is technically much more complicated, and we hope that we shall be able to simplify these technicalities.

In [J], C. Jantzen has parametrized the irreducible representations of the classical groups by irreducible representations supported by the "cuspidal lines"

$$\pi \leftrightarrow (\pi_1, \ldots, \pi_\ell).$$

(see [J] for precise result). This correspondence has a number of nice properties. For example, π is square integrable if and only if all π_i are square integrable. Further interesting (but much harder) question is: is π unitary if and only if all π_i are unitary. Our paper gives a very weak support to expect that the answer could be positive. Also [Ha1] gives a weak support. Let us note that both papers use methods depending essentially only on the reducibility point, not the actual inducing representation. We can pose the following interesting question: is the unitarity determined only by the reducibility points (which is the case for square integrability)?

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2 Notation and Preliminaries

We shall fix a local non-archimedean field F such that $\mathrm{char}(F) \neq 2$. The normalized absolute value on F will be denoted by $| \ \ |_F$.

If we have the group of rational points of a connected reductive group $\mathcal{G}$ over F, we shall denote by $\mathcal{R}(\mathcal{G})$ the Grothendieck group of the category $\mathrm{Alg}_{f.l.}(\mathcal{G})$ of all smooth representations of $\mathcal{G}$ of finite length. We have a natural mapping, called the semi simplification, from $\mathrm{Alg}_{f.l.}(\mathcal{G})$ into $\mathcal{R}(\mathcal{G})$:

$$\pi \mapsto \mathrm{s.s.}(\pi).$$

There is a natural ordering $\leq$ on $\mathcal{R}(\mathcal{G})$. When we write $\pi_1 \leq \pi_2$ for π_1, π_2 from $\mathrm{Alg}_{f.l.}(G)$, we shall actually mean the inequality between their images in $\mathcal{R}(\mathcal{G})$. Also, the irreducible representations of $\mathcal{G}$ will be also considered as elements of $\mathcal{R}(\mathcal{G})$.

We shall use very often Frobenius reciprocities. There are two forms of it. Fix for a moment a parabolic subgroup $P = MN$ of $\mathcal{G}$, and admissible representations π and σ of G and M respectively. Denote the Jacquet module of π with respect to $P = MN$ by $r_P^G(\pi)$. The first form of the Frobenius reciprocity says that there is a canonical isomorphism

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma)) \cong \mathrm{Hom}_M(r_P^G(\pi), \sigma).$$

Fix a maximal split torus $A_\emptyset$ in $\mathcal{G}$. Suppose that parabolic subgroup P contains $A_\emptyset$. We can find Levi subgroup M of P containing $A_\emptyset$ (such M is unique). Denote by $\bar{P}$ the opposite parabolic subgroup (this is the unique parabolic subgroup containing $A_\emptyset$, whose Levi subgroup is M and which satisfies $P \cap \bar{P} = M$). The second form of Frobenius reciprocity is

$$\mathrm{Hom}_G(\mathrm{Ind}_P^G(\sigma), \pi) \cong \mathrm{Hom}_M(\sigma, r_{\bar{P}}^G(\pi))$$

(see [C1]).

Now we shall recall the notation of the representation theory of general linear groups over F. We shall follow the usual notation of the Bernstein-Zelevinsky theory (following mainly [Z1]). Let

$$\nu : GL(n,F) \to \mathbb{R}^\times, \quad g \mapsto |\det(g)|_F.$$

The set of equivalence classes of all irreducible essentially square integrable modulo center[5] representations of all $GL(n,F)$, $n \geq 1$, will be denoted by

$$D.$$

For $\delta \in D$ there exists a unique $e(\delta) \in \mathbb{R}$ and a unique unitarizable representation δ^u (which is square integrable modulo center), such that

$$\delta = \nu^{e(\delta)} \delta^u.$$

For smooth representations π_1 and π_2 of $GL(n_1,F)$ and $GL(n_2)$ respectively, $\pi_1 \times \pi_2$ will denote the smooth representation of $GL(n_1+n_2,F)$ parabolically induced by $\pi_1 \otimes \pi_2$ from the appropriate maximal standard parabolic subgroup (for us, the standard parabolic subgroups will be those parabolic subgroups, which contain the subgroup of the upper triangular matrices). Parabolic induction that we use in the paper will always be normalized (it carries unitarizable representations to the unitarizable ones).

Let $\pi_1 \otimes \cdots \otimes \pi_k$ and π be admissible representations of $GL(n_1,F) \times \cdots \times GL(n_k,F)$ and $GL(n_1+\cdots+n_k,F)$ respectively. The second form of the Frobenius reciprocity tells here

$$\begin{aligned}\operatorname{Hom}_{GL(n_1+\cdots+n_k,F)}&(\pi_1 \times \cdots \times \pi_k, \pi)\\ &\cong \operatorname{Hom}_{GL(n_1,F)\times\cdots\times GL(n_k,F)}(\pi_k \otimes \cdots \otimes \pi_1, r_P^{GL(n_1+\cdots+n_k,F)}(\pi)),\end{aligned}$$

where P denotes the standard parabolic subgroup which has $GL(n_1,F) \times \cdots \times GL(n_k,F)$ for Levi factor.

We consider

$$R = \underset{n\geq 0}{\oplus} \mathcal{R}(GL(n,F))$$

as a graded group. Since parabolic indiction is exact functor, $\times$ lifts naturally to a $\mathbb{Z}$-bilinear mapping $R \times R \to R$, which we denote again by $\times$. This $\mathbb{Z}$-bilinear mapping factors through the tensor product, and the factoring homomorphism will be denoted by $m : R \otimes R \to R$.

Let π be an irreducible smooth representation $GL(n,F)$. The sum of the semi simplifications of the Jacquet modules with respect to the standard parabolic subgroups which have Levi subgroups $GL(k,F) \times GL(n-k,F)$, $0 \leq k \leq n$, defines an element of $R \otimes R$ in a natural way (see [Z1] for more details). Jacquet modules that we consider in this paper are

[5] Irreducible representations which became square integrable modulo center after twist by a (not necessarily) unitary character of the group.

normalized. We extend additively this mapping to the whole R, and denote the extension by

$$m^* : R \to R \otimes R.$$

In this way R becomes a graded Hopf algebra.

For an irreducible representation π of $GL(n, F)$ there exist irreducible cuspidal representations $\rho_1, \dots, \rho_k$ of general linear groups such that π is isomorphic to a subquotient of $\rho_1 \times \dots \times \rho_k$. The multiset of equivalence classes $(\rho_1, \dots, \rho_k)$ is called the cuspidal support of π (it depends only on the the equivalence class of π).

We shall recall now of the Langlands classification for the general linear groups. We denote by M(D) the set of all finite multisets in D. For $d = (\delta_1, \delta_2, \dots, \delta_k) \in M(D)$ take a permutation p of $\{1, \dots, k\}$ such that

$$e(\delta_{p(1)}) \geq e(\delta_{p(2)}) \geq \dots \geq e(\delta_{p(k)}).$$

Then the representation

$$\delta_{p(1)} \times \delta_{p(2)} \times \dots \times \delta_{p(k)},$$

has a unique irreducible quotient which will be denoted by

$$L(d).$$

The mapping $d \mapsto L(d)$ defines a bijection between $M(D)$ and the set of all equivalence classes of the irreducible smooth representations of all the general linear groups over F. We shall use the formula for the contragredient in the Langlands classification:

$$L(\delta_1, \delta_2, \dots, \delta_k)\tilde{} \cong L(\tilde{\delta}_1, \tilde{\delta}_2, \dots, \tilde{\delta}_k).$$

Let

$$\mathcal{C}$$

denotes the set of all equivalence classes of irreducible cuspidal representations of all $GL(n, F)$, $n \geq 1$. A segment in $\mathcal{C}$ is the set of form

$$\Delta = [\rho, \nu^k \rho] = \{\rho, \nu\rho, \dots, \nu^k \rho\},$$

where $\rho \in \mathcal{C}, k \in \mathbb{Z}_{\geq 0}$. The set of all such segments will be denoted by

$$\mathcal{S}.$$

For a segment $\Delta = [\rho, \nu^k \rho] = \{\rho, \nu\rho, \dots, \nu^k \rho\} \in \mathcal{S}$, the representation

$$\nu^k \rho \times \nu^{k-1} \rho \times \dots \times \nu\rho \times \rho$$

contains a unique irreducible subrepresentation, which will be denoted by

$$\delta(\Delta)$$

and a unique irreducible quotient, which will be denoted by

$$\mathfrak{s}(\Delta).$$

The representation $\delta(\Delta)$ is an essentially square integrable representation modulo center. In this way we get a bijection between $\mathcal{S}$ and D. Further, $\mathfrak{s}(\Delta) = L(\rho, \nu\rho, \dots, \nu^k\rho)$ and

$$m^*(\delta([\rho, \nu^k\rho])) = \sum_{i=-1}^{k} \delta([\nu^{i+1}\rho, \nu^k\rho]) \otimes \delta([\rho, \nu^i\rho]),$$

$$m^*(\mathfrak{s}([\rho, \nu^k\rho])) = \sum_{i=-1}^{k} \mathfrak{s}([\rho, \nu^i\rho]) \otimes \mathfrak{s}([\nu^{i+1}\rho, \nu^k\rho]).$$

Using the above bijection between D and $\mathcal{S}$, we can express Langlands classification in terms of finite multisets $M(\mathcal{S})$ in $\mathcal{S}$:

$$L(\Delta_1, \dots, \Delta_k) := L(\delta(\Delta_1), \dots, \delta(\Delta_k)).$$

Now we shall recall of the Zelevinsky classification. For $(\Delta_1, \dots, \Delta_k) \in M(\mathcal{S})$ chose a permutation p of $\{1, \dots, k\}$ such that

$$e(\delta(\Delta_{p(1)})) \geq e(\delta(\Delta_{p(2)})) \geq \cdots \geq e(\delta(\Delta_{p(k)})).$$

Then the representation

$$\mathfrak{s}(\Delta_{p(1)}) \times \mathfrak{s}(\Delta_{p(2)}) \times \cdots \times \mathfrak{s}(\Delta_{p(k)}),$$

has a unique irreducible subrepresentation. This subrepresentation is denoted by

$$Z(\Delta_1, \dots, \Delta_k).$$

Again the mapping $d \mapsto Z(d)$ defines a bijection between $M(\mathcal{S})$ and the set of all equivalence classes of the irreducible smooth representations of all general linear groups over F.

The ring R is a polynomial ring over D. Therefore, the ring homomorphism $\pi \mapsto \hat{\pi}$ on R determined by the requirement that $\delta(\Delta) \mapsto \mathfrak{s}(\Delta)$, $\Delta \in \mathcal{S}$, is uniquely determined by this condition. It is an involution, and it is called the Zelevinsky involution. It is a special case of an involution which exists for any connected reductive group, called the Aubert involution. A very important property of this involution (as well as of the Aubert involution) is that it carries irreducible representations to the irreducible ones ([A], [ScSt]).

For $\Delta = [\rho, \nu^k\rho] \in \mathcal{S}$ denote

$$\Delta^- = [\rho, \nu^{k-1}\rho],$$

and for $d = (\Delta_1, \dots, \Delta_k) \in M(\mathcal{S})$ denote

$$d^- = (\Delta_1^-, \dots, \Delta_k^-).$$

Then the ring homomorphism $\mathcal{D} : R \to R$ defined by the requirement that $\delta(\Delta)$ goes to $\delta(\Delta^-)$ for all $\Delta \in \mathcal{S}$, is called the derivative. This is a positive mapping (carries non-negative elements to the non-negative ones; [Z1]). Let $\pi \in R$ and $\mathcal{D}(\pi) = \sum \mathcal{D}(\pi)_n$, where $\mathcal{D}(\pi)_n$ is in the n-th grading group of R. If k is the lowest index such that $\mathcal{D}(\pi)_k \neq 0$, then $\mathcal{D}(\pi)_k$ is called the highest derivative of π, and denoted by h.d.(π). Obviously, the highest derivative is multiplicative. Further

$$\text{h.d.}(Z(\Delta_1, \dots, \Delta_k)) = Z(\Delta_1^-, \dots, \Delta_k^-)$$

(see [Z1]).

We shall now recall of the basic notation of the representation theory of the classical p-adic groups. We shall follow mainly the notation of [T5] and [MœT]. We fix a Witt tower $V \in \mathcal{V}$ of symplectic vector spaces over F, or of orthogonal vector spaces starting with an anisotropic space of odd dimension. We denote by $S(V)$ the group of isometries of $V \in \mathcal{V}$ of determinant 1 (which is automatically satisfied in the symplectic case). The group of split rank n is denoted by S_n. The direct sum of Grothendieck groups $\mathcal{R}(S_n)$, $n \geq 0$, is dented by $R(S)$.

The standard maximal parabolic subgroup of S_n whose Levi factor is a direct product of $GL(k,F)$ and a classical group S_{n-k} will be denoted by $P_{(k)}$. We take $P_{(0)} = S_n$. Analogously as in the case of general linear groups, using parabolic induction we define $\pi \rtimes \sigma$ for a smooth representations π and σ of a general linear group (over F) and S_m respectively (see [T5] for the split case). In the same way as in the case of general linear groups, $\rtimes$ lifts to a mapping $R \times R(S) \to R$, again denoted by $\rtimes$. Factorization through $R \otimes R(S)$ is dented by μ. Thus $R(S)$ is a R-module.

The Jacquet module of a representation π of S_n for $P_{(k)}$ is denoted by $s_{(k)}(\pi)$. For an irreducible representation π of S_n, the sum of semi simplifications of $s_{(k)}(\pi)$, $0 \leq k \leq n$, is denoted by

$$\mu^*(\pi).$$

We consider $\mu^*(\pi) \in R \otimes R(S)$ and extend additively μ^* to the mapping on whole $R(S)$:

$$\mu^* : R(S) \to R \otimes R(S).$$

In this way $R(S)$ becomes R-comodule.

Note that $R \otimes R(S)$ is in a natural way $R \otimes R$-module (we denote the multiplication again by $\rtimes$). Let $\sim : R \to R$ be the contragredient map and $\kappa : R \otimes R \to R \otimes R$, $\sum x_i \otimes y_i \mapsto y_i \otimes x_i$. If we denote

$$M^* = (m \otimes \text{id}_R) \circ (\sim \otimes m^*) \circ \kappa \circ m^*,$$

then

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma) \tag{2.1}$$

for $\pi \in R$ and $\sigma \in R(S)$ (or for admissible representations π and σ of $GL(n,F)$ and S_m respectively).

We several times use the following simple fact:

Lemma 2.1. *Let τ be a representation of some $GL(m, F)$ and let*

$$m^*(\tau) = \sum x \otimes y.$$

Then, the sum of the irreducible subquotients of the form $* \otimes 1 \leq M^*(\tau)$ is

$$\sum x \times \tilde{y} \otimes 1,$$

and the sum of the irreducible subquotients of the form $1 \otimes *$ in $M^*(\tau)$ is $1 \otimes \tau$.

Proof. The lemma follows directly from the formula $M^* = (m \otimes \mathrm{id}_R) \circ (\tilde{} \otimes m^*) \circ \kappa \circ m^*$. □

We shall use several times the formula for

$$M^*(\delta([\nu^a\rho, \nu^c\rho])) \quad = \quad \sum_{s=a-1}^{c} \sum_{t=i}^{c} \delta([\nu^{-s}\tilde{\rho}, \nu^{-a}\tilde{\rho}]) \;\times\; \delta([\nu^{t+1}\rho, \nu^c\rho]) \;\otimes\; \delta([\nu^{s+1}\rho, \nu^t\rho]).$$

We can differently index the summation, and write down the above formula in the following form:

$$M^*(\delta([\nu^a\rho, \nu^c\rho])) = \sum_{j=0}^{c-a+1} \sum_{i=j}^{c-a+1} \delta([\nu^{i-c}\tilde{\rho}, \nu^{-a}\tilde{\rho}]) \times \delta([\nu^{c+1-j}\rho, \nu^c\rho]) \otimes \delta([\nu^{c+1-i}\rho, \nu^{c-j}\rho]).$$

Now we shall recall of the Langlands classification in this case. Denote

$$D_+ = \{\delta \in D; e(\delta) > 0\}.$$

Let T be the set of all equivalence classes of tempered representations of S_n, for all $n \geq 0$. For $((\delta_1, \delta_2, \ldots, \delta_k), \tau) \in M(D_+) \times T$ take a permutation p of $\{1, \ldots, k\}$ such that

$$\delta_{p(1)} \geq \delta_{p(2)} \geq \cdots \geq \delta_{p(k)}$$

Then the representation

$$\delta_{p(1)} \times \delta_{p(2)} \times \cdots \times \delta_{p(k)} \rtimes \tau$$

has a unique irreducible quotient, which will be denoted by

$$L(\delta_1, \delta_2, \ldots, \delta_k; \tau).$$

The mapping

$$((\delta_1, \delta_2, \ldots, \delta_k), \tau) \mapsto L(\delta_1, \delta_2, \ldots, \delta_k; \tau)$$

defines a bijection from the set $M(D_+) \times T$ onto the set of all the equivalence classes of irreducible smooth representations of all S_n, $n \geq 0$.

Denote

$$\mathcal{S}_+ = \{\Delta \in \mathcal{S}; e(\Delta) > 0\}.$$

Then we can define Langlands classification $(a, \tau) \mapsto L(a, \tau)$ using $M(\mathcal{S}_+) \times T$ for parameters (instead of $M(D_+) \times T$).

Let τ and ω be irreducible representations of $GL(p, F)$ and S_q respectively, and π an admissible representation of S_{p+q}. Then the first form of the Frobenius reciprocity tells

$$\mathrm{Hom}_{S_{p+q}}(\pi, \tau \rtimes \omega) \cong \mathrm{Hom}_{GL(p,F)\times S_q}(s_{(p)}(\pi), \tau \otimes \omega),$$

while the second one tells

$$\mathrm{Hom}_{S_{p+q}}(\tau \rtimes \omega, \pi) \cong \mathrm{Hom}_{GL(p,F)\times S_q}(\tilde{\tau} \otimes \omega, s_{(p)}(\pi))$$

(τ in the above formula must be irreducible). We could write the above formulas not necessarily for the maximal parabolic subgroups (as we did in the case of general linear groups).

Fix irreducible unitarizable cuspidal representations ρ and σ of $GL(p, F)$ and of S_q respectively. If $\nu^\alpha\rho \rtimes \sigma$ reduces for some $\alpha \in \mathbb{R}$, then $\rho \cong \tilde{\rho}$. Conversely, if $\rho \cong \tilde{\rho}$, then we have always reduction for unique $\alpha \geq 0$. This reducibility point will be denoted by

$$\alpha_{\rho,\sigma}.$$

In all known examples holds $\alpha_{\rho,\sigma} \in (1/2)\mathbb{Z}$. F. Shahidi has proved this to be the case if σ is generic (see [Sh1] and [Sh2]). This is expected to hold in general.

At the end we shall recall of the generalized Steinberg representations.

Proposition 2.2. *Let ρ and σ be as above. Suppose $\tilde{\rho} \cong \rho$. Assume that $\alpha := \alpha_{\rho,\sigma} > 0$. Then, the representation*

$$\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^{\alpha+1}\rho \times \nu^\alpha\rho \rtimes \sigma$$

has a unique irreducible subrepresentation, which we denote by $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho]; \sigma)$ $(n \geq 0)$. This subrepresentation is square integrable. We have

$$\mu^*\left(\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho]; \sigma)\right) = \sum_{k=-1}^{n} \delta([\nu^{\alpha+k+1}\rho, \nu^{\alpha+n}\rho]) \otimes \delta([\nu^\alpha\rho, \nu^{\alpha+k}\rho]; \sigma)$$

and $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho]; \sigma)\tilde{} \cong \delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho]; \tilde{\sigma})$. Further

$$\mu^*\left(L(\nu^{\alpha+n}\rho. \dots, \nu^{\alpha+1}\rho, \nu^\alpha\rho; \sigma)\right) =$$
$$\sum_{k=-1}^{n} L(\nu^{-(\alpha+n)}\rho, \dots, \nu^{-(\alpha+k+2)}\rho, \nu^{-(\alpha+k+1)}\rho) \otimes L(\nu^{\alpha+k}\rho. \dots, \nu^{\alpha+1}\rho, \nu^\alpha\rho; \sigma)$$

3 A lemma on Langlands parameters

Let ρ be an irreducible unitarizable cuspidal representation of $GL(p,F)$ and σ an irreducible cuspidal representation of S_q. Suppose that $\alpha \in \frac{1}{2}\mathbb{Z}$, $\alpha > 0$, is such that

$$\nu^\alpha \rho \rtimes \sigma$$

reduces (then $\tilde{\rho} \cong \rho$).

For a sequence $\Delta_1, \dots .\Delta_l$ we say that they are descending if $e(\Delta_1) \geq \cdots \geq e(\Delta_l)$.

Lemma 3.1. *Let $n \geq 1$. Fix an integer b satisfying $0 \leq b \leq n-1$. Let $\Delta_1, \dots,\ \Delta_k$ be a sequence of decreasing non-empty segments such that*

$$\Delta_1 + \dots +\ \Delta_k = (\nu^{\alpha+b+1}\rho, \dots, \nu^{\alpha+n-1}\rho, \nu^{\alpha+n}\rho).$$

Let $\Delta_{k+1}, \dots, \Delta_l$, $k < l$, be a sequence of decreasing segments satisfying

$$\Delta_{k+1} + \cdots +\ \Delta_l = (\nu^\alpha\rho, \nu^{\alpha+1}\rho, \dots, \nu^{\alpha+b}\rho),$$

such that $\Delta_{k+1}, \dots, \Delta_{l-1}$ are non-empty. Denote

$$a = (\Delta_1, \dots, \Delta_{k-1}),$$

$$b = (\Delta_{k+2}, \dots, \Delta_{l-1}).$$

Then in $R(S)$ we have:

if $k+1 < l$, then

$$\begin{aligned} L(a + (\Delta_k)) \rtimes L(b; \delta(\Delta_l, \sigma)) = \\ L(a + (\Delta_k, \Delta_{k+1}) + b; \delta(\Delta_l, \sigma)) + \\ L(a + (\Delta_k \cup \Delta_{k+1}) + b; \delta(\Delta_l, \sigma)); \end{aligned} \quad (3.1)$$

if $k+1 = l$, then

$$L(a + (\Delta_k)) \rtimes \delta(\Delta_{k+1}, \sigma) = L(a + (\Delta_k); \delta(\Delta_{k+1}, \sigma)) + L(a; \delta(\Delta_k \cup \Delta_{k+1}, \sigma)). \quad (3.2)$$

Proof. To shorten notation, write (3.1) as

$$\pi = \pi_1 + \pi_2$$

and (3.2) as

$$\pi' = \pi'_1 + \pi'_2.$$

Observe that we are in the regular situation, i.e. $\nu^{\alpha+n}\rho \times \cdots \times \nu^\alpha\rho \rtimes \sigma$ is a regular representation. Therefore, representations π and π' are multiplicity one representations. Further, $\pi_1 \ncong \pi_2$ and $\pi'_1 \ncong \pi'_2$.

We shall first prove that π_1 and π_2 (resp. π_1' and π_2') are subquotients of π (resp. π').

In the case $k+1<l$, we have an epimorphism

$$\big(\delta(\Delta_1)\times\cdots\times\delta(\Delta_k)\big)\rtimes\big(\delta(\Delta_{k+1})\times\cdots\times\delta(\Delta_{l-1})\rtimes\delta(\Delta_l,\sigma)\big)\to\pi.$$

Therefore, π_1 is a subquotient of π.

In the case $k+1=l$, we also have an epimorphism

$$\big(\delta(\Delta_1)\times\cdots\times\delta(\Delta_{k-1})\times\delta(\Delta_k)\big)\rtimes\delta(\Delta_{k+1},\sigma)\to\pi'.$$

So again π_1' is a subquotient of π'.

For the second subquotient we shall use some standard properties of the Langlands classification. Observe that we have

$$L(\Delta_1,\dots,\Delta_{l-1},\delta(\Delta_l,\sigma))\hookrightarrow\delta(\tilde{\Delta}_1)\times\cdots\times\delta(\tilde{\Delta}_{l-1})\rtimes\delta(\Delta_l,\sigma)$$

and the representation on the right hand side has a unique irreducible subrepresentation. By Frobenius reciprocity

$$\delta(\tilde{\Delta}_1)\otimes\cdots\otimes\delta(\tilde{\Delta}_{l-1})\otimes\delta(\Delta_l,\sigma)$$

is in a Jacquet module of $L(\Delta_1,\dots,\Delta_{l-1},\delta(\Delta_l,\sigma))$. Moreover, the representation

$$L(\Delta_1,\dots,\Delta_{l-1},\delta(\Delta_l,\sigma))$$

can be characterized as an irreducible representation which has $\delta(\tilde{\Delta}_1)\otimes\cdots\otimes\delta(\tilde{\Delta}_{l-1})\otimes\delta(\Delta_l,\sigma)$ in the Jacquet module.

In the case $k+1<l$, look at

$$L(\Delta_{k+1},\dots,\Delta_{l-1},\delta(\Delta_l,\sigma))\hookrightarrow\delta(\tilde{\Delta}_{k+1})\times\cdots\times\delta(\tilde{\Delta}_{l-1})\rtimes\delta(\Delta_l,\sigma).$$

From this we conclude that

$$L(\Delta_{k+1},\dots,\Delta_{l-1},\delta(\Delta_l,\sigma))\hookrightarrow\delta(\tilde{\Delta}_{k+1})\rtimes L(\Delta_{k+2},\dots,\Delta_{l-1},\delta(\Delta_l,\sigma)).$$

Frobenius reciprocity implies that

$$\delta(\tilde{\Delta}_{k+1})\otimes L(\Delta_{k+2},\dots,\Delta_{l-1},\delta(\Delta_l,\sigma))$$

is in $\mu^*(L(\Delta_{k+1},\dots,\Delta_{l-1},\delta(\Delta_l,\sigma)))$.

Observe now that

$$L(\Delta_1,\dots,\Delta_k)\hookrightarrow\delta(\Delta_k)\times\delta(\Delta_{k-1})\times\cdots\times\delta(\Delta_1),$$

and the left hand side representation is the unique irreducible subrepresentation of the right-hand side. From this we have

$$L(\Delta_1,\dots,\Delta_k)\hookrightarrow\delta(\Delta_k)\times L(\Delta_1,\dots,\Delta_{k-1}).$$

Thus

$$\delta(\Delta_k) \otimes L(\Delta_1, \dots, \Delta_{k-1})$$

is in a Jacquet module of $L(\Delta_1, \dots, \Delta_k)$. Now the formula for M^* implies that

$$L(\tilde{\Delta}_1, \dots, \tilde{\Delta}_{k-1}) \otimes \delta(\Delta_k)$$

is in $M^*(L(\Delta_1, \dots, \Delta_k))$.

The above discussion implies for the case $k+1 < l$ that

$$L(\tilde{\Delta}_1, \dots, \tilde{\Delta}_{k-1}) \times \delta(\tilde{\Delta}_{k+1}) \otimes \delta(\Delta_k) \rtimes L(\Delta_{k+2}, \dots, \Delta_{l-1}, \delta(\Delta_l, \sigma))$$

is in a Jacquet module of $L(\Delta_1, \dots, \Delta_k) \rtimes L(\Delta_{k+1}, \dots, \Delta_{l-1}, \delta(\Delta_l, \sigma))$. From this follows that the last representation has an irreducible subquotient γ which has

$$\delta(\tilde{\Delta}_1) \otimes \dots \otimes \delta(\tilde{\Delta}_{k-1}) \otimes \delta(\tilde{\Delta}_{k+1}) \otimes \delta(\tilde{\Delta}_k) \otimes \delta(\tilde{\Delta}_{k+2}) \otimes \dots \otimes (\tilde{\Delta}_{l-1}) \otimes \delta(\Delta_l, \sigma)$$

in the Jacquet module. Then γ must have

$$\delta(\tilde{\Delta}_1) \otimes \dots \otimes \delta(\tilde{\Delta}_{k-1}) \otimes \delta(\tilde{\Delta}_{k+1} \cup \tilde{\Delta}_k) \otimes \delta(\tilde{\Delta}_{k+2}) \otimes \dots \otimes (\tilde{\Delta}_{l-1}) \otimes \delta(\Delta_l, \sigma)$$

in a Jacquet module. This implies

$$\gamma \cong L(\Delta_1, \dots, \Delta_{k-1}, \Delta_{k+1} \cup \Delta_k, \Delta_{k+2}, \dots, \Delta_{l-1}; \delta(\Delta_l, \sigma)),$$

i.e. $\gamma \cong \pi_2$.

Look now at the case $k+1 = l$. Clearly, $1 \otimes \delta(\Delta_l, \sigma) \leq \mu^*(\delta(\Delta_l, \sigma))$. From

$$L(\tilde{\Delta}_1, \dots, \tilde{\Delta}_{k-1}) \otimes \delta(\Delta_k) \leq M^*(L(\Delta_1, \dots, \Delta_k))$$

follows that

$$L(\tilde{\Delta}_1, \dots, \tilde{\Delta}_{k-1}) \otimes \delta(\Delta_k) \rtimes \delta(\Delta_l, \sigma)$$

is in the Jacquet module of $\pi' = L(\Delta_1, \dots, \Delta_{k-1}, \Delta_k) \rtimes \delta(\Delta_l, \sigma)$. But then is also

$$L(\tilde{\Delta}_1, \dots, \tilde{\Delta}_{k-1}) \otimes \delta(\Delta_k \cup \Delta_l, \sigma)$$

in the Jacquet module of π'. From this follows that also $\delta(\tilde{\Delta}_1) \otimes \dots \delta(\tilde{\Delta}_{k-1}) \otimes \delta(\Delta_k \cup \Delta_l, \sigma)$ is in the Jacquet module of π', which implies that $\pi_2' = L(\Delta_1, \dots, \Delta_{k-1}; \delta(\Delta_k \cup \Delta_l, \sigma))$ is a subquotient of $\pi' = L(\Delta_1, \dots, \Delta_{k-1}, \Delta_k) \rtimes \delta(\Delta_l, \sigma)$.

Now we shall see that these are all subquotients of π and π'. Looking at possible Langlands parameters of irreducible subquotients, one directly sees that that the induced representation $\nu^{\alpha+b+1}\rho \times \dots \times \nu^{\alpha+n}\rho$ has length 2^{n-b-1}. One gets easily considering Langlands parameters (using [T6]) that the lengths of $\nu^{\alpha}\rho \times \dots \times \nu^{\alpha+b}\rho \rtimes \sigma$ and $\nu^{\alpha}\rho \times \dots \times \nu^{\alpha+n}\rho \rtimes \sigma$ are 2^{b+1} and 2^{n+1} respectively. Since $2 \cdot 2^{n-b-1} \cdot 2^{b+1} = 2^{n+1}$ and $\nu^{\alpha}\rho \times \ldots \times \nu^{\alpha+n}\rho \rtimes \sigma$ is a multiplicity one representation, we see that there can not be more irreducible subquotients then π_1, π_2 in π, and π_1', π_2' in π'. This ends the proof of the lemma. □

4 Non–unitarizability

Let $\alpha \in \frac{1}{2}\mathbb{Z}_{>0}$ satisfy, as before, that $\nu^\alpha\rho \rtimes \sigma$ reduces. The goal of this paper and of sequel of this paper would be to prove that each irreducible subquotient of

$$\nu^\alpha\rho \times \cdots \times \nu^{\alpha+n}\rho \rtimes \sigma, \quad n \geq 1,$$

different from $L(\nu^\alpha\rho, \nu^{\alpha+1}\rho, \ldots, \nu^{\alpha+n}\rho; \sigma)$ and $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho]; \sigma)$, is not unitarizable.

Note that there are $2^{n+1} - 2$ such non-unitarizable subquotients (this follows easily from [T6]). In this paper we shall prove the above claim for 2^n irreducible subquotients.

We shall denote by γ an irreducible subquotients of $\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^\alpha\rho \rtimes \sigma$, different from $L(\nu^\alpha\rho, \nu^{\alpha+1}\rho, \ldots, \nu^{\alpha+n}\rho; \sigma)$ and $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho]; \sigma)$. We can write an irreducible subquotient as

$$\gamma = L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}; \sigma))$$

for some $k \geq 0$, where $\Delta_1, \ldots, \Delta_{k+1}$ is a sequence of decreasing segments such that

$$\Delta_1 + \ldots + \Delta_k + \Delta_{k+1} = (\nu^\alpha\rho, \ldots, \nu^{\alpha+n}\rho),$$

and that $\Delta_1, \ldots, \Delta_k$ are non-empty. Since γ is different from $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho]; \sigma)$, we have

$$k \geq 1.$$

Difference from $L(\nu^\alpha\rho, \nu^{\alpha+1}\rho, \ldots, \nu^{\alpha+n}\rho; \sigma)$ implies

$$\Delta_{k+1} \neq \emptyset \text{ or } \Delta_{k+1} = \emptyset \text{ and card } (\Delta_i) > 1 \text{ for some } 1 \leq i \leq k.$$

We shall split our study of γ into several cases. We shall consider first

4.1 The case of card$(\Delta_k) > 1$ and $\Delta_{k+1} = \emptyset$

Write

$$\Delta_k = [\nu^\alpha\rho, \nu^c\rho].$$

Then

$$\alpha < c.$$

Denote (as before)

$$a = (\Delta_1, \Delta_2, \ldots, \Delta_{k-1}).$$

We shall prove in this section the following

Proposition 4.1. *If card*$(\Delta_k) > 1$*, then*

$$L(a + (\Delta_k); \sigma)$$

is not unitarizable.

Consider $L(a+(\Delta_k)) \rtimes \sigma$. By the previous lemma, in the Grothendieck group we have

$$L(a+(\Delta_k)) \rtimes \sigma = L(a+(\Delta_k);\sigma) + L(a;\delta(\Delta_k;\sigma)). \tag{4.1}$$

We shall denote, as in the proof of the previous lemma, these representations by π', π'_1 and π'_2. Thus in $R(S)$ we have

$$\pi' = \pi'_1 + \pi'_2.$$

We need to prove that π'_1 is not unitarizable.

Denote

$$\Delta_u = [\nu^{-\alpha}\rho, \nu^{\alpha}\rho]$$

and

$$\Delta = \left[\nu^{-\alpha}\rho, \nu^{c}\rho\right].$$

Consider now

$$\delta(\Delta_u) \rtimes \pi' \tag{4.2}$$

By (4.1) we know that in R(S) this representation is

$$\delta(\Delta_u) \rtimes \pi' = \delta(\Delta_u) \rtimes \pi'_1 + \delta(\Delta_u) \rtimes \pi'_2 \tag{4.3}$$

Lemma 4.2. *We have*

$$L(a+(\Delta)) \times \nu^{\alpha}\rho \le \delta(\Delta_u) \times L(a+(\Delta_k)).$$

Proof. First observe that the segment $\{\nu^{\alpha}\rho\}$ is not linked with any segment entering a (since card $(\Delta_k) > 1$), and it is not linked to Δ. Therefore, the representation $L(a+(\Delta)) \times \nu^{\alpha}\rho$ is irreducible (see [R] and [Z1]).

Since in general $L(\Delta'_1, \Delta'_2, \ldots, \Delta'_m)^t = Z(\Delta'_1, \Delta'_2, \ldots, \Delta'_m)$ (see [R]), it is enough to prove the lemma for Zelevinsky classification.

The highest (non-trivial) derivative of $\mathfrak{s}(\Delta_u) \times Z(a+(\Delta_k))$ is $\mathfrak{s}(\Delta_u^-) \times Z(a^- + (\Delta_k^-))$. One can easily see that one subquotient of the last representation is $Z(a^- + (\Delta^-))$. Therefore, there must exist an irreducible subquotient of $\mathfrak{s}(\Delta_u) \times Z(a+(\Delta_k))$ whose highest derivative is $Z(a^- + (\Delta^-))$. The support and highest derivative determine completely the irreducible representation, and it is $Z(a+(\Delta, \{\nu^{\alpha}\rho\})) = Z(a+(\Delta)) \times \nu^{\alpha}\rho$. □

Now using the above lemma we get

$$L(a+(\Delta)) \rtimes \delta(\nu^{\alpha}\rho, \sigma) \le L(a+(\Delta)) \times \nu^{\alpha}\rho \rtimes \sigma \le \delta(\Delta_u) \times L(a+(\Delta_k)) \rtimes \sigma = \delta(\Delta_u) \rtimes \pi'.$$

Thus

$$L(a+(\Delta)) \rtimes \delta(\nu^{\alpha}\rho, \sigma) \le \delta(\Delta_u) \rtimes \pi'. \tag{4.4}$$

Lemma 4.3. *(i) Each irreducible quotient of*

$$L(a+(\Delta)) \rtimes \delta(\nu^\alpha\rho,\sigma)$$

has in a Jacquet module

$$L(\tilde{a}+(\tilde{\Delta})) \otimes \delta(\nu^\alpha\rho,\sigma),$$

and the last representation is not in the Jacquet module of $\delta(\Delta_u) \rtimes \pi_2'$.

(ii) If π_1' *is unitarizable, then each irreducible quotient of*

$$L(a+(\Delta)) \rtimes \delta(\nu^\alpha\rho,\sigma)$$

is a subrepresentation of $\delta(\Delta_u) \rtimes \pi_1'$.

Proof. The fact that

$$\theta := L(\tilde{a}+(\tilde{\Delta})) \otimes \delta(\nu^\alpha\rho,\sigma)$$

is in the Jacquet modules of each quotient follows from the second adjointness (see the second section). Further, using (4.4) and (4.3), from (i) we get directly (ii).

Suppose that $\theta = L(\tilde{a}+(\tilde{\Delta})) \otimes \delta(\nu^\alpha\rho,\sigma)$ is in the Jacquet module of

$$\delta(\Delta_u) \rtimes \pi_2' = \delta(\Delta_u) \rtimes L(a;\delta(\Delta_k;\sigma)).$$

Observe

$$\delta(\Delta_u) \rtimes \pi_2' \leq \delta(\Delta_u) \times L(a) \rtimes \delta(\Delta_k;\sigma).$$

Recall that

$$\mu^*(\delta(\Delta_k;\sigma)) = \sum_{i=-1}^{c-\alpha} \delta([\nu^{\alpha+i+1}\rho,\nu^c\rho]) \otimes \delta([\nu^\alpha\rho,\nu^{\alpha+i}\rho];\sigma).$$

We shall analyze how we can get θ in the Jacquet module of $\delta(\Delta_u) \times L(a) \rtimes \delta(\Delta_k;\sigma)$. We shall use the formula for μ^* of that representation (using M^*), which is

$$M^*(\delta(\Delta_u)) \times M^*(L(a)) \rtimes \mu^*(\delta(\Delta_k;\sigma)).$$

We shall turn our attention to $\nu^{-c}\rho$, which is in the cuspidal support of the left hand side tensor factor of θ. From the above formula for $\mu^*(\delta(\Delta_k;\sigma))$, we see that $\nu^{-c}\rho$ cannot come from this term. It cannot also come from $M^*(L(a))$, since $\nu^c\rho$ (and also $\nu^{-c}\rho$) is not in the cuspidal support of $L(a)$. Therefore, $\nu^{-c}\rho$ must come from $M^*(\delta(\Delta_u))$. But it cannot come also from this factor, since neither $\nu^{-c}\rho$ nor $\nu^c\rho$ is in the cuspidal support of $\delta(\Delta_u)$ (here we use that $\operatorname{card}(\Delta_k) > 1$). So we got a contradiction. This completes the proof. □

We shall need the following technical

Lemma 4.4. *We have*

$$\delta(\Delta_u) \otimes \pi_1' \not\leq \mu^*(L(a) \times \delta(\Delta) \rtimes \delta(\nu^\alpha\rho,\sigma)).$$

Proof. Suppose that the claim of the lemma does not hold. Then

$$\delta(\Delta_u) \otimes \pi_1' \leq M^*(L(a)) \times M^*(\delta(\Delta)) \rtimes \mu^*(\delta(\nu^\alpha\rho, \sigma))).$$

Now we shall analyze how we can get $\delta(\Delta_u) \otimes \pi_1'$ from above product on the right hand side. Recall $\pi_1' = (L(a + (\Delta_k); \sigma)$.

Since the cuspidal support of $L(a) = L(\Delta_1, \Delta_2, \dots, \Delta_{k-1})$ is disjoint with the cuspidal support of $\delta(\Delta_u)$ $(= \delta(\tilde{\Delta}_u))$, Lemma 2.1 implies that from the first factor we must take

$$1 \otimes L(a).$$

Since

$$\mu^*(\delta(\nu^\alpha\rho, \sigma))) = 1 \otimes \delta(\nu^\alpha\rho, \sigma)) + \nu^\alpha\rho \otimes \sigma,$$

to get right hand factor in $\delta(\Delta_u) \otimes \pi_1'$, we need to take here

$$\nu^\alpha\rho \otimes \sigma$$

(use Lemma 3.1).

To get left hand side factor in $\delta(\Delta_u) \otimes \pi_1'$, we shall need to take from $M^*(\delta(\Delta))$ a term which is not of the form $1 \otimes *$ (look at $\nu^{-\alpha}\rho$ term in the cuspidal support). Write $M^*(\delta(\Delta))$:

$$M^*(\delta([\nu^{-\alpha}\rho, \nu^c\rho])) \quad = \quad \sum_{i=-(\alpha+1)}^{c} \sum_{j=i}^{c} \delta([\nu^{-i}\rho, \nu^\alpha\rho]) \ \times \ \delta([\nu^{j+1}\rho, \nu^c\rho]) \ \otimes \ \delta([\nu^{i+1}\rho, \nu^j\rho]).$$

Since $\nu^c\rho$ is not in the cuspidal support of $\delta(\Delta_u)$, we need to take from the above sum a term with $j = c$, i.e. a term of the form

$$\delta([\nu^{-i}\rho, \nu^\alpha\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^c\rho]).$$

We already noted that it must be $-i \leq \alpha$. This implies $\delta(\Delta_u) \leq \delta([\nu^{-i}\rho, \nu^\alpha\rho]) \times \nu^\alpha\rho$ for some $-i \leq \alpha$. This is impossible, since the cuspidal supports of the left hand side and right hand sides are different (look at $\nu^\alpha\rho$). This contradiction ends the proof of the lemma. □

Now we shall end the proof of the Proposition 4.1.

Suppose that $\gamma = \pi_1'$ is unitarizable. Then (ii) of the above Lemma implies that we have a non-trivial intertwining:

$$L(a + (\Delta)) \rtimes \delta(\nu^\alpha\rho, \sigma) \to \delta(\Delta_u) \rtimes \pi_1'$$

Frobenius reciprocity implies

$$\delta(\Delta_u) \otimes \pi_1' \leq \mu^*(L(a + (\Delta)) \rtimes \delta(\nu^\alpha\rho, \sigma)).$$

This further implies

$$\delta(\Delta_u) \otimes \pi_1' \leq \mu^*(L(a) \times \delta(\Delta) \rtimes \delta(\nu^\alpha\rho, \sigma)).$$

This contradicts the last lemma. Therefore, π_1' cannot be unitarizable. So the proof of the proposition is complete.

4.2 The case of card$(\Delta_{k+1}) = 1$

In this case, the proof of the non–unitarizability will "dually" follow from the proof of the non–unitarizability of the representations with card$(\Delta_k) > 1$ and $\Delta_{k+1} = \emptyset$. We will apply Aubert involution to achieve this. First, we will prove that the Aubert dual of the representation with card$(\Delta_{k+1}) = 1$ is a representation satisfying card $(\Delta_k) > 1$ and $\Delta_{k+1} = \emptyset$, and conversely (keeping the notation from the previous sections). Then, we "dualize" the whole procedure from the case card$(\Delta_k) > 1$ and $\Delta_{k+1} = \emptyset$.

We will denote the Aubert involution by

$$\pi \mapsto \hat{\pi}.$$

Lemma 4.5. *The Aubert involution transfers the representation of the form*

$$\pi := L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}, \sigma))$$

where

$$card(\Delta_{k+1}) = 1$$

onto the set of representation of the form

$$\pi' = L(\Delta'_1, \ldots, \Delta'_s; \sigma)$$

with

$$card(\Delta'_s) > 1 \ .$$

Proof. One can first observe that both sets of representations considered in the lemma have the same cardinality (which is 2^{n-1}), and that they are disjoint.

Observe that for π as in the lemma, by Lemma 3.1 we have

$$\pi = L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}, \sigma)) \leq L(\Delta_1, \ldots, \Delta_k) \rtimes \delta(\Delta_{k+1}, \sigma).$$

This implies

$$\hat{\pi} \leq L(\Delta_1, \ldots, \Delta_k)\hat{\ } \rtimes L(\Delta_{k+1}, \sigma).$$

Therefore, $\hat{\pi}$ is a representation of the form $L(\Delta'_1, \ldots, \Delta'_s; \sigma)$.

Observe now that

$$\pi = L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}, \sigma)) \leq L(\Delta_1, \ldots, \Delta_k, \Delta_{k+1}) \rtimes \sigma$$

$$\leq L(\Delta_1, \ldots, \Delta_k \backslash \{\nu^{\alpha+1}\rho\}) \times L(\{\nu^{\alpha+1}\rho\}, \Delta_{k+1}) \rtimes \sigma.$$

This implies

$$\hat{\pi} \leq L(\Delta_1, \ldots, \Delta_k \backslash \{\nu^{\alpha+1}\rho\})\hat{\ } \times \delta([\nu^{\alpha}\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma.$$

Now Lemma 3.1, together with the first conclusion, implies that $\hat{\pi}$ is of the form $\pi' = L(\Delta'_1, \ldots, \Delta'_s; \sigma)$ with card$(\Delta'_s) > 1$. Since the sets of π and π' as in the lemma have the same cardinality, the proof of the lemma is now complete. □

Proposition 4.6. *Let*

$$card(\Delta_{k+1}) = 1.$$

Then representation

$$\pi := L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}, \sigma))$$

is not unitarizable.

Proof. We denote

$$L(\Delta_1, \ldots, \widehat{\Delta_k; \delta}(\Delta_{k+1}, \sigma)) = L(\Delta_1^{(d)}, \ldots, \Delta_{k^{(d)}}^{(d)}; \sigma).$$

We assume that segments $\Delta_1^{(d)}, \ldots, \Delta_{k^{(d)}}^{(d)}$ are descending. We know $\mathrm{card}(\Delta_{k^{(d)}}^{(d)}) > 1$.

Denote

$$a^{(d)} = (\Delta_1^{(d)}, \ldots, \Delta_{k^{(d)}-1}^{(d)}),$$
$$\Delta_{k^{(d)}}^{(d)} = [\nu^\alpha \rho, \nu^{c^{(d)}} \rho],$$
$$\Delta_u^{(d)} = [\nu^{-\alpha} \rho, \nu^\alpha \rho],$$
$$\Delta^{(d)} = [\nu^{-\alpha} \rho, \nu^{c^{(d)}} \rho].$$

By Lemma 4.2 we know that

$$L(a^{(d)} + (\Delta^{(d)})) \times \nu^\alpha \rho \leq \delta(\Delta_u^{(d)}) \times L(a^{(d)} + (\Delta_{k^{(d)}}^{(d)})),$$

which implies

$$L(a^{(d)} + (\Delta^{(d)})) \rtimes \delta(\nu^\alpha \rho; \sigma) \leq \delta(\Delta_u^{(d)}) \times L(a^{(d)} + (\Delta_{k^{(d)}}^{(d)})) \rtimes \sigma.$$

Let $\tau^{(d)}$ be an irreducible quotient of

$$L(a^{(d)} + (\Delta^{(d)})) \rtimes \delta(\nu^\alpha \rho; \sigma).$$

By Lemma 4.3 we know that $\tau^{(d)}$ is not a subquotient of $\delta(\Delta_u^{(d)}) \rtimes L(a^{(d)} + \delta(\Delta_{k^{(d)}}^{(d)}; \sigma))$. It is a subquotient of $\delta(\Delta_u^{(d)}) \rtimes L(a^{(d)} + (\Delta_{k^{(d)}}^{(d)}); \sigma)$.

From this follows that $(\tau^{(d)})\hat{}$ is a subquotient of $\delta(\Delta_u^{(d)})\hat{} \rtimes L(a^{(d)} + (\Delta_{k^{(d)}}^{(d)}); \sigma)\hat{}$. Suppose that $L(a^{(d)} + (\Delta_{k^{(d)}}^{(d)}); \sigma)\hat{}$ is unitarizable. Then Frobenius reciprocity implies

$$\delta(\Delta_u^{(d)})\hat{} \otimes L(a^{(d)} + (\Delta_{k^{(d)}}^{(d)}); \sigma)\hat{} \leq \mu^*((\tau^{(d)})\hat{}).$$

Now behavior of Jacquet modules under the Aubert involution implies

$$\delta(\Delta_u^{(d)})\tilde{} \otimes L(a^{(d)} + (\Delta_{k^{(d)}}^{(d)}); \sigma) \leq \mu^*((\tau^{(d)})).$$

But

$$\mu^*((\tau^{(d)})) \leq \mu^*(L(a^{(d)} + (\Delta^{(d)})) \rtimes \delta(\nu^\alpha \rho; \sigma)).$$

This would imply

$$\delta(\Delta_u^{(d)})\tilde{} \otimes L(a^{(d)} + (\Delta_{k^{(d)}}^{(d)}); \sigma) \leq \mu^*(L(a^{(d)} + (\Delta^{(d)})) \rtimes \delta(\nu^\alpha \rho; \sigma)).$$

By Lemma 4.4, this is not possible. This completes the proof of Proposition 4.6. □

Remark 4.7. We can in the same way prove Propositions 4.1 and 4.6 for the unitary groups. The only difference is that all the time when we were using contragredient representation, say π, in the case of unitary groups we need to use the representation $g \mapsto \tilde{\pi}(\theta(g))$, where θ is a non-trivial F-automorphism of the separable quadratic extension F' defining the series of the unitary groups that we consider (see [MœT] for more details).
Remark 4.8. The extension of Propositions 4.1 and 4.6 to the case of the full even–orthogonal groups is also possible. Although this group is not connected, our arguments can be extended to this case; we just make a comment about all the ingredients needed for the proof. The transitivity of parabolic induction and of the Jacquet modules is proved in $[BZ]$ in the context of the l–groups and their closed subgroups satisfying the conditions which are fulfilled in the case of the full even–orthogonal groups. The structure formula (2.1), proved in [T5] in the context of (connected) odd orthogonal and symplectic groups, was proved for the full even–orthogonal groups by Ban in [B]. The Langlands classification takes the analogous form and the only thing left to consider is the uniqueness of the positive reducibility point in the generalized rank 1 case for the cuspidal representation. We were unable to find the correct reference, so, for the sake of completeness, we give the proof. The notation for the parabolic induction for $O(2n, F)$ is analogous, which reflects the same structure of standard parabolic subgroups.

Lemma 4.7. *Let ρ be a cuspidal, selfcontragredient, irreducible representation of $GL(m_\rho, F)$, and σ an irreducible cuspidal representation of $O(2n, F)$. If the representation $\rho\nu^\alpha \rtimes \sigma$ reduces for some $\alpha > 0$, this α is a unique non-negative real number with this property.*

Proof. We prove this lemma by studying the restriction of this representation to $SO(2n, F)$. We use [MVW], Lemma 5 on p.60 which describes such restrictions. The reasoning is quite analogous to Lemma 2.5 of [LMT]. Assume that α is a point of reducibility. We now comment the case $n \geq 2$ (split $O(2, F)$ does not have cuspidal representations). Direct calculation shows that the following holds:

$$(\rho\nu^\alpha \rtimes \sigma)|_{SO} \cong \rho\nu^\alpha \rtimes (\sigma_| SO).$$

Assume firstly that $\sigma|_{SO}$ is irreducible. Then, the reducibility of the left–hand side above (as $O(2(n + m_\rho), F)$ representations), forces the right–hand side above to be reducible (as $SO(n+m_\rho), F)$ representation), and the claim follows from the uniqueness of the reducibility point in the connected case.

Now, let $\sigma|_{SO} = \sigma' \oplus \sigma''$. In general, for any irreducible representation π of $GL(m_\rho, F)$ the following holds: $\pi \rtimes \sigma$ is irreducible if and only if $\pi \rtimes \sigma'$ is irreducible and $\pi \rtimes \sigma' \ncong \pi \rtimes \sigma''$. Both implications directly follow from the lemma of [MVW] mentioned above (we use that $\mathrm{Ind}_{SO}^{O}(\pi \rtimes \sigma') \cong \pi \rtimes \sigma$). This means $\rho\nu^\alpha \rtimes \sigma$ is reducible if and only if $\nu^\alpha\rho \rtimes \sigma'$ is reducible or if $\nu^\alpha\rho \rtimes \sigma'$ is irreducible, then $\rho\nu^\alpha \rtimes \sigma' \cong \rho\nu^\alpha \rtimes \sigma''$. We examine the latter possibility. Then, since $\rho\nu^\alpha \rtimes \sigma'' \cong (\rho\nu^\alpha \rtimes \sigma')^\epsilon$, (notation from [MVW]), the representation $\mathrm{Ind}_{SO}\rho\nu^\alpha \rtimes \sigma'$ reduces and is isomorphic to the direct sum of two representations which differ by the sign character. On the other hand, $\mathrm{Ind}_{SO}\rho\nu^\alpha \rtimes \sigma' \cong \rho\nu^\alpha \rtimes \sigma$, and, since $\alpha > 0$, this representation has a unique irreducible quotient, and cannot be a direct sum. So, again we have that $\rho\nu^\alpha \rtimes \sigma'$ reduces, and α is unique.

The case of Siegel parabolic is similar, but simpler. □

5 Bibliography

[A] A.-M. Aubert, Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longeur finie d'un groupe réductif p-adique, *Trans. Amer. Math. Soc.*, 347 (1995), 2179-2189 (and Erratum, *Trans. Amer. Math. Soc.*, 348 (1996), 4687-4690).

[B] D. Ban, Parabolic induction and Jacquet modules of representations of $O(2n, F)$, *Glas. Mat. Ser. III* 34(54) (1999), 147–185.

[BD] J. Bernstein,rédigé par P. Deligne, *Le "centre" de Bernstein*, "Représentations des Groupes Réductifs sur un Corps Local" written by J.-N. Bernstein, P. Deligne, D. Kazhdan and M.-F. Vignéras, Hermann, Paris, 1984.

[BZ] I. N. Bernstein, A.V. Zelevinsky, Induced representations of reductive $\mathfrak{p}$-adic groups. I, *Ann. Sci. École Norm. Sup. (4)*, 10, (1977) (4), 441–472.

[C1] W. Casselman, *Introduction to the theory of admissible representations of reductive p-adic groups* preprint.

[C2] W. Casselman, *A new non-unitarity argument for p-adic representations*, Jour. Fac. Sci. Tokyo, vol. 28, 1982, 907-928

[Ha1] M. Hanzer, The unitarizability of the Aubert dual of the strongly positive discrete series, Israel Journal of Mathematics

[Ha2] M. Hanzer, Unitarizability of a certain class of irreducible representations of classical groups, *Manuscripta Math.* vol. 127, 2008, no.3, 275-307

[HoMr] R. Howe and C.C. Moore, Asymptotic properties of unitary representations, J. Functional Analysis, vol. 32, 1979, 72-96

[J] C. Jantzen, On supports of induced representations for symplectic and odd-orthogonal groups, Amer J. Math., vol. 119, 1997, 1213-1262

[LMT] E. Lapid, G. Muić, M. Tadić, On the generic unitary dual of quasisplit classical groups, *Int. Math. Res. Not.* no. 26, 2004, 1335–1354.

[MVW] C. Mœglin, M-F. Vignéras, J-L. Waldspurger, Correspondances de Howe sur un corps p-adique, Lecture Notes in Mathematics, 1291. Springer-Verlag, Berlin, 1987.

[R] F. Rodier, *Représentations de $GL(n, k)$ où k est un corps p-adique*, Séminaire Bourbaki no 587 (1982), Astérisque, vol. 92-93, 1982, 201-218

[MœT] C. Mœglin and M. Tadić, Construction of discrete series for classical p-adic groups, J. Amer. Math. Soc., vol. 15, 2002, 715-786

[Mu] G. Muić, *Construction of the Steinberg Type Representations for Reductive p-adic Groups*, Math. Z., vol. 253, 2006, 635-652.

[SaT] P.J. Sally and M. Tadić, Induced representations and classifications for $\mathrm{GSp}(2, F)$ and $\mathrm{Sp}(2, F)$, Mémoires Soc. Math. France, vol 52, 1993, 75-133

[ScSt] P. Schneider and U. Stuhler, Representation theory and sheaves on the Bruhat-Tits building, Publ. Math. IHES, 1997, 97-191

[Sh1] F. Shahidi, A proof of Langlands conjecture on Plancherel measures; complementary series for p-adic groups, Ann. of Math., vol. 132, 1990, 273-330

[Sh2] F. Shahidi, F. Twisted endoscopy and reducibility of induced representations for p-adic groups, Duke Math. J., vol 66, 1992, 1-41

[T1] M. Tadić, Classification of unitary representations in irreducible representations of a general linear group (non-archimedean case), *Ann. Scient. Éc. Norm. Sup.* 19 (1986), 335-382.

[T2] M. Tadić, Geometry of dual spaces of reductive groups (non-archimedean case), J. Analyse Math., vol 51, 1988, 139-181

[T3] M. Tadić, An external approach to unitary representations, *Bulletin Amer. Math. Soc.* 28, No. 2 (1993), 215-252.

[T4] M. Tadić, Representations of p-adic symplectic groups, Compositio Math., vol 90, 123-181, 1994

[T5] M. Tadić, Structure arising from induction and Jacquet modules of representations of classical p–adic groups, Journal of Algebra, vol. 177, 1995, 1–33

[T6] M. Tadić, On regular square integrable representations of p-adic groups, Amer. J. Math., vol. 120, 1998, 159-210

[T 7] M. Tadić, Square integrable representations of classical p-adic groups corresponding to segments, Representation Theory, vol. 3, 1999, 58-89

[T8] M. Tadić, On classification of some classes of irreducible representations of classical groups, in book Representations of real and p-adic groups, 95-162, Singapore University Press and World Scientific, Singapore, 2004

[T9] M. Tadić, On reducibility and unitarizability for classical p-adic groups, some general results, Canadian Journal of Math. (to appear)

[Z1] A.V. Zelevinsky, Induced representations of reductive p-adic groups II, *Ann. Scient. Éc. Norm. Sup.* 13 (1980), 165-210.

[Z2] A.V. Zelevinsky, p-adic analogue of the Kazhdan-Lusztig conjecture, Funct. Anal. Appl., vol 15, 83-92, 1981