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On the Generic Unitary Dual of Quasisplit Classical Groups

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1 Introduction

Let G be a classical group, that is, either a symplectic, orthogonal, or unitary group, defined over a p-adic field or an Archimedean field. The study of admissible representations of G (especially in the p-adic case) was carried out extensively by many authors, including, among others, Ban, Goldberg, Jantzen, Mœglin, Shahidi, and the present authors. In particular, the square-integrable, tempered, and generic representations, as well as the reducibility points of generalized principal series were classified, under certain assumptions, in terms of supercuspidal representations [6, 7, 16, 19]. Our knowledge about the unitary dual of G is much less complete. The classification of the unitary dual of GL_n was achieved by Vogan in the Archimedean case [32] and by the third-named author in the p-adic case [27]. For split classical groups, Barbasch determined the unitary dual in the complex case [2] and Barbasch and Moy determined the unramified part of the unitary dual in the p-adic case [3]. The purpose of this paper is to classify the *generic* unitary representations for quasisplit classical groups (both in the Archimedean and in the p-adic case) in terms of their Langlands data.

The point of departure of our work is the key fact, proved by Vogan [30] and Kostant [12] in the Archimedean case and by the second-named author [18] in the p-adic case, that the irreducible generic representations are precisely the irreducible standard modules with generic Langlands data. The problem then becomes to analyze the structure of the various complementary series.

To explain the main result of this paper, let G_n be a classical quasisplit group of rank n defined over a local field F of characteristic 0. (See Section 2 for the exact setup.)

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In the case of unitary groups, we write E for the corresponding quadratic extension of F, and let θ be the nontrivial Galois involution of E/F. In all other cases, set $E = F$ and $\theta = 1$. Denote by $|\cdot|$ the normalized absolute value[1] of E and by ν the character on $GL_m(E)$ defined by $\nu(\cdot) = |\det(\cdot)|$. If $\sigma \in \mathrm{Irr}(GL_m(F))$, define

$$\sigma^*(g) = \theta(\tilde{\sigma}), \tag{1.1}$$

where $\tilde{\sigma}$ is the contragredient representation of σ.

For any essentially square-integrable representation δ of $GL_m(E)$, let $e(\delta)$ be the unique $e \in \mathbb{R}$ such that $\delta^u = \delta\nu^{-e} = \delta \otimes \nu^{-e}$ is unitary.

Let π be an irreducible generic representation of G_n. By the aforementioned result of Vogan, Kostant, and Muić, we can write π uniquely as

$$\pi \simeq \delta_1 \times \cdots \times \delta_k \rtimes \tau, \quad e(\delta_1) \geq \cdots \geq e(\delta_k) > 0, \tag{1.2}$$

where the δ_i's are essentially square-integrable of $GL_{m_i}(E)$ and τ is a tempered generic representation of $G_{n-m_1-\cdots-m_k}$. Here $\times$ and $\rtimes$ denote parabolic induction (see Section 2).

For any square-integrable representation δ of $GL_m(E)$, denote by $\mathcal{E}_\pi(\delta)$ the multi-set of exponents $e(\delta_i)$ for those i such that $\delta_i^u \simeq \delta$.

Our main result is the following theorem.

Theorem 1.1. The unitarizable generic representations of G_n are given by the representations of the form (1.2) satisfying the following conditions with respect to any discrete series representation δ of $GL_m(E)$:

(1) $\mathcal{E}_\pi(\delta^*) = \mathcal{E}_\pi(\delta)$, that is, π is Hermitian;

(2) if either $\delta \not\simeq \delta^*$ or $\delta\nu^{1/2} \rtimes 1$ is reducible, then $0 < \alpha < 1/2$ for all $\alpha \in \mathcal{E}_\pi(\delta)$;

(3) if $\delta^* \simeq \delta$ and $\delta\nu^{1/2} \rtimes 1$ is irreducible, then $\mathcal{E}_\pi(\delta)$ satisfies Barbasch's conditions, that is, $\mathcal{E}_\pi(\delta) = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l\}$ with

$$0 < \alpha_1 \leq \cdots \leq \alpha_k < \frac{1}{2} \leq \beta_1 < \cdots < \beta_l < 1 \tag{1.3}$$

such that

(a) $\alpha_i + \beta_j \neq 1$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, l$,

(b) $\#\{1 \leq i \leq k : \alpha_i > 1 - \beta_1\}$ is even if $l > 0$,

(c) $\#\{1 \leq i \leq k : 1 - \beta_j > \alpha_i > 1 - \beta_{j+1}\}$ is odd for $j = 1, \ldots, l-1$,

(d) $k + l$ is even if $\delta \rtimes \tau$ is reducible. □

[1] Note that in the complex case, $|a + bi| = a^2 + b^2$ for $a, b \in \mathbb{R}$.

We remark that the reducibility of $\delta\nu^{1/2} \rtimes \mathbf{1}$ can be expressed in terms of L-functions (cf. Section 2) and is also related to functoriality. Finally, note that the dependence on τ is only through (3d).

One implication of the theorem is the following corollary.

Corollary 1.2. The generic unitary dual is pathwise connected to the tempered generic unitary dual. More precisely, given π generic and unitarizable, there exists a continuous path π_t, $t \in [0, 1]$, in the generic unitary dual such that $\pi_0 \simeq \pi$ and π_1 is tempered. □

The resemblance between our classification and that of Barbasch and Moy is not purely aesthetical. In fact, unramified representations become generic under the Iwahori-Matsumoto involution, and the latter is known to preserve unitarity in certain cases.

The structure of the paper is as follows. In Section 2, we recall some results about classical groups and their representations, R-groups, and standard modules and their reducibility. The unitarity part of Theorem 1.1 is proved in Section 3 by a rather straightforward argument. The exhaustion part is proved in Section 4 using a series of reductions. The method is analogous to the one used in [27], but is more complicated in some aspects. The basic case emanating from the reduction is finally treated using an argument of Vogan.

We hope that our result will shed some light on the much more complicated problem of classifying the full unitary dual.

2 Generic representations, standard modules, and self-duality type

2.1

Let $\mathcal{S} = \{G_n\}_{n=0}^{\infty}$ be any one of the following families of groups:

(1) the split special orthogonal group in $2n + 1$ variables,
(2) the symplectic group in $2n$ variables,
(3) the split orthogonal group in $2n$ variables,
(4) the quasisplit but nonsplit special orthogonal group in $2n+2$ variables which splits over a given quadratic extension F'/F,
(5) the quasisplit unitary group in $2n + 1$ variables defined by the quadratic extension E/F,
(6) the quasisplit unitary group in $2n$ variables defined by the quadratic extension E/F.

(See [17, Section 1] for a thorough discussion of classical groups).

The family $\mathcal{S}$ will be fixed throughout. Note that in (3), we use the full, rather than the special, orthogonal group. This will make the discussion below more uniform. The groups G_n are the isomorphism groups (or subgroups of index two thereof) of the corresponding symplectic, orthogonal, or Hermitian space V_n. The space V_{n+1} is obtained by adjoining a hyperbolic plane to V_n, and V_0 is anisotropic of dimension 0, 1, or 2. Thus, $G_0 \subset G_1 \subset G_2 \subset \cdots$.

In the last two cases, we let θ be the nontrivial Galois involution of E/F and $\omega_{E/F}$ the quadratic character of F^* attached to E by class field theory. In all other cases, we set $E = F$ and $\theta = \mathbf{1}$.

In each case, G_n is of rank n over F. It has precisely n maximal parabolic subgroups up to conjugation and their Levi subgroups are isomorphic to $GL_k(E) \times G_{n-k}$, $k = 1, \ldots, n$ (see [17]).

We will use the notations $\times$ and $\rtimes$ to denote induction of representations as in [29].

The Langlands classification still holds in the context of $O(2n)$ [1] and takes the form (1.2). The adaptation of the main result of [18] from $SO(2n)$ to $O(2n)$ is also a simple application of Mackey's theory (cf. [20, Proposition 2.1]).

We will fix a family of nondegenerate characters χ_n on the maximal unipotent subgroups of G_n which is compatible with respect to restriction. A representation π of G_n is *generic* if it has a (smooth) Whittaker model with respect to χ_n. (This depends on the choice of χ_n.) Note also that in the $O(2n)$ case, there is no uniqueness of (smooth) Whittaker model. However, if π is irreducible and generic, then each of the possibly two irreducible summands of the restriction of π to $SO(2n)$ is generic.

2.2 Twisted self-duality types

Let $\mathcal{D}_n$ be the set of (infinitesimal) equivalence classes of square-integrable (irreducible) representations of $GL_n(E)$. Set $\mathcal{D} = \cup_n \mathcal{D}_n$.

We define the following representations of the L-group of $GL_n(E)$:

$$r = \begin{cases} \wedge^2, & \text{case } (1), \\ \mathrm{sym}^2, & \text{cases } (2), (3), (4), \\ \Psi, & \text{case } (5), \\ \Psi \otimes \omega_{E/F}, & \text{case } (6), \end{cases} \qquad r_2 = \begin{cases} \mathrm{sym}^2, & \text{case } (1), \\ \wedge^2, & \text{cases } (2), (3), (4), \\ \Psi \otimes \omega_{E/F}, & \text{case } (5), \\ \Psi, & \text{case } (6). \end{cases} \tag{2.1}$$

(See [8] for the definition of Ψ, which gives rise to the so-called generalized Asai L-function.) In any case, $L(s,\delta,r)$ and $L(s,\delta,r_2)$ are well defined and we have (see [8, 24])

$$L\big(s,\delta\otimes\theta(\delta)\big)=L(s,\delta,r)L\big(s,\delta,r_2\big) \tag{2.2}$$

for any $\delta\in\mathcal{D}$. We set

$$\mathcal{D}^{\theta}=\big\{\delta\in\mathcal{D}:\theta(\delta)\simeq\tilde{\delta}\big\}. \tag{2.3}$$

If $\delta\in\mathcal{D}^{\theta}$, we say that δ is *twisted self-dual*. It follows from (2.2) that we have a partition of $\mathcal{D}^{\theta}$ into two disjoint subsets $\mathcal{D}^{\mathcal{S}}$ and $\mathcal{D}^{n\mathcal{S}}$ defined by

$$\begin{aligned}\mathcal{D}^{\mathcal{S}}&=\big\{\delta\in\mathcal{D}:L(0,\delta,r)=\infty\big\},\\ \mathcal{D}^{n\mathcal{S}}&=\big\{\delta\in\mathcal{D}:L(0,\delta,r_2)=\infty\big\}.\end{aligned} \tag{2.4}$$

The conditions $\delta\in\mathcal{D}^{\mathcal{S}}$ and $\mathcal{D}^{n\mathcal{S}}$ determine the *twisted self-duality type* of δ. They are also related to functoriality and twisted endoscopy (cf. [8, 24]). Note that, in the cases (2), (3), and (4), if $\delta\in\mathcal{D}_m\cap\mathcal{D}^{n\mathcal{S}}$, then m is necessarily even. Also, note that $\delta\in\mathcal{D}^{\theta}$ (resp., $\delta\in\mathcal{D}^{\mathcal{S}}$, $\mathcal{D}^{n\mathcal{S}}$) if and only if $\tilde{\delta}\in\mathcal{D}^{\theta}$ (resp., $\tilde{\delta}\in\mathcal{D}^{\mathcal{S}}$, $\mathcal{D}^{n\mathcal{S}}$).

Let $\mathfrak{C}$ be the set of equivalence classes of unitary supercuspidal representations of $GL_n(E)$, $n=1,2,\dots$, if E is p-adic. If $E=\mathbb{R}$, let $\mathfrak{C}$ be the set of unitary characters of $GL_1(E)$. If $E=\mathbb{C}$, let $\mathfrak{C}$ be the set of unramified unitary characters of $GL_1(E)$ (i.e., those which factor through ν). Set $\mathfrak{C}^{\theta}=\mathcal{D}^{\theta}\cap\mathfrak{C}$.

A *segment* I (cf. [36]) consists of $\sigma\in\mathfrak{C}$ and $a,b\in\mathbb{R}$ such that $b-a\in\mathbb{Z}_{\geq 0}$ (or $b-a\in\mathbb{Z}$ if $E=\mathbb{C}$). It will be denoted by $I=[\sigma\nu^{a},\sigma\nu^{b}]$. For any segment I, we define $\delta(I)$ as follows. In the non-Archimedean case, it is the irreducible quotient of $\sigma\nu^{a}\times\cdots\times\sigma\nu^{b}$. If $E=\mathbb{C}$, it is the quasicharacter $z\mapsto(z/|z|)^{b-a}\sigma(z)\nu^{(a+b)/2}(z)$. If $E=\mathbb{R}$, it is either the quasicharacter $\sigma\nu^{a}$ if $b=a$ or the unique irreducible quotient of $\sigma\nu^{a}\times\sigma\,\mathrm{sgn}^{b-a+1}\nu^{b}$, where sgn denotes the signum character of R^*. In all cases, $\delta(I)$ is essentially square-integrable, and all essentially square-integrable, representations of $GL_n(E)$ are obtained this way. In particular,

$$\mathcal{D}=\left\{\delta(\sigma,m):\sigma\in\mathfrak{C},\ m\in\frac{1}{2}\mathbb{Z}_{\geq 0}\left(\text{or } m\in\frac{1}{2}\mathbb{Z}\text{ if } E=\mathbb{C}\right)\right\}, \tag{2.5}$$

where we set $\delta(\sigma,m)=\delta([\sigma\nu^{-m},\sigma\nu^{m}])$. It will sometimes be convenient to define $\delta(\sigma,-1/2)$ to be the trivial representation of $GL_0(E)=\{1\}$. Note that for $E=\mathbb{R}$, we have $\delta(\mathrm{sgn},m)\simeq\delta(\mathbf{1},m)$ for $m>0$.

We will say that $\delta, \delta' \in \mathcal{D}$ are *adjacent* if we can write $\delta = \delta(\sigma, m)$ and $\delta' = \delta(\sigma, m')$, where $\sigma \in \mathfrak{C}$ and $|m - m'| = 1/2$.

The following lemma follows from the recipe for L-functions of square-integrable representations given in [8, 24] in the p-adic case. In the Archimedean case, one notes that for any $\pi \in \mathcal{D}_2$, $L(s, \pi, \wedge^2) = L(s, \omega_\pi)$, where ω_π is the central character of π and for any $\chi \in \mathcal{D}_1$, $L(s, \chi, \Psi) = L(s, \chi|_{F^*})$.

Lemma 2.1. If $F = \mathbb{C}$, then $\mathcal{D}^\theta = \{1\}$. Suppose that $F \neq \mathbb{C}$ and let $\sigma \in \mathfrak{C}$. Then $\delta(\sigma, m)^* = \delta(\sigma^*, m)$ for any $m \in (1/2)\mathbb{Z}_{\geq 0}$ (or $m \in (1/2)\mathbb{Z}$ if $E = \mathbb{C}$). Thus, $\delta(\sigma, m) \in \mathcal{D}^\theta$ if and only if $\sigma \in \mathfrak{C}^\theta$. Moreover, if $\sigma \in \mathcal{D}^{\mathcal{S}}$ (resp., $\sigma \in \mathcal{D}^{n\mathcal{S}}$), then $\delta(\sigma, m) \in \mathcal{D}^{\mathcal{S}}$ (resp., $\delta(\sigma, m) \in \mathcal{D}^{n\mathcal{S}}$) precisely when $m \in \mathbb{Z}$. In particular, if $\delta, \delta' \in \mathcal{D}^\theta$ are adjacent, then $\delta \in \mathcal{D}^{\mathcal{S}}$ if and only if $\delta' \in \mathcal{D}^{n\mathcal{S}}$. □

2.3 R-groups

We let $\mathfrak{D}$ (resp., $\mathfrak{T}$) be the set of (equivalence classes of) generic discrete-series (resp., tempered) irreducible representations of all groups in $\mathcal{S}$.

The description of R-groups for classical groups is recorded in the following lemma (for which the assumption of genericity is not actually needed).

Lemma 2.2. Let $\delta_i \in \mathcal{D}$, $i = 1, \ldots, k$, and $\sigma \in \mathfrak{D}$. Let I be a set of representatives for the i's for which $\delta_i \rtimes \sigma$ is reducible up to the equivalence of the δ_i's. Then the intertwining algebra $\mathrm{End}(\delta_1 \times \cdots \times \delta_k \rtimes \sigma)$ is generated by R_i, $i \in I$, where R_i is the operator induced from the normalized intertwining operator on $\delta_i \rtimes \sigma$. The representation $\delta_1 \times \cdots \times \delta_k \rtimes \sigma$ decomposes into $2^{|I|}$ mutually inequivalent irreducible representations which are the joint eigenspaces of the R_i's, $i \in I$. □

This is a consequence of [7] for the odd orthogonal and symplectic cases and of [9] for the unitary cases. The nonsplit even orthogonal case follows the same pattern. In the even (nonconnected) orthogonal case, the statements are equivalent to the seemingly more complicated [10, Theorem 3.3]. (This is a simple exercise in Mackey's theory.) Alternatively, they follow from the results of [11].

Corollary 2.3. Let $\delta \in \mathcal{D}^\theta$ and $\tau \in \mathfrak{T}$. Suppose that τ is a constituent of $\pi = \delta_1 \times \cdots \times \delta_k \rtimes \sigma$ with $\delta_i \in \mathcal{D}$, $i = 1, \ldots, k$, and $\sigma \in \mathfrak{D}$. Then the following conditions are equivalent:

(1) $\delta \rtimes \tau$ is irreducible;
(2) the normalized intertwining operator of $\delta \rtimes \tau$ is a scalar;
(3) at least one of the following conditions holds:

(i) $\delta \rtimes \sigma$ is irreducible,

(ii) $\delta \simeq \delta_i$ for some i;

(4) $\delta \rtimes \tau'$ is irreducible for any irreducible constituent τ' of π;

(5) the numbers of irreducible constituents of π and of $\delta \rtimes \pi$ coincide;

(6) the irreducible constituents of $\delta \rtimes \pi$ are of the form $\delta \times \tau'$, where τ' is an irreducible constituent of π. □

Proof. Clearly, (4) implies (1); (4), (5), and (6) are equivalent; and (1) implies (2). Moreover, (3) and (5) are equivalent by Lemma 2.2. Finally, suppose that (2) holds. By Lemma 2.2, $\mathrm{End}(\delta \rtimes \pi)$ is generated by the image I of $\mathrm{End}(\pi)$ under induction and the operator N which is induced from the normalized intertwining operator of $\delta \rtimes \sigma$, if the latter is reducible. Clearly, I acts by scalars on $\delta \rtimes \tau'$ for any irreducible constituent τ' of π. It follows that $\mathrm{End}(\delta \rtimes \tau')$ is generated by the restriction of N to $\delta \rtimes \tau'$. Hence, $\delta \rtimes \tau'$ decomposes into one or two irreducible components, according to whether or not N acts as a scalar on $\delta \rtimes \tau'$. By our assumption, N acts as a scalar on $\delta \rtimes \tau$, and hence, $\delta \rtimes \tau$ is irreducible. Thus, $\delta \rtimes \pi$ decomposes into strictly less than twice the number of components π does. We conclude (5) by Lemma 2.2. ■

Corollary 2.4. Suppose that $\delta_1, \delta_2 \in \mathcal{D}^\theta$ with $\delta_1 \not\simeq \delta_2$ and $\tau \in \mathfrak{T}$. Assume that $\delta_1 \rtimes \tau'$ is irreducible for some constituent τ' of $\delta_2 \rtimes \tau$. Then $\delta_1 \rtimes \tau$ is irreducible. □

For $\delta \in \mathcal{D}$ and $\tau \in \mathfrak{T}$, we denote by $L(s, \pi \times \tau)$ the tensor product L-function as defined by Shahidi [23]. (In the $O(2n)$ case, we set $L(s, \delta \times \tau) = L(s, \delta \times \tau')$ for any constituent τ' of the restriction of τ to $SO(2n)$. This is well defined.)

Lemma 2.5. Suppose that $\delta \in \mathcal{D}$ and $\tau \in \mathfrak{D}$. Then the representation $\delta \rtimes \tau$ is reducible if and only if $\delta \in \mathcal{D}^{\mathcal{S}}$ and $L(0, \delta \times \tau) \neq \infty$. □

Proof. In the connected case, this follows from the description of the Plancherel measure in [23, Corollary 3.6]. For the split even orthogonal case, we argue as follows. Suppose that $\delta \in \mathcal{D}_m$ and $\tau \in \mathfrak{D}_n$. First consider the case $n = 0$. Then the restriction of $\delta \rtimes \tau$ to $SO(2m)$ has two constituents $I_P(\delta)$ and $I_{P'}(\delta)$, where P and P' are the two parabolic subgroups of the type GL_m. Thus, $\delta \rtimes \tau$ is irreducible if and only if $I_P(\delta)$ is irreducible and $I_P(\delta) \not\simeq I_{P'}(\delta)$. Now, $I_P(\delta)$ is reducible if and only if m is even and $\delta \in \mathcal{D}^{\mathcal{S}}$. If this is not the case, then $I_P(\delta) \simeq I_{P'}(\delta)$ if and only if m is odd and $\delta \in \mathcal{D}^\theta$. Thus, the case $n = 0$ holds. Suppose now that $n > 0$ and the restriction τ' of τ to $SO(2m)$ is irreducible. Then $\delta \rtimes \tau$ is irreducible if and only if its restriction $\delta \rtimes \tau'$ is irreducible, since the latter has exactly one irreducible generic constituent. On the other hand, $\delta \rtimes \tau'$ is reducible precisely when $\delta \in \mathcal{D}^{\mathcal{S}}$ and $L(0, \delta \times \tau') \neq \infty$. Finally, suppose that $\tau = \tau_1 \oplus \tau_2$, where $\tau_1 \not\simeq \tau_2$ and τ_2 is the twist of τ_1 by the outer involution of $SO(2m)$. Then $\delta \rtimes \tau$ is irreducible if and only if $\delta \rtimes \tau_1$ is irreducible and $\delta \rtimes \tau_2 \not\simeq \delta \rtimes \tau_1$. Now, $\delta \rtimes \tau_1$ is reducible if and only if m is even, $\delta \in \mathcal{D}^{\mathcal{S}}$, and $L(0, \delta \times \tau_1) \neq \infty$. On the other hand, if $\delta \rtimes \tau_1$ is irreducible, then $\delta \rtimes \tau_2 \simeq \delta \rtimes \tau_1$ if and only if m is odd and $\delta \in \mathcal{D}^{\theta}$. It remains to observe that if m is odd, then $L(0, \delta \times \tau_1) \neq \infty$, since the Plancherel measure for $\delta \rtimes \tau_1$ is not zero at 0 because the data is not self-associate. ■

2.4 Reducibility of standard modules

The following result is well known (see, e.g., [36] in the p-adic case, [15, 25] for $E = \mathbb{R}$, and [33] for $E = \mathbb{C}$).

Lemma 2.6. Let δ_1, δ_2 be essentially square-integrable representations of $GL_{n_i}(E)$. Suppose that $\eta = |e(\delta_1) - e(\delta_2)| < 2$. Then $\delta_1 \times \delta_2$ is irreducible unless either $\delta_1^u \simeq \delta_2^u$ and $\eta = 1$ or δ_1^u and δ_2^u are adjacent and $\eta = 3/2$. □

We now turn to classical groups. We will often use the following fact which follows from [18] in the p-adic case and from [12, 30] in the Archimedean case.

(Q) The Langlands quotient of a reducible standard module is not generic.

This will enable us to use [6, Propositions 5.3 and 5.4].

Lemma 2.7. Suppose that $\delta \in \mathcal{D}$ and $\tau \in \mathfrak{T}$. Then $\delta\nu^s \rtimes \tau$ is irreducible for $0 < s < 1$ except, possibly, at $s = 1/2$. Moreover, $\delta\nu^{1/2} \rtimes \tau$ is reducible if and only if $\delta \in \mathcal{D}^{n\mathcal{S}}$. □

Proof. By [6, Proposition 5.3], $\delta\nu^s \rtimes \tau$ is reducible at $s > 0$ if and only if

$$L\big(1 - s, \tilde{\delta} \otimes \tilde{\tau}\big) L\big(1 - 2s, \tilde{\delta}, r_2\big) = \infty. \tag{2.6}$$

The first factor is regular for $s < 1$ by [6, Theorem 4.1]. To analyze the second factor, we note that the computation of [8, 24] shows that $L(s, \tilde{\delta}, r_2)$ cannot have a pole for $-1 < s < 1$, except for $s = 0$ if $\delta \in \mathcal{D}^{n\mathcal{S}}$. The lemma follows. ■

We will say that a standard module π is in *general position* if $e(\delta_i) \notin (1/2)\mathbb{Z}$ for all i and $e(\delta_i) \pm e(\delta_j) \notin (1/2)\mathbb{Z}$ for all $i \neq j$.

The following lemma also follows from the reducibility criterion [6, Proposition 5.4] (or from [26]).

Lemma 2.8. Let π be a generic standard module of the form (1.2). Then π is irreducible if and only if both of the following conditions are satisfied:

(1) $\delta_i \rtimes \tau$ is irreducible for all i;

(2) $\delta_i \times \delta_j$ and $\delta_i^* \times \delta_j$ are irreducible for all i, j.

In particular, this holds if π is in general position. □

Finally, we will need the following lemma.

Lemma 2.9. If $\delta \in \mathcal{D}^{n\mathcal{S}}$ and $\delta\nu \rtimes \tau$ is reducible, then τ is fully induced. □

Proof. By (2.6),

$$L(0, \tilde{\delta} \times \tilde{\tau}) L(-1, \tilde{\delta}, r_2) = \infty. \tag{2.7}$$

From the description of $L(s, \delta, r_2)$ (see [8, 24]) and the fact that $L(0, \tilde{\delta}, r_2) = \infty$, it easily follows that $L(-1, \tilde{\delta}, r_2) \neq \infty$. Hence, $L(0, \tilde{\delta} \times \tilde{\tau}) = \infty$. Embed τ in $\delta_1 \times \cdots \times \delta_k \rtimes \tau'$, where $\delta_i \in \mathcal{D}$ and $\tau' \in \mathfrak{D}$. We have (see [6, 22])

$$L(s, \tilde{\delta} \times \tilde{\tau}) = L(s, \tilde{\delta} \times \tilde{\tau'}) \prod_i L(s, \tilde{\delta} \times \delta_i) L(s, \tilde{\delta} \times \delta_i^*). \tag{2.8}$$

We observe that $L(0, \tilde{\delta} \times \tilde{\tau'}) \neq \infty$, otherwise, this would contradict [18, Theorem 3.1] (which is still valid in the Archimedean case). Thus, either $L(0, \tilde{\delta} \times \delta_i) = \infty$ or $L(0, \tilde{\delta} \times \delta_i^*) = \infty$ for some i. In any case, this implies that $\delta \simeq \delta_i$, since $\delta \in \mathcal{D}^\theta$. Assume without loss of generality that $i = 1$. Applying Corollary 2.3 with $\delta = \delta_1$ and k replaced by $k - 1$, we infer that $\tau = \delta_1 \rtimes \tau_1$ for some constituent τ_1 of $\delta_2 \times \cdots \times \delta_k \rtimes \tau'$. ■

3 Unitarizability

Let $\mathcal{G}_n$ be the set of equivalence classes of representations of G_n which satisfy the conditions of Theorem 1.1. Set $\mathcal{G} = \bigcup \mathcal{G}_n$. Let $\mathcal{U}$ be the class of unitarizable, irreducible generic representations of G_n, $n = 0, 1, 2, \ldots$. The statement of Theorem 1.1 is that $\mathcal{G} = \mathcal{U}$. In this section, we will show that $\mathcal{G} \subset \mathcal{U}$.

Lemma 3.1. Any $\pi \in \mathcal{G}_n$ is irreducible. □

Proof. We will use Lemma 2.8. The irreducibility of $\delta_i \rtimes \tau$ follows from Lemma 2.7 since $e(\delta_i) < 1$ for all i. The irreducibility of $\delta_i \times \delta_j$ follows from Lemma 2.6 since $0 < e(\delta_i)$, $e(\delta_j) < 1$. Suppose on the contrary that $\delta_i \times \delta_j^*$ is reducible. Then by Lemma 2.6, either $\delta_i^u \simeq (\delta_j^u)^*$ and $e(\delta_i) + e(\delta_j) = 1$ or δ_i^u and δ_j^u are adjacent and $e(\delta_i) + e(\delta_j) = 3/2$. The first case contradicts condition (2) or (3a) of Theorem 1.1. In the second case, at most one of δ_i^u and δ_j^u is in $\mathcal{D}^{\mathcal{S}}$ by Lemma 2.1 so that one exponent is less than $1/2$ and the other is less than 1, in contradiction. ■

Lemma 3.2. If $\pi \in \mathcal{G}_n$ is not tempered, then π can be written as $\rho \rtimes \pi'$, where $\pi' \in \mathcal{G}_{n'}$, with $n' < n$, and ρ is of the form

(1) $\delta\nu^\alpha \times \delta^*\nu^\alpha$ with $0 < \alpha < 1/2$,

(2) $\delta\nu^\alpha$ with $\delta \in \mathcal{D}^{nS}$ and $0 < \alpha < 1/2$,

(3) $\delta\nu^\alpha$ with $\delta \in \mathcal{D}^S$, where $\delta \rtimes \tau$ is irreducible and $0 < \alpha < 1$ is the unique element of $\mathcal{E}_\pi(\delta)$, or

(4) $\delta\nu^\alpha \times \delta\nu^\beta$ with $\delta \in \mathcal{D}^S$, where $0 < \alpha \le \beta < 1 - \alpha$ and all elements of $\mathcal{E}_{\pi'}(\delta)$ lie outside the interval $[\alpha, 1 - \alpha]$. □

Proof. By the assumption, we have $\mathcal{E}_\pi(\delta) \neq \varnothing$ for some $\delta \in \mathcal{D}$. If $\delta \notin \mathcal{D}^\theta$, then conditions (1) and (2) of Theorem 1.1 imply that the first alternative of the lemma holds. If $\delta \in \mathcal{D}^{nS}$, then once again, condition (2) implies that the second alternative holds. Suppose now that $\delta \in \mathcal{D}^S$ and let $\alpha_1, \dots, \alpha_k$, $\beta_1 \dots, \beta_l$ be as in (1.3). If $k + l = 1$, then by (3d) the third alternative of the lemma holds with $\alpha = \alpha_1$. Suppose then that $k + l > 1$. If $l = 0$, then $k \ge 2$ and the fourth alternative of the lemma holds with $\alpha = \alpha_{k-1}$ and $\beta = \alpha_k$. Suppose that $l > 0$. Then by (3b), the number $\#\{1 \le i \le k : \alpha_i > 1 - \beta_1\}$ is even. If it is nonzero, then it is at least two, and the fourth alternative of the lemma holds with $\alpha = \alpha_{k-1}$, $\beta = \alpha_k$. Otherwise, $\alpha_k < 1 - \beta_1$ and the fourth alternative of the lemma holds with $\alpha = \alpha_k$ and $\beta = \beta_1$. The case $k = 0$ is excluded by (3c) and the assumption $k + l > 1$. ■

To prove unitarity, we will use the following well-known principle.

Lemma 3.3. Suppose that π_t, $0 \le t \le 1$, is given by

$$\pi_t = \delta_1\nu^{\alpha_1(t)} \times \cdots \times \delta_k\nu^{\alpha_k(t)} \rtimes \tau, \tag{3.1}$$

where $\alpha_i : [0, 1] \to \mathbb{R}$ are continuous, $\tau_i \in \mathcal{D}$, and $\tau \in \mathfrak{T}$. Suppose that π_t is irreducible and Hermitian for $0 \le t < 1$, and that π_0 is unitarizable. Then π_t is unitarizable for all $t \in [0, 1)$ and so is every subquotient of π_1. □

Proof. The Hermitian structure on π_t is given (up to a scalar) by the intertwining operator corresponding to the longest Weyl element (suitably normalized so that it is holomorphic, but still nonzero, at each point under consideration). Thus, for $0 \le t < 1$, we get a continuous family of nondegenerate Hermitian structures which is definite at 0 (on any K-type). Hence, it is definite for all $0 \le t < 1$, which implies that π_t is unitarizable. The last statement follows from a theorem of Miličić [14] (cf. [28]). ■

We will prove unitarity by induction on n, separating into the cases described in Lemma 3.2.

In the first case, we may write $\pi \simeq \delta\nu^{\alpha} \times \delta\nu^{-\alpha} \rtimes \pi'$ since π is irreducible. Hence, π is unitary.

In this second (resp., third) case, $\delta\nu^{\alpha} \rtimes \pi' \in \mathcal{G}_n$ (and in particular, is irreducible) for all $0 < \alpha < 1/2$ (resp., $0 < \alpha < 1$). By Lemma 2.8, Corollary 2.3, and Lemma 2.5, $\delta \rtimes \pi'$ is irreducible, and unitarity follows from Lemma 3.3.

Finally, in the fourth case, we consider $\pi_s = \delta'\nu^s \rtimes \pi'$, where $\delta' = \delta\nu^{(\alpha+\beta)/2} \times \delta\nu^{-(\alpha+\beta)/2}$. For $0 \le s \le s_0 = (\beta - \alpha)/2$, it is easy to see that

$$\delta\nu^{(\alpha+\beta)/2+s} \times \delta\nu^{(\alpha+\beta)/2-s} \rtimes \pi' \in \mathcal{G}_n \tag{3.2}$$

and hence, is irreducible and isomorphic to π_s. We may then appeal once again to Lemma 3.3 to conclude that $\pi_{s_0} = \pi$ is unitary.

We mention that, given Theorem 1.1, Corollary 1.2 also follows easily from Lemma 3.2 using induction on n.

4 Nonunitarity

We now turn to the converse inclusion $\mathcal{G} \supset \mathcal{U}$ of Theorem 1.1.

From now on, let π be a generic irreducible representation of G_n which is written in the form (1.2).

4.1 First reductions

Lemma 4.1. Suppose that $\pi \in \mathcal{U}$ is of the form $\pi = \delta\nu^{\alpha} \times \delta^*\nu^{\alpha} \rtimes \pi'$ for some $\delta \in \mathcal{D}$. Then $\alpha < 1/2$ and $\pi' \in \mathcal{U}$. Moreover, if $\pi \notin \mathcal{G}$, then $\pi' \notin \mathcal{G}$. □

Proof. Since π is irreducible, we can write $\pi \simeq \delta\nu^{\alpha} \times \delta\nu^{-\alpha} \rtimes \pi'$. Now, $\delta\nu^{\alpha} \times \delta\nu^{-\alpha} \otimes \pi'$ is a Hermitian representation of the corresponding Levi subgroup. By the principle of "unitary parabolic reduction" (see [28, page 234]), we obtain that $\delta\nu^{\alpha} \times \delta\nu^{-\alpha} \otimes \pi'$ is unitarizable and irreducible, that is, both $\delta\nu^{\alpha} \times \delta\nu^{-\alpha}$ and π' are unitarizable and irreducible. The first statement follows from the classification of the generic unitary dual of GL_n (cf. [4, 27] in the p-adic case, [25, 32, 33] for the Archimedean case). It is also simple to verify that if $\pi' \in \mathcal{G}$, then $\pi \in \mathcal{G}$. ■

Corollary 4.2. If $\pi' \in \mathcal{U} \setminus \mathcal{G}$, then there exists $\pi \in \mathcal{U} \setminus \mathcal{G}$ such that $\mathcal{E}_{\pi}(\delta) = \varnothing$ for any $\delta \notin \mathcal{D}^{\theta}$. □

Proof. Since π' is unitary, it is Hermitian, and hence, $\mathcal{E}_{\pi'}(\delta) = \mathcal{E}_{\pi'}(\delta^*)$ for all $\delta \in \mathcal{D}$. It follows from Lemma 4.1 that if $\delta \notin \mathcal{D}^{\theta}$, then all exponents in $\mathcal{E}_{\pi'}(\delta)$ are less than $1/2$ and δ can be eliminated. ■

Henceforth, we will always assume that π satisfies the conclusion of Corollary 4.2.

Next, note that for a standard module to be irreducible is an open condition on the parameters. On the set of irreducible standard modules, unitarity is an open condition. By Lemma 2.8, any standard module in general position is irreducible. The set of $\pi \in \mathcal{U}$ in general position forms an open dense subset $\mathcal{U}^0$ of $\mathcal{U}$. On the other hand, inside $\mathcal{U}^0$, the condition to be in $\mathcal{G}$ is a closed condition on the parameters. We conclude the following.

Lemma 4.3. Suppose that $\pi' \in \mathcal{U} \setminus \mathcal{G}$. Then there exists $\pi \in \mathcal{U}^0 \setminus \mathcal{G}$ "near" π'. [2] □

4.2 Further reduction

Lemma 4.4. Suppose that $\pi' \in \mathcal{U}^0$ and that $\mathcal{E}_{\pi'}(\delta) \subset (0, 1)$ for all $\delta \in \mathcal{D}$. Assume that $\pi' = \delta\nu^\alpha \times \delta\nu^\beta \rtimes \pi$, where $\delta \in \mathcal{D}^{\mathcal{S}}$, $0 < \alpha \leq \beta < 1/2$, and $\mathcal{E}_\pi(\delta) \cap [1 - \alpha, 1 - \beta] = \varnothing$. Then $\pi \in \mathcal{U}^0$. □

Proof. Set $\pi_s = \delta'\nu^s \rtimes \pi$, where $\delta' = \delta\nu^{(\alpha+\beta)/2} \times \delta\nu^{-(\alpha+\beta)/2}$. It is easy to see from Lemma 2.8, Lemma 2.6, and our conditions on the exponents that

$$\delta\nu^{(\alpha+\beta)/2+s} \times \delta\nu^{(\alpha+\beta)/2-s} \rtimes \pi \tag{4.1}$$

is irreducible for $0 \leq s \leq s_0 = (\beta - \alpha)/2$, and hence, it is isomorphic to π_s. Since $\pi_{s_0} \simeq \pi' \in \mathcal{U}$ by assumption, we infer from Lemma 3.3 that $\pi_0 \in \mathcal{U}$. By appealing to Lemma 4.1, we deduce that $\pi \in \mathcal{U}$. ■

Lemma 4.5. Suppose that $\pi \in \mathcal{U}^0 \setminus \mathcal{G}$ and that $\mathcal{E}_\pi(\delta) \subset (0, 1)$ for all $\delta \in \mathcal{D}^\theta$. Then condition (2) of Theorem 1.1 is violated for some $\delta \in \mathcal{D}^{n\mathcal{S}}$. □

Proof. Suppose that this is not the case. By Corollary 4.2, it suffices to prove that the conditions (3a), (3b), (3c), and (3d) are satisfied for each $\delta \in \mathcal{D}^{\mathcal{S}}$. Condition (3a) follows since $\pi \in \mathcal{U}^0$. Let $\mathcal{E}_\pi(\delta) = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l\}$ with $0 < \alpha_1 < \cdots < \alpha_k < 1/2 < \beta_1 < \cdots < \beta_l < 1$. By applying Lemma 4.4 repeatedly, we can assume that each of the intervals $I_j = (1 - \beta_{j+1}, 1 - \beta_j)$, $j = 0, \ldots, l$, (where we set $\beta_0 = 1/2$, $\beta_{l+1} = 1$) contains at most one element of $\mathcal{E}_\pi(\delta)$. We need to show that $I_0 \cap \mathcal{E}_\pi(\delta) = \varnothing$, $I_j \cap \mathcal{E}_\pi(\delta) \neq \varnothing$ for $j = 1, \ldots, l-1$, and that if $I_l \cap \mathcal{E}_\pi(\delta) \neq \varnothing$, then $\delta \rtimes \tau$ is irreducible. Suppose first that $I_0 \cap \mathcal{E}_\pi(\delta) = \{\alpha\}$. Then by Lemma 3.3 we can deform α to β_1 (without hitting a reducibility point), and this contradicts Lemma 4.1. Similarly, if $I_j \cap \mathcal{E}_\pi(\delta) = \varnothing$ for $1 \leq j < l$, then we can deform

[2] By "near" we mean that we slightly perturb the exponents of $\delta_1, \ldots, \delta_k$ in (1.2).

β_j to β_{j+1}, and again, this contradicts Lemma 4.1. Finally, suppose that $I_l \cap \mathcal{E}_\pi(\delta) = \{\alpha\}$. We can deform β_1 to α_k and use once again Lemma 4.1. Repeating this procedure, we may assume without loss of generality that $k = 1$ and $l = 0$. Since by our assumption the conditions (2) are satisfied, we can also get rid of any pairs of exponents of $\mathcal{E}_\pi(\delta)$ with $\delta \in \mathcal{D}^{n\mathcal{S}}$. Thus, we can assume that $\mathcal{E}_\pi(\delta)$ is at most a singleton for all $\delta \in \mathcal{D}^\theta$. It remains to prove that $\delta \rtimes \tau$ is irreducible whenever $\mathcal{E}_\pi(\delta) \neq \varnothing$. Fix such δ. We can deform the exponent of any $\delta' \neq \delta$ to 0. This will change τ, but not the irreducibility of $\delta \rtimes \tau$, by Corollary 2.4. Thus, we may assume that $\mathcal{E}_\pi(\delta) \neq \varnothing$ for all $\delta' \neq \delta$. Now we have a representation of the form $\delta\nu^\alpha \rtimes \tau$, $\delta \in \mathcal{D}^\theta$, $\tau \in \mathfrak{T}$, which is unitarizable for some $0 < \alpha < 1/2$. The Hermitian structure is given by the normalized intertwining operator. Hence, the latter is a scalar at $\alpha = 0$. By Corollary 2.3, $\delta \rtimes \tau$ is irreducible as required. ■

We call $\pi \in \mathcal{U}$ "bad" if either of the following conditions holds:

(1) there exists $\delta \in \mathcal{D}^\mathcal{S}$ such that $\mathcal{E}_\pi(\delta) \not\subset [0, 1]$,

(2) there exists $\delta \in \mathcal{D}^{n\mathcal{S}}$ such that $\mathcal{E}_\pi(\delta) \not\subset [0, 1/2]$.

Clearly, a bad π is not in $\mathcal{G}$. Conversely, by Lemma 4.5, if $\mathcal{G} \subsetneq \mathcal{U}$, then there exists a bad $\pi \in \mathcal{U}^0$.

4.3 Final reduction

To state the main reduction step, we introduce some more notation. We will assume in this subsection that $F \neq \mathbb{C}$ (until Section 4.4). Let $\sigma \in \mathfrak{C}^\theta$. (If $E = \mathbb{R}$, then $\sigma = \mathbf{1}$ or sgn; if $E = \mathbb{C}$, then $\sigma = \mathbf{1}$.) For π of the form (1.2), we denote by $\mathcal{E}_\pi([\sigma])$ the multiset of pairs $(m_i, e(\delta_i)) \subset (1/2)\mathbb{Z} \times \mathbb{R}$ for those δ_i of the form $\delta_i = \delta(\sigma_i, m_i)$.

Lemma 4.6. Let $\pi' \in \mathcal{U}^0$. Suppose that $(m, \alpha) \in \mathcal{E}_\pi([\sigma])$ with $\alpha > 1/2$. Let k be the half-integer between α and $\alpha + 1/2$. Then there exists $\pi \in \mathcal{U}$ such that

$$\mathcal{E}_\pi([\sigma]) = \mathcal{E}_{\pi'}([\sigma]) \setminus \{(m, \alpha)\}$$
$$\cup \begin{cases} \left\{\left(m - \frac{1}{2}, \alpha - \frac{1}{2}\right), (m + k - 1, k - \alpha), \left(m - \frac{1}{2} + k, \alpha + \frac{1}{2} - k\right)\right\}, & m > 0, \\ \left\{(k - 1, \alpha + 1 - k), \left(k - \frac{3}{2}, \alpha + \frac{1}{2} - k\right)\right\}, & m = 0, \\ \left\{\left(m + \frac{1}{2}, \alpha - \frac{1}{2}\right), (m - k + 1, k - \alpha), \left(m + \frac{1}{2} - k, \alpha + \frac{1}{2} - k\right)\right\}, & m < 0. \end{cases} \tag{4.2}$$

□

Proof. Write $\pi' = \delta([\sigma\nu^{\alpha-m}, \sigma\nu^{\alpha+m}]) \rtimes \pi_1$. If $m > 0$, then the representation

$$\pi_2 = \delta\big(\big[\sigma\nu^{1+\alpha-m-2k}, \sigma\nu^{\alpha+m-1}\big]\big) \times \delta\big(\big[\sigma\nu^{1-\alpha-m}, \sigma\nu^{m+2k-\alpha-1}\big]\big) \rtimes \pi' \tag{4.3}$$

obtained by "multiplying" π' by a complementary series of $GL_n(E)$ is unitarizable. It is well known that

$$\delta\big(\big[\sigma\nu^{1+\alpha-m-2k}, \sigma\nu^{\alpha+m-1}\big]\big) \times \delta\big(\big[\sigma\nu^{\alpha-m}, \sigma\nu^{\alpha+m}\big]\big) \tag{4.4}$$

contains the generic irreducible representation

$$\delta\big(\big[\sigma\nu^{1+\alpha-m-2k}, \sigma\nu^{\alpha+m}\big]\big) \times \delta\big(\big[\sigma\nu^{\alpha-m}, \sigma\nu^{\alpha+m-1}\big]\big)\big\{\times \sigma\,\mathrm{sgn}\,\nu^{\alpha-1/2}\big\} \tag{4.5}$$

as a subquotient (see, e.g., [36] for the p-adic case, [25] for $E = \mathbb{R}$, and [34, Section 5.7] for $F = \mathbb{C}$). The factor inside the $\{\cdot\}$ appears only if $m = 1/2$ and $E = \mathbb{R}$. Hence, π_2 contains the generic subquotient

$$\begin{aligned}\pi = {} & \delta\big(\big[\sigma\nu^{1+\alpha-m-2k}, \sigma\nu^{\alpha+m}\big]\big) \times \delta\big(\big[\sigma\nu^{\alpha-m}, \sigma\nu^{\alpha+m-1}\big]\big)\big\{\times \sigma\,\mathrm{sgn}\,\nu^{\alpha-1/2}\big\} \\ & \times \delta\big(\big[\sigma\nu^{1-\alpha-m}, \sigma\nu^{m+2k-\alpha-1}\big]\big) \rtimes \pi_1.\end{aligned} \tag{4.6}$$

By Lemma 2.8, π is irreducible. Similarly, if $m = 0$, we multiply π' by

$$\delta\big(\big[\sigma\nu^{\alpha-2k+2}, \sigma\nu^{\alpha-1}\big]\big) \times \delta\big(\big[\sigma\nu^{1-\alpha}, \sigma\nu^{2k-\alpha-2}\big]\big) \tag{4.7}$$

and we get $\delta([\sigma\nu^{\alpha-2k+2}, \sigma\nu^{\alpha}]) \times \delta([\sigma\nu^{1-\alpha}, \sigma\nu^{2k-\alpha-2}]) \rtimes \pi_1$ as the irreducible generic quotient. The case $m < 0$ (for $E = \mathbb{C}$) is similar. ■

Lemma 4.7. Suppose that $\pi \in \mathcal{U} \setminus \mathcal{G}$. Then there exists $\pi' \in \mathcal{U}^0 \setminus \mathcal{G}$ and $(m, \alpha) \in \mathcal{E}_{\pi'}([\sigma])$ such that $1/2 < \alpha < 3/2$, $\delta(\sigma, m) \in \mathcal{D}^{n\mathcal{S}}$, and all other exponents in $\bigcup_\delta \mathcal{E}_{\pi'}(\delta)$ are in $(0, 1/2)$. □

Proof. By the above, we may assume that π is bad. Thus, there exists $\sigma \in \mathfrak{C}^\theta$ and $(m, \alpha) \in \mathcal{E}_\pi([\sigma])$ such that $\alpha > 1/2$, and moreover $\alpha > 1$ if $\delta(\sigma, m) \in \mathcal{D}^\mathcal{S}$. Applying Lemma 4.6 (several times if necessary), we may assume that $1/2 < \alpha < 3/2$. If $\delta(\sigma, m) \in \mathcal{D}^\mathcal{S}$ (in which case $\alpha > 1$), we apply Lemma 4.6 once more to obtain $\pi' \in \mathcal{U}$ with $(m - 1/2, \alpha - 1/2) \in \mathcal{E}_{\pi'}([\sigma])$ (or $(1/2, \alpha - 1/2)$ if $m = 0$). Thus, by Lemma 2.1, we may assume that $\delta(\sigma, m) \in \mathcal{D}^{n\mathcal{S}}$. To obtain the last property, we continue to apply Lemma 4.6 repeatedly. Finally, by a small perturbation of the parameters, we may always assume, in addition, that $\pi' \in \mathcal{U}^0$. ■

Lemma 4.8. Under the same assumption, there exists $\pi' \in \mathcal{U}^0 \setminus \mathcal{G}$ and $(m, \alpha) \in \mathcal{E}_{\pi'}([\sigma])$ such that $1/2 < \alpha < 1$, $\delta(\sigma, m) \in \mathcal{D}^{nS}$, and all other exponents in $\bigcup_\delta \mathcal{E}_{\pi'}(\delta)$ are in $(0, 1/2)$. $\square$

Proof. By Lemma 4.7, we only need to consider the case $1 < \alpha < 3/2$. Set $\delta = \delta(\sigma, m)$ and let δ_i, $i = 1, \dots k$, be the elements of $\mathcal{D}^\theta$ adjacent to δ. (Thus, $k \leq 3$, with an equality if $m = 1/2$ and $E = \mathbb{R}$.) Note that no two δ_i's are adjacent. Using Lemma 3.3, we can deform any exponent of δ in the interval $(0, \alpha - 1)$ to 0. Similarly, we can eliminate any exponent of δ_i in the interval $(0, 3/2 - \alpha)$. Moreover, the proof of Lemma 4.4 shows that we can eliminate any pair of exponents of δ_i in $(3/2 - \alpha, 1/2)$. Thus, we may assume that each of $\mathcal{E}_\pi(\delta_i)$ is at most a singleton $\{\beta_i\}$ and that $\beta_i \in (3/2 - \alpha, 1/2)$ if it exists. Finally, by using parabolic reduction as in the proof of Lemma 4.1, we can assume that τ is not fully induced.

We write

$$\pi = \mathsf{X}_{j \in J} \delta_j \nu^{\beta_j} \times \delta(\sigma, m) \nu^\alpha \rtimes \pi_1, \tag{4.8}$$

where $J \subset \{1, \dots, k\}$ and π_1 is such that $\mathcal{E}_{\pi_1}(\delta_i) = \varnothing$ for any $i = 1, \dots, k$ and $\mathcal{E}_{\pi_1}(\delta) \subset (\alpha - 1, 1/2)$. Lemmas 2.8, 2.6, and 2.9 imply that the representation

$$\pi_t = \mathsf{X}_{j \in J} \delta_j \nu^{\beta_j + t} \times \delta(\sigma, m) \nu^{\alpha - t} \rtimes \pi_1 \tag{4.9}$$

is irreducible for $t \leq 1/2$. Thus, by Lemma 3.3, $\pi' = \pi_{1/2} \in \mathcal{U}$, and we have $\alpha - 1/2 \in \mathcal{E}_{\pi'}([\sigma])$. It remains to apply Lemma 4.6 to π' and each $\delta_j \nu^{\beta_j + 1/2}$, $j \in J$. $\blacksquare$

Proposition 4.9. Suppose that $\pi \in \mathcal{U} \setminus \mathcal{G}$. Then there exists $\pi' \in \mathcal{U}$ of the form $\delta\nu^\alpha \rtimes \tau$, where $\tau \in \mathfrak{T}$, $1/2 < \alpha < 1$, and $\delta \in \mathcal{D}^{nS}$. $\square$

Proof. By Lemma 4.8, we can assume that there is a unique $(m, \alpha) \in \mathcal{E}_\pi([\sigma])$ with $\alpha > 1/2$ and that $\alpha < 1$ and $\delta(\sigma, m) \in \mathcal{D}^{nS}$. By the same argument as before, we can eliminate all $\delta(\sigma, m')$ with $m' \neq m$ by deforming their exponents to 0. We can also eliminate any exponent of $\delta(\sigma, m)$ in $(0, 1 - \alpha)$ by deforming it to 0, and finally eliminate any pair of exponents in $(1 - \alpha, 1/2)$ by deforming one to the other. Thus, we are reduced to the case where $\mathcal{E}_\pi([\sigma]) \setminus \{(m, \alpha)\}$ consists of (at most) a single element (m, β) with $1 - \alpha < \beta < 1/2$. If no such β exists, we are done. Otherwise, we may assume, again by deforming, that α is very close to $1/2$. (This will change β of course.) We now deform β to $1/2$ and consider the generic subquotient π' of $\delta\nu^{1/2} \rtimes \tau$. Since $\delta\nu^{1/2} \rtimes \tau$ is reducible, π' is not the Langlands quotient (Q). By [5, Proposition IV.4.13 and Lemma XI.2.13], the Langlands parameter of π' is strictly smaller than that of $\delta\nu^{1/2} \rtimes \tau$ with respect to the ordering defined by the roots. It follows that the exponents of π' are less than or equal to $1/2$.[3] Moreover, if $\mathcal{E}_{\pi'}(\delta) \neq \varnothing$, then $\mathcal{E}_{\pi'}(\delta') = \varnothing$ for any $\delta' \neq \delta$ and $\mathcal{E}_{\pi'}(\delta) = \{\gamma\}$ with $\gamma < 1/2$. Since π' is an irreducible standard module (again by (Q)), we can, as before, deform the exponents to 0. (If $\mathcal{E}_{\pi'}(\delta) \neq \varnothing$, we need to choose α so that $\alpha < 1-\gamma$.) We remain (possibly after changing τ) with a representation of the required form. ■

4.4 The case $F = \mathbb{C}$

Assume now that $F = \mathbb{C}$. Recall that $\mathcal{D}^\theta = \{\mathbf{1}\}$. We claim that Proposition 4.9 still holds in this case. We first claim that $\mathcal{E}_\pi(\mathbf{1}) \subset (0,1)$ for any $\pi \in \mathcal{U}$. Indeed, suppose that $\alpha \in \mathcal{E}_\pi(\mathbf{1})$ with $\alpha > 1/2$ and let $k \geq 1$ be the nearest integer to α. We multiply π by the complementary series $\nu^{\alpha-k} \times \nu^{k-\alpha}$ and note that the generic subquotient of $\nu^{\alpha-k} \times \nu^{k-\alpha} \times \nu^\alpha$ is $\delta(\mathbf{1}, -k/2)\nu^{\alpha-k/2} \times \delta(\mathbf{1}, -k/2)\nu^{\alpha-k/2} \times \nu^{k-\alpha}$. Thus, as in Lemma 4.6, we can replace ν^α by $\delta(\mathbf{1}, -k/2)\nu^{\alpha-k/2} \times \delta(\mathbf{1}, -k/2)\nu^{\alpha-k/2} \times \nu^{k-\alpha}$. We may replace $\delta(\mathbf{1}, -k/2)\nu^{\alpha-k/2}$ by its contragredient (i.e., inverse) $\delta(\mathbf{1}, k/2)\nu^{k/2-\alpha}$ which is the Hermitian dual of $\delta(\mathbf{1}, -k/2)\nu^{\alpha-k/2}$. Thus, by parabolic reduction, $\delta(\mathbf{1}, -k/2)\nu^{\alpha-k/2} \times \delta(\mathbf{1}, -k/2)\nu^{k/2-\alpha}$ is a unitary representation of $GL_2(\mathbb{C})$, which implies that $|\alpha - k/2| < 1/2$. This is possible only if $\alpha < 1$ as required. Also, this way we can eliminate any exponent greater than $1/2$. If $\mathbf{1} \in \mathcal{D}^{\mathcal{S}}$, then we may appeal to Lemma 4.5. If $\mathbf{1} \in \mathcal{D}^{n\mathcal{S}}$ (i.e., $G_n = SO(2n+1, \mathbb{C})$), we conclude Proposition 4.9 as follows. First, we eliminate all exponents greater than $1/2$ except for a single one—call it α_0. We can deform all exponents less than $1 - \alpha_0$ to 0. We can deform any two exponents in $(1 - \alpha_0, 1/2)$ to each other and use Lemma 4.4 to get rid of them. Thus, we remain with only one possible exponent β except α_0, which lies in $(1 - \alpha_0, 1/2)$. We can also assume that τ is not induced, that is, $\tau = \mathbf{1}$. Deforming β to $1/2$ and looking at the tempered subquotient of $\nu^{1/2} \rtimes \mathbf{1}$ (of $SO(3, \mathbb{C})$), we will obtain Proposition 4.9.

4.5 Conclusion of proof

The proof of Theorem 1.1 will be completed once we show the following proposition.

Proposition 4.10. Suppose that $\delta \in \mathcal{D}^{n\mathcal{S}}$ and $\tau \in \mathfrak{T}$. Then, for $1/2 < \alpha < 1$, $\delta\nu^\alpha \rtimes \tau$ is not unitary. □

The rest of the section is devoted to the proof of Proposition 4.10. In the case of $O(2n)$, we will work instead with $SO(2n)$ and replace τ by any irreducible constituent of its restriction to $SO(2n)$. Note that in this case $\delta \in \mathcal{D}_m$ with m even.

[3] In fact, one can show that π' is tempered.

We identify the spaces $\delta\nu^s \rtimes \tau$ with $\mathcal{H} = \delta \rtimes \tau$ as K-representations with the usual inner product, where K is a maximal compact subgroup of G. Let

$$R(s) = R(\delta \otimes \tau, s) : \delta\nu^s \rtimes \tau \longrightarrow \delta\nu^{-s} \rtimes \tau \tag{4.10}$$

be the normalized intertwining operator defined using Shahidi's normalization [23], that is,

$$R(s) = \frac{L(s+1, \delta \otimes \tau)\varepsilon(s, \delta \otimes \tau, \psi)}{L(s, \delta \otimes \tau)} \frac{L\big(2s+1, \delta, r_2\big)\varepsilon\big(2s, \delta, r_2, \psi\big)}{L\big(2s, \delta, r_2\big)} M(s), \tag{4.11}$$

where $M(s)$ is the unnormalized intertwining operator. For $\mathrm{Re}(s) > 0$, the operator $R(s)$ is holomorphic and nonzero [6]. Also, the operator $R(s)$ is Hermitian for $s \in \mathbb{R}$.

Let $\lambda_\chi(\varphi, s)$ be the Jacquet integral which defines a Whittaker functional for $\delta\nu^s \rtimes \tau$. By [21] (p-adic case) and [35, Section 15.6] (Archimedean case), it is an entire function. Recall the functional equation

$$\lambda_\chi(s, \varphi) = C_\chi(\delta \otimes \tau, s)\lambda_\chi\big(M(s)\varphi, -s\big), \tag{4.12}$$

where $C_\chi(\delta \otimes \tau, s)$ is the local coefficient defined by Shahidi and given by (see [23])

$$C_\chi(\delta \otimes \tau, s) = \frac{L(1-s, \delta \otimes \tau)\varepsilon(s, \delta \otimes \tau, \psi)}{L(s, \delta \otimes \tau)} \frac{L\big(1-2s, \delta, r_2\big)\varepsilon\big(2s, \delta, r_2, \psi\big)}{L\big(2s, \delta, r_2\big)} \tag{4.13}$$

(up to an immaterial factor). It follows that

$$\lambda_\chi(s, \varphi) = D_\chi(\delta \otimes \tau, s)\lambda_\chi\big(R(s)\varphi, -s\big), \tag{4.14}$$

where

$$D_\chi(\delta \otimes \tau, s) = \frac{L(1-s, \delta \otimes \tau)}{L\big(1+s, \delta \otimes \tau\big)} \frac{L\big(1-2s, \delta, r_2\big)}{L\big(1+2s, \delta, r_2\big)}. \tag{4.15}$$

In particular, $D_\chi(\delta \otimes \tau, s)$ has a simple pole at $s = 1/2$ since $\delta \in \mathcal{D}^{nS}$.

We fix our attention on a given K-type κ, that is, we consider the κ-isotypic part $\mathcal{H}^\kappa$ of $\mathcal{H}$. Let $R(s) = R(1/2) + (s - 1/2) \times R'(1/2) + \cdots$ be the Taylor expansion of $R(s)$ near $1/2$ (operating on $\mathcal{H}^\kappa$). Note that the image of $R(1/2)$ is the Langlands quotient so that, in particular, $\operatorname{Ker} R(1/2) \neq 0$ (for an appropriate κ). Let P be the orthogonal projection in $\mathcal{H}$ onto $\operatorname{Ker} R(1/2)$.

Lemma 4.11. The operator $PR'(1/2)$ is nonzero on $\operatorname{Ker} R(1/2)$. □

Proof. By (Q), $\operatorname{Ker} R(1/2)$ is generic. It follows that we can choose $v \in \operatorname{Ker} R(1/2)$ such that $\lambda_\chi(1/2, v) \neq 0$. The relation (4.14) yields that $\lambda_\chi(R'(1/2)v, -1/2) \neq 0$. On the other hand, $\lambda_\chi(\cdot, -1/2)$ is trivial on $\operatorname{Im} R(1/2)$, because, once again by (Q), the Langlands quotient of $\delta\nu^{1/2} \rtimes \tau$ is not generic. It follows that $R'(1/2)v \notin \operatorname{Im} R(1/2)$. However, $\operatorname{Im} R(1/2) = (\operatorname{Ker} R(1/2))^\perp = \operatorname{Ker} P$ since $R(1/2)$ is Hermitian. Thus, $PR'(1/2)v \neq 0$ and the lemma follows. ■

We now use a very special case of an argument of Vogan (cf. [31]). Since $\delta\nu^s \rtimes \tau$ is unitary for $0 < s < 1/2$, $\epsilon R(s)$ is definite for $0 < s < 1/2$, where $\epsilon = \pm 1$. (In fact, $\epsilon = 1$, cf. [13], but this is unimportant for us.) We infer that $\epsilon R(1/2)$ is positive definite and that $\epsilon R'(1/2)$ is negative semidefinite on $\operatorname{Ker} R(1/2)$ or, more precisely, $\epsilon PR'(1/2)P$ is negative semidefinite. Clearly, if $v \notin \operatorname{Ker} R(1/2)$, then $\epsilon(R(1/2)v, v) > 0$, and hence, $\epsilon(R(s)v, v) > 0$ for s near $1/2$. On the other hand, let $v \in \operatorname{Ker} R(1/2)$ be such that $PR'(1/2)Pv = PR'(1/2)v \neq 0$. This is possible by Lemma 4.11. Then we have $\epsilon(R'(1/2)v, v) = \epsilon(PR'(1/2)Pv, v) < 0$. Thus, $\epsilon(R(s)v, v) < 0$ for s slightly bigger than $1/2$. We infer that $R(s)$ cannot be definite for $s = 1/2 + \eta$ for a small $\eta > 0$. Thus, $\delta\nu^s \rtimes \tau$ is not unitarizable.

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