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ON REDUCIBILITY OF PARABOLIC INDUCTION

Marko Tadić

Introduction

Reducibility of parabolically induced representations plays an important role in a number of problems of representation theory of reductive groups (among others, in classifying of irreducible square integrable, tempered and unitary representations). If a parabolically induced representation of a reductive p-adic group reduces, then all Jacquet modules reduces. They reduce in a way compatible with the transitivity of Jacquet modules. Using this simple observation, one gets a possibility of proving irreducibility of parabolically induced representations. To be able to apply this approach to irreducibility of the parabolic induction, one needs to have an information about Jacquet modules of parabolically induced representations. A general result and formula about their composition series is provided by a result of J. Bernstein and A.V. Zelevinsky, and W. Casselman.

Classical groups are particularly convenient for application of this method, since we have a rather good information about part of the representation theory of their Levi subgroups. Namely, general linear groups are factors of their Levi subgroups. This enables us to apply the Bernstein-Zelevinsky theory to representations of Levi subgroups.

In this paper, we apply the above approach to the problem of determining reducibility of parabolically induced representations of $Sp(n, F)$ and $SO(2n + 1, F)$ (F is a non-archimedean local field, char $F \neq 2$). Also we show how to identify the irreducible subquotients. We show how reducibility of certain generalized principal series (and some other interesting parabolically induced representations) can be reduced to the reducibility in the cuspidal case. When the cuspidal reducibility is known, we get explicit answers (see the end of the introduction for an account of these explicit results, as well as the eleventh section; if the representations are supported in the minimal parabolic subgroups, then the cuspidal reducibility is well-known rank one reducibility, which have been known for decades).

A very satisfactory theory of reducibility for general linear groups was created by Bernstein and Zelevinsky ([Z]). A number of cuspidal reducibility for other classical groups have been determined recently by F. Shahidi. Our paper is not directed to cuspidal reducibility (although in the tenth section it is shown how one can get them in some simple situations, which include some new cases).

A method for determining reducibility based on Jacquet modules has been already applied in a number of papers ([T4], [J1], [SaT], [J2], [J3] among others). The problem with this method is that there exist points when the method, in its simplest form, can not decide the reducibility. There are very few such points, but they exist. I shall call them delicate cases (one can give them precise definition, but we shall not do that in this paper). In this paper we show how one can also use the method in such situations.

We shall denote by S_n either the group $Sp(n, F)$ or $SO(2n+1, F)$. Take a maximal parabolic subgroup $P = MN$ of S_n. Then the Levi factor M is isomorphic to $GL(k, F) \times S_{n-k}$. An irreducible admissible representation π of M can be decomposed as $\tau \otimes \sigma$. By $\tau \rtimes \sigma$ we denote the parabolically induced representation from $\tau \otimes \sigma$. We consider in this paper reducibility of $\tau \rtimes \sigma$ when τ is any twist by a (not necessarily unitary) character of a generalized Steinberg representation of a general linear group and σ is an irreducible cuspidal representation, or conversely. We are also interested in the situation when τ or σ are representations which have the opposite asymptotic properties of generalized Steinberg representations (in the case of general linear groups, these are the segment representations of Zelevinsky).

To get explicit information from the Bernstein-Zelevinsky and Casselman result about composition series of Jacquet modules of given parabolically induced representation(s), requires certain calculation (mainly in the Weyl groups). These calculations were done in [T6] for classical groups $Sp(n)$ and $SO(2n+1)$. There we have constructed a structure which provides us with a simple combinatorial algorithm for calculation of these composition series. To avoid repetition of the calculation done in [T6], we apply that structure for calculation of composition series.

The first section recalls briefly the notation and results regarding general linear groups that we use in this paper. In the second section we present the notation for groups $Sp(n, F)$ and $SO(2n+1, F)$. Third section gives simple criteria for determining reducibility and irreducibility of parabolically induced representations. These criteria apply to any connected reductive group over F. They are very simple. Therefore, we did not consider necessary to state them explicit in the first version of this paper (preprint "On reducibility of parabolic induction" in Mathematica Goettingensis, no. 19, 1993). In the fourth section we deal with reducibility of $\tau \rtimes \sigma$ when σ is cuspidal and when we are in the unitary situation (what means that τ is an irreducible square integrable representation of a general linear group). We consider in this section the case when involved cuspidal representations have generic reducibilities (we shall say shorter, in the case of generic cuspidal reducibilities). For the definition of generic cuspidal reducibility see the beginning of the fourth section. In this situation there are no delicate cases. These results give alternative proofs of some implications of Shahidi's paper [Sh2], from cuspidal reducibilities to square integrable reducibilities. They also give some new cases not covered by Shahidi's results, and an alternative proof that the duality in the cuspidal case implies the duality in the square integrable case ([Sh2]). Shahidi's proof is based on analysis of L-functions. Using the results of the tenth section, we can get new reducibility results in positive characteristic.

The fifth section treats one delicate case when τ is cuspidal. The first case when such a situation occurs is $Sp(2, F)$ (the representation is unramified). This case was settled by F. Rodier using Macdonald's explicit formulas for zonal spherical functions, and also by C. Jantzen using the Hecke algebra method. Both methods are based on the fact that one is dealing with a very simple and well understood inducing representation. Our method is based on the type of cuspidal reducibility, and therefore applies everywhere where we have this type of cuspidal reducibility. In the seventh and the eighth sections we treat our most general cases of reducibility of $\tau \rtimes \sigma$ when τ is cuspidal, in the setting of generic cuspidal reducibilities. In the seventh section there is a situation when we need the delicate case which was treated in the fifth section (no new delicate cases appear here). We also give a complete description of Langlands parameters of the irreducible subquotients. The sixth section settles one delicate case when σ is cuspidal. In the ninth section we treat our most general case when σ is cuspidal (we do not have new delicate cases here).

In the tenth section we show how to treat some simple cuspidal reducibilities.

The eleventh section is the most interesting one, particularly if one wants to see applications and the power of the method that we have developed in this paper. We write down some of the most interesting concrete consequences of the general results that we proved in preceding sections. Theorems 11.1 and 11.2 describe reducibility points of the degenerate principal series and generalized principal representations $\chi \rtimes 1_{Sp(n,F)}$, $\chi \rtimes \mathrm{St}_{Sp(n,F)}$, $\chi \rtimes 1_{SO(2n+1,F)}$ and $\chi \rtimes \mathrm{St}_{SO(2n+1,F)}$ when χ is any character of $F^\times$ (Langlands parameters of irreducible subquotients are obtained in the seventh and eighth sections). To give an idea of these reducibility results, we shall recall here the reducibility points from Theorem 11.1 for the first two representations: we have reducibility of $\chi \rtimes 1_{Sp(n,F)}$ (or $\chi \rtimes \mathrm{St}_{Sp(n,F)}$) if and only if $\chi^2 = 1_{F^\times}$ or $\chi = \nu^{\pm(n+1)} 1_{F^\times}$ (see the first two sections for notation). Theorem 11.2 contains a similar description of reducibilities for the other two representations. Further, in Theorems 11.3 and 11.4 we describe the reducibility points of the degenerate principal series and generalized principal series representations $\chi 1_{GL(n,F)} \rtimes 1$ and $\chi \mathrm{St}_{GL(n,F)} \rtimes 1$, both of $Sp(n,F)$ and $SO(2n+1,F)$ (for $SO(2n+1,F)$ we assume char $F = 0$, because we use one result of D. Goldberg). At this point, let us note that some of the reducibilities of the degenerate principal series were settled before this paper. The case of $\chi 1_{GL(n,F)} \rtimes 1$ for $Sp(n,F)$ is the topic of [Gu] and [KuRa]. S. Kudla and S. Rallis describe also irreducible subquotients (even in this case, our result is not completely covered by theirs, since we do not assume char $F = 0$, but only char $F \neq 2$). Reducibilities of degenerate principal series representations considered in the above theorems were obtained by C. Jantzen in regular and in low rank cases ([J1] and [J2], ranks two and three, char $F = 0$; he also described the irreducible subquotients in such situations). Using the methods of this paper and continuing from the results that we have obtained here, C. Jantzen obtained in a recent paper [J3] (among others) reducibility points and irreducible subquotients of all degenerate principal series of groups $Sp(n,F)$ and $SO(2n+1,F)$ which are induced from maximal parabolic subgroups (no new delicate cases show up here).

Each irreducible square integrable representation δ of a general linear group is isomorphic to the unique irreducible square integrable subquotient of $\nu^{-(m-1)/2}\rho \times \nu^{-(m-1)/2+1}\rho \times \cdots \times \nu^{(m-1)/2}\rho$ where ρ is an irreducible unitarizable cuspidal representation of some $GL(p,F)$ (see the first section for notation). Then we shall write $\delta \cong \delta(\rho,m)$. Each irreducible essentially square integrable representation of a general linear group is of the form $\nu^\alpha \delta(\rho,m)$ for some $\alpha \in \mathbb{R}$, m and ρ as above. Assume char $F = 0$. Let p be odd and greater than 1. Theorems 11.6 and 11.8 say that the representation $\nu^\alpha \delta(\rho,m) \rtimes 1$ of $Sp(mp,F)$ (resp. $SO(2mp+1,F)$) reduces if and only if $\rho \cong \tilde{\rho}$ and $\alpha \in \{(-m+1)/2, (-m+1)/2+1, (-m+1)/2+2, \ldots, (m-1)/2\}$ (resp. $\rho \cong \tilde{\rho}$ and $\alpha \in \{-m/2, -m/2+1, -m/2+2, \ldots, m/2\}$). The case $p = 1$ is covered by Theorems 11.3 and 11.4. In particular, these reducibility criteria completely determine the reducibility points of the representations $\delta \rtimes 1$ of $Sp(\ell,F)$ and $SO(2\ell+1,F)$ when ℓ is odd and δ is any irreducible essentially square integrable representation of $GL(\ell,F)$ (Corollaries 11.7 and 11.9). Similar results hold for the segment representations of Zelevinsky. We also describe when we have reducibility for the representations $\nu^\alpha\delta(\rho, m) \rtimes 1$ of $Sp(2m, F)$ and $SO(4m+1, F)$, where ρ is an irreducible cuspidal representation of $GL(2, F)$ (Theorems 11.10 and 11.11). At the end, we describe in Theorem 11.13 reducibilities of $\chi 1_{GL(n,F)} \rtimes \sigma$ and $\chi \mathrm{St}_{GL(n,F)} \rtimes \sigma$ where σ is any irreducible cuspidal representation of $Sp(1, F) = SL(2, F)$ (one can describe such reducibilities for $SO(2n+1, F)$-groups also). For the last result we only assume char $F \neq 2$. There are also other possible applications.

It is interesting to note that the method presented in this paper gives all reducibility points of the representations $\nu^\alpha\delta(\rho, m) \rtimes 1$ of $Sp(pm, F)$ when ρ is an irreducible unitarizable cuspidal representation of $Sp(p, F)$ with a non-trivial central character, and $\alpha \in \mathbb{R}$. We do not need to assume char $F = 0$, since we do not use Shahidi's results in the proofs (the simple cuspidal reducibilities considered in the tenth section are enough for this).

F. Shahidi proved in [Sh2] a duality between parabolic inductions in the case of the groups $Sp(n, F)$ and $SO(2n+1, F)$, when one is inducing (unitary) irreducible square integrable representations of $GL(n, F)$ (char $F = 0$). In the twelfth section we show how this duality can be extended (in a suitable form) to the non-unitary case (Theorem 12.1). More precisely, we make a partition of the set of all (classes of) irreducible essentially square integrable representations of $GL(n, F)$, say into X and Y. Then for $\pi \in X$ both parabolically induced representations $\pi \rtimes 1$, of $Sp(n, F)$ and $SO(2n+1)$, are irreducible. On Y we have a duality, one representation is reducible if and only if the other one is irreducible. The set of all unitarizable classes in Y is exactly the set of all selfcontragredient irreducible square integrable representations of $GL(n, F)$. This is the place where Shahidi showed the duality (one needs to assume $n \geq 2$ in this case).

In the thirteenth section we consider reducibilities of some generalized principal series representations in the case of non-generic cuspidal reducibilities. First we consider in this section reducibility problems similar to those ones of the seventh and the eighth sections (in the new setting). In the case of non-generic cuspidal reducibilities, there exist square integrable representations of a new type, closer to the Zelevinsky segment representations then to the square integrable representations of general linear groups (see Lemma 7.1 of [T7]; Jacquet modules of that representations may have on GL-factors Zelevinsky segment representations). We find reducibility points of representations parabolically induced from such representations, tensored with a cuspidal representation of a general linear group ((iii) of Proposition 13.1). Lemma 13.3 deals with the reducibility of a representation parabolically induced by an even more unusual square integrable representation than the above ones (Jacquet modules of this square integrable representation are not irreducible). Note that our method applies to the setting of these new cuspidal reducibilities without essential changes.

Conversations with D. Goldberg, C. Jantzen, P.J. Sally and F. Shahidi were helpful in the process of clarifying ideas on which this paper is based. C. Jantzen's remarks helped a lot in improving the style of the paper. M. Reeder and C. Mœglin showed me examples of non-generic cuspidal reducibilities ([Mg], [Rd3]; see also [T7]). We thank them all for their help.

We did this work during our stay in Göttingen as a guest of SFB 170 in 1993 (the present paper is an expanded version of the preprint from June 1993 in Mathematica Goettingensis with the same title). We want to thank SFB 170 for their kind hospitality and support which enabled completion of this work. We are also thankful to the Hong Kong University of Science and Technology, where we investigated reducibilities of parabolically induced representations in the case of non-generic cuspidal reducibilities.

1. General linear groups

We shall use standard notation of the representation theory of p-adic general linear groups. This notation is introduced mainly in [Z]. We shall briefly recall of that notation. For more details and for proofs of the facts that we shall present in this section one should consult [Z], and also [Ro2].

Fix a local non-archimedean field F. We shall assume that characteristic of F is different from two. The modulus of F is denoted by $|\ \ |_F$. The character $|\det(\ \ )|_F$ of $GL(n,F)$ is denoted by ν. We fix a minimal parabolic subgroup $P_{\min}^{GL}$ of $GL(n,F)$ consisting of all upper triangular matrices in $GL(n,F)$. Parabolic subgroups of $GL(n,F)$ that contain $P_{\min}$ will be called standard parabolic subgroups.

For $p_i \times p_i$ matrices X_i, $i = 1,\dots,k$, the quasi-diagonal $(p_1 + \cdots + p_k) \times (p_1 + \cdots + p_k)$ matrix which has on the quasi-diagonal the matrices $X_1, \cdots, X_k$, is denoted by q-diag $(X_1,\cdots,X_k)$.

Let $\alpha = (n_1,\dots,n_k)$ be an ordered partition of n. Denote

$$M_\alpha^{GL} = \{\text{q-diag}\,(X_1,\dots,X_k), X_i \in GL(n_i,F)\},$$
$$P_\alpha^{GL} = M_\alpha^{GL} P_{\min}^{GL}.$$

The unipotent radical of P_α^{GL} will be denoted by N_α^{GL}. We identify M_α^{GL} with $GL(n_1,F)\times \cdots \times GL(n_k,F)$ in an obvious way.

For admissible representations π_i of $GL(n_i,F)$,

$$\pi_1 \times \pi_2$$

denotes the representation of $GL(n_1+n_2,F)$ which is parabolically induced by $\pi_1 \otimes \pi_2$ from $P_{(n_1,n_2)}^{GL} = M_{(n_1,n_2)}^{GL} N_{(n_1,n_2)}^{GL}$. If additionally π_3 is an admissible representation of $GL(n_3,F)$, then

$$\pi_1 \times (\pi_2 \times \pi_3) \cong (\pi_1 \times \pi_2) \times \pi_3. \tag{1-1}$$

For a reductive group G over F denote by $\mathfrak{R}(G)$ the Grothendieck group of the category of all admissible representations of G of finite length. There is a natural mapping from the objects of the category to $\mathfrak{R}(G)$. We call this mapping semi simplification, and denote it by s.s.. The image of s.s. determines a cone in $\mathfrak{R}(G)$. In this way we get a natural partial order $\leq$ on $\mathfrak{R}(G)$. In this paper we shall keep the following convention: when we write $\pi_1 \leq \pi_2$ for two representations of G of finite length, it will mean the inequality between semi simplifications s.s.$(\pi_1) \leq$ s.s.(π_2). Further, for each finite set $\pi_1,\dots,\pi_k$ in $\mathfrak{R}(G)$, there exists $\inf(\pi_1,\dots,\pi_k)$ (the highest lower bound).

Set $R_n = \mathfrak{R}(GL(n,F))$ and $R = \oplus_{n\geq 0} R_n$. One lifts in a natural way the multiplication which we have defined above, to a multiplication $\times$ on R. The induced mapping from

$R\otimes R$ to R is denoted by m. In this way R becomes a commutative (associative) ring with identity.

Let $\alpha = (n_1,\dots,n_k)$ be an ordered partition of n and let π be an admissible representation of $GL(n,F)$ of finite length. The (normalized) Jacquet module of π with respect to the parabolic subgroup P_α^{GL} will be denoted by $r_\alpha(\pi)$. We shall consider s.s.$(r_\alpha(\pi)) \in R_{n_1}\otimes\cdots\otimes R_{n_k}$ in a natural way. Define

$$m^*(\pi)=\sum_{k=0}^{n} \text{s.s.}(r_{(k,n-k)}(\pi)) \in R\otimes R.$$

One lifts m^* $\mathbb{Z}$-linearly to a mapping from R to $R\otimes R$. With such comultiplication R is a Hopf algebra (see [Z]).

Take an admissible representation π of $GL(n,F)$. Suppose that π is a subquotient of $\rho_1\times\rho_2\times\cdots\times\rho_k$ where ρ_i are irreducible cuspidal representations of general linear groups. Then the multiset $(\rho_1,\rho_2,\dots,\rho_k)$ will be called the support of π. If additionally we have an admissible representation σ of a reductive group G over F, then $\pi\otimes\sigma$ is a representation of $GL(n,F)\times G$ and we define GL-support of $\pi\otimes\sigma$ to be the support of π, i.e., $(\rho_1,\rho_2,\dots,\rho_k)$.

The support of an irreducible representation π of $GL(n,F)$ always exists (it is uniquely determined, see [Z]). Further, if some irreducible subquotient $\rho_1'\otimes\cdots\otimes\rho_{k'}'$ of some $r_\alpha(\pi)$ is cuspidal, then

$$(\rho_1',\dots,\rho_{k'}')$$

is the support of π.

Let ρ be an irreducible cuspidal representation of a general linear group and let n be a non-negative integer. The set $[\rho,\nu^n\rho]=\{\rho,\nu\rho,\nu^2\rho,\dots,\nu^n\rho\}$ is called a segment of cuspidal representations of general linear groups. The representation $\nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu\rho\times\rho$ has a unique irreducible subrepresentation which we denote by $\delta([\rho,\nu^n\rho])$, and a unique irreducible quotient which we denote by $\mathfrak{s}([\rho,\nu^n\rho])$ (Zelevinsky segment representation). Thus

$$\delta([\rho,\nu^n\rho])\hookrightarrow \nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu\rho\times\rho \twoheadrightarrow \mathfrak{s}([\rho,\nu^n\rho]). \tag{1-2}$$

If $k>\ell$, we take formally $[\nu^k\rho,\nu^\ell\rho]=\emptyset$. We take $\delta(\emptyset)=\mathfrak{s}(\emptyset)$ to be identity of R. We have

$$m^*(\delta([\rho,\nu^n\rho]))=\sum_{k=-1}^{n}\delta([\nu^{k+1}\rho,\nu^n\rho])\otimes\delta([\rho,\nu^k\rho]), \tag{1-3}$$

$$m^*(\mathfrak{s}([\rho,\nu^n\rho]))=\sum_{k=-1}^{n}\mathfrak{s}([\rho,\nu^k\rho])\otimes\mathfrak{s}([\nu^{k+1}\rho,\nu^n\rho]) \tag{1-4}$$

([Z]). Suppose that ρ is a representation of $GL(p,F)$. Denote $(p,p,\dots,p)\in\mathbb{Z}^\ell$ by $(p)^\ell$. Then

$$r_{(p)^{n+1}}(\delta([\rho,\nu^n\rho]))=\nu^n\rho\otimes\nu^{n-1}\rho\otimes\cdots\otimes\nu\rho\otimes\rho, \tag{1-5}$$

$$r_{(p)^{n+1}}(\mathfrak{s}([\rho,\nu^n\rho]))=\rho\otimes\nu\rho\otimes\cdots\otimes\nu^{n-1}\rho\otimes\nu^n\rho. \tag{1-6}$$

The representations on the right hand side in above two formulas also characterize representations $\delta([\rho,\nu^n\rho])$ and $\mathfrak{s}([\rho,\nu^n\rho])$ as irreducible subquotients of $\nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu\rho\times\rho$ which have them for subquotients of corresponding Jacquet modules. The set of all segments of cuspidal representations of general linear groups will be denoted by $\mathcal{S}$.

For an irreducible essentially square integrable representation δ of $GL(m,F)$, one can find a unique $e(\delta)\in\mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. Set $\delta^u=\nu^{-e(\delta)}\delta$. Then $\delta=\nu^{e(\delta)}\delta^u$, where $e(\delta)\in\mathbb{R}$ and δ^u is unitarizable. We denote by D the set of all equivalence classes of irreducible essentially square integrable representations of $GL(m,F)$'s for all $m\geq 1$. Let $d=(\delta_1,\dots,\delta_k)\in M(D)$, where $M(D)$ denotes the set of all finite multisets in D. Choose a permutation ξ of the set $\{1,2,\dots,k\}$ such that $e(\delta_{\xi(1)})\geq e(\delta_{\xi(2)})\geq\dots\geq e(\delta_{\xi(k)})$. The representation $\delta_{\xi(1)}\times\delta_{\xi(2)}\times\dots\times\delta_{\xi(k)}$ has a unique irreducible quotient which we denote by $L(d)$. Then $d\mapsto L(d)$ is the Langlands classification for general linear groups. We shall write $L(d)=L((\delta_1,\dots,\delta_k))$ simply as $L(\delta_1,\dots,\delta_k)$. Note that $\mathfrak{s}([\rho,\nu^n\rho])=L(\rho,\nu\rho,\nu^2\rho,\dots,\nu^n\rho)$.

We shall now describe the Langlands classification in a slightly different way. We shall also describe the parameterization introduced by A.V. Zelevinsky in [Z]. Denote by $M(\mathcal{S})$ the set of all finite multisets in $\mathcal{S}$. Let $a=(\Delta_1,\dots\Delta_k)\in M(\mathcal{S})$. Choose a permutation ξ of $\{1,2,\dots,k\}$ such that $e(\delta(\Delta_{\xi(1)}))\geq e(\delta(\Delta_{\xi(2)}))\geq\dots\geq e(\delta(\Delta_{\xi(k)}))$. Introduce representations

$$\lambda(a)=\delta(\Delta_{\xi(1)})\times\delta(\Delta_{\xi(2)})\times\dots\times\delta(\Delta_{\xi(k)}),$$
$$\zeta(a)=\mathfrak{s}(\Delta_{\xi(1)})\times\mathfrak{s}(\Delta_{\xi(2)})\times\dots\times\mathfrak{s}(\Delta_{\xi(k)}).$$

The representation $\lambda(a)$ (resp. $\zeta(a)$) has a unique irreducible quotient (resp. a unique irreducible subrepresentation) which we denote by $L(a)$ (resp. $Z(a)$). We shall denote often $L((\Delta_1,\dots\Delta_k))$ (resp. $Z((\Delta_1,\dots\Delta_k)))$ simply by $L(\Delta_1,\dots\Delta_k)$ (resp. $Z(\Delta_1,\dots\Delta_k)$).

Let $a=(\Delta_1,\dots\Delta_k)\in M(\mathcal{S})$. Suppose that there exist $1\leq i<j\leq k$ so that $\Delta_i\cup\Delta_j\in\mathcal{S}$ and $\Delta_i\cup\Delta_j\notin\{\Delta_i,\Delta_j\}$. Define

$$a'=(\Delta_1.\dots,\Delta_{i-1},\Delta_i\cup\Delta_j,\Delta_{i+1},\dots,\Delta_{j-1},\Delta_i\cap\Delta_j,\Delta_{j+1},\dots,\Delta_k)$$

(if $\Delta_i\cap\Delta_j=\emptyset$, then we omit $\Delta_i\cap\Delta_j$ in the above definition of a'). Then we shall write

$$a'\prec a.$$

Generate by $\prec$ a partial order $\leq$ on $M(\mathcal{S})$. Then we have the following theorem from the Bernstein-Zelevinsky theory (for the Langlands classification apply the Zelevinsky involution).

1.1. Theorem. *Let $a,b\in M(\mathcal{S})$.*

(i) *$L(b)$ (resp. $Z(b)$) is a subquotient of $\lambda(a)$ (resp. $\zeta(a)$), if and only if $b\leq a$.*
(ii) *The multiplicity of $L(a)$ (resp. $Z(a)$) in $\lambda(a)$ (resp. $\zeta(a)$) is one.*
(iii) *If $b\leq a$ and if b is minimal in $M(\mathcal{S})$, then the multiplicity of $L(b)$ (resp. $Z(b)$) in $\lambda(a)$ (resp. $\zeta(a)$) is one.* □

We shall use often the following fact: if $L(b)$ (resp. $Z(b)$) is a subquotient of $\lambda(a)$ (resp. $\zeta(a)$), then

$$\operatorname{supp} L(a) = \operatorname{supp} L(b) \tag{1-7}$$

(note that $\operatorname{supp} L(a) = \operatorname{supp} Z(a)$ and $\operatorname{supp} L(b) = \operatorname{supp} Z(b)$).

1.2. Remark. Let $\Delta \in \mathcal{S}$. Then $\delta(\Delta) = L(\Delta)$. Therefore we could work only with notation $L(\Delta)$ as F. Rodier did in [Ro2]. For our purposes we find this confusing at some situations and this is a reason that we have separate notation for $L(\Delta)$ (another reason is importance of these representations). A similar situation is with representations $\mathfrak{s}(\Delta) = Z(\Delta)$.

2. Groups $Sp(n, F)$ and $SO(2n+1)$

We shall briefly recall in this section the notation for classical groups $Sp(n, F)$ and $SO(2n+1)$ introduced in [T5] and [T6] (see these two papers for more details and proofs). For a (square) matrix g denote by ${}^t g$ (resp. ${}^\theta g$) the transposed matrix of g (resp. the transposed matrix of g with respect to the second diagonal).

Denote by J_n the $n \times n$ matrix having 1's on the second diagonal and all other entries 0. The identity $n \times n$ matrix is denoted by I_n. Set

$${}^\dagger S = \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} {}^t S \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix},$$

where S is $2n \times 2n$ matrix. The group $Sp(n, F)$ is the group of all $2n \times 2n$ matrices over F which satisfy ${}^\dagger S S = I_{2n}$. We take $Sp(0, F)$ to be the trivial group.

The group $SO(2n+1, F)$ is the group of all $(2n+1) \times (2n+1)$ matrices X over F of determinant one, which satisfy ${}^\theta X X = I_{2n+1}$.

In the sequel, we denote by S_n either the group $Sp(n, F)$ or $SO(2n+1, F)$. We fix the minimal parabolic subgroup $P_{\min}$ in S_n consisting of all upper triangular matrices in the group. Parabolic subgroups of S_n that contain $P_{\min}$ we shall call standard parabolic subgroups.

Let $\alpha = (n_1, \dots, n_k)$ be an ordered partition of some non-negative integer $m \leq n$ into positive integers. If $m = 0$, then the only partition of 0 (empty partition) will be denoted by (0). Set $M_\alpha = \left\{ \text{q-diag}\,(g_1, \cdots, g_k, h, {}^\theta g_k^{-1}, \cdots, {}^\theta g_1^{-1}); g_i \in GL(n_i, F), h \in S_{n-m} \right\}$ Then $P_\alpha = M_\alpha P_{\min}$ is a standard parabolic subgroup of S_n. The unipotent radical of P_α is denoted by N_α. We identify M_α with $GL(n_1, F) \times \ldots \times GL(n_k, F) \times S_{n-m}$ in an obvious way: $\text{q-diag}\,(g_1, \cdots, g_k, h, {}^\theta g_k^{-1}, \cdots, {}^\theta g_1^{-1}) \mapsto \text{q-diag}\,(g_1, \cdots, g_k, h)$.

Let π be an admissible representation of $GL(m, F)$ and let σ be an admissible representation of S_n. We denote by

$$\pi \rtimes \sigma$$

the parabolically induced representation of S_{m+n} from $P_{(m)}$ of $\pi \otimes \sigma$. Here $\pi \otimes \sigma$ maps $\text{q-diag}(g, h, {}^\theta g^{-1}) \in M_{(n)}$ to $\pi(g) \otimes \sigma(h)$. Denote the contragredient representation of τ by $\tilde{\tau}$. The following proposition only expresses well-known facts about parabolic induction in terms of our notation.

2.1. Proposition. *For admissible representations π, π_1, π_2 of general linear groups and for an admissible representation σ of S_m we have $\pi_1 \rtimes (\pi_2 \rtimes \sigma) \cong (\pi_1 \times \pi_2) \rtimes \sigma$, and $(\pi \rtimes \sigma)^{\sim} \cong \tilde{\pi} \rtimes \tilde{\sigma}$.*

Proof. Proposition 4.1 of [T5] (see the proof of that proposition), and Proposition 6.1 of [T6] imply the proposition. □

Set $R_n(S) = \mathfrak{R}(S_n)$ and $R(S) = \oplus_{n \geq 0} R_n(S)$. Lift the multiplication $\rtimes$ to a multiplication $\rtimes : R \times R(S) \to R(S)$ in the usual way. Denote the contragredient involution on R and $R(S)$ by $^{\sim}$.

Again the following proposition expresses well-known facts in terms of our notation (it follows from a well-known fact about parabolic induction from associate representations and the fact that $\pi \otimes \sigma$ and $\tilde{\pi} \otimes \sigma$ are associate, what follows from Theorem 2. of [GfKa]). As the referee noted, the proposition follows also from the commutativity of parabolic induction and the description of the contragredient representations of classical groups in [MgVW].

2.2. Proposition. *For $\pi \in R$ and $\sigma \in R(S)$ we have the equality $\pi \rtimes \sigma = \tilde{\pi} \rtimes \sigma$ in $R(S)$ (i.e. the equality holds in the Grothendieck groups).*

Proof. Proposition 4.2 of [T5] and Proposition 6.2 of [T6]. □

Let σ be an admissible representation of S_n and let $\alpha = (n_1, \dots, n_k)$ be an ordered partition of $0 \leq m \leq n$. The Jacquet module of σ for P_α is denoted by $s_\alpha(\sigma)$. If σ has a finite length, then we shall consider $\text{s.s.}(s_\alpha(\sigma)) \in R_{n_1} \otimes \dots R_{n_k} \otimes R_{n-m}(S)$.

Let π_i be admissible representations of $GL(n_i, F)$ for $i = 1, 2, \dots, k$, let τ be a similar representation of S_q and let σ be a similar representation of $S_{n_1+\dots+n_k+q}$. Denote $\alpha = (n_1, \dots, n_k)$. Then Frobenius reciprocity in this setting tells

$$\text{(F-R)} \qquad \text{Hom}_{S_{n_1+\dots+n_k+q}}(\sigma, \pi_1 \times \dots \times \pi_k \rtimes \tau) \cong \text{Hom}_{M_\alpha}(s_\alpha(\sigma), \pi_1 \otimes \dots \otimes \pi_k \otimes \tau).$$

We now introduce a $\mathbb{Z}$-linear mapping $\mu^* : R(S) \to R \otimes R(S)$ which is defined on the basis of irreducible admissible representations by

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}(s_{(k)}(\sigma)).$$

Consider $R \otimes R(S)$ as an $R \otimes R$-module in an obvious way: $(\sum_i r_i' \otimes r_i'') \rtimes (\sum_j r_j \otimes s_j) = \sum_i \sum_j (r_i' \times r_j) \otimes (r_i'' \rtimes s_j)$. Denote by $\kappa : R \otimes R \to R \otimes R$ the mapping defined by $\kappa(\sum_i x_i \otimes y_i) = \sum_i y_i \otimes x_i$.

2.3. Theorem. *Set $M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ \kappa \circ m^*$. Then for $\pi \in R$ and $\sigma \in R(S)$ we have*

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma).$$

Proof. Theorems 5.4 and 6.5 of [T6]. □

Let π be an admissible representation of $GL(p,F)$ of finite length and let σ be a cuspidal representation of S_q of finite length. If τ is a subquotient of $\pi \rtimes \sigma$, then we define

$$s_{GL}(\tau)$$

to be $s_{(p)}(\tau)$. The above Jacquet module will be called the Jacquet module of GL-type. This Jacquet module is particularly interesting for us because it has the following property: s.s.$(s_{GL}(\tau)) = r \otimes \sigma$ for some $r \in R_p, r \geq 0$ (one can easily deduce it from the above theorem, since $\mu^*(\sigma) = 1 \otimes \sigma$).

Set $D_+ = \{\delta \in D; e(\delta) > 0\}$. Let $T(S)$ be the set of all equivalence classes of irreducible admissible tempered representations of S_n's for all $n \geq 0$. Take $t = ((\delta_1, \dots, \delta_n), \tau) \in M(D_+) \times T(S)$ ($M(D_+)$ denotes the set of all finite multisets in D_+). Choose a permutation ξ of the set $\{1, 2, \dots, n\}$ such that $e(\delta_{\xi(1)}) \geq e(\delta_{\xi(2)}) \geq \dots \geq e(\delta_{\xi(n)})$. The representation $\delta_{\xi(1)} \times \delta_{\xi(2)} \times \dots \times \delta_{\xi(n)} \rtimes \tau$ has a unique irreducible quotient which we denote by $L(t)$. This is the Langlands classification for groups S_m. We shall write $L(t) = L(((\delta_1, \dots, \delta_n), \tau))$ simply as $L((\delta_1, \dots, \delta_n), \tau)$ or $L(\delta_1, \dots, \delta_n, \tau)$.

2.4. Proposition. *Let ρ be an irreducible unitarizable cuspidal representation of the group $GL(p,F)$ and let σ be a similar representation of S_m. Suppose that $\nu^\alpha\rho \rtimes \sigma$ reduces for some $\alpha > 0$. Then:*
(i) $\rho \cong \tilde{\rho}$ (we shall say that ρ is selfcontragredient).
(ii) The representation $\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \dots \times \nu^{\alpha+1}\rho \times \nu^\alpha\rho \rtimes \sigma$, $n \geq 0$, has a unique irreducible subrepresentation which we denote by $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$. We have

$$\mu^*(\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)) = \sum_{k=-1}^{n} \delta([\nu^{\alpha+k+1}\rho, \nu^{\alpha+n}\rho]) \otimes \delta([\nu^\alpha\rho, \nu^{\alpha+k}\rho], \sigma) \tag{2-1}$$

(we take formally $\delta(\emptyset, \sigma) = \sigma$). The representation $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$ is square integrable and we have $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)\tilde{} \cong \delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \tilde{\sigma})$.
(iii) The representation $\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \dots \times \nu^{\alpha+1}\rho \times \nu^\alpha\rho \rtimes \sigma$, $n \geq 0$, has a unique irreducible quotient which we denote by $\mathfrak{s}([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$. We have

$$\mu^*(\mathfrak{s}([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)) = \sum_{k=-1}^{n} \mathfrak{s}([\nu^{-\alpha-n}\rho, \nu^{-\alpha-k-1}\rho]) \otimes \mathfrak{s}([\nu^\alpha\rho, \nu^{\alpha+k}\rho], \sigma) \tag{2-2}$$

(we take formally $\mathfrak{s}(\emptyset, \sigma) = \sigma$). Clearly, $\mathfrak{s}([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma) = L(\nu^\alpha\rho, \nu^{\alpha+1}\rho, \dots, \nu^{\alpha+n}\rho, \sigma)$. The representation $\mathfrak{s}([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$ can be characterized as a unique irreducible subquotient π of $\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \dots \times \nu^{\alpha+1}\rho \times \nu^\alpha\rho \rtimes \sigma$ which satisfies

$$\nu^{-\alpha-n}\rho \otimes \nu^{-\alpha-(n-1)}\rho \otimes \dots \otimes \nu^{-\alpha-1}\rho \otimes \nu^{-\alpha}\rho \otimes \sigma \leq s_{(p)^{n+1}}(\pi). \tag{2-3}$$

Furthermore, for $\pi = \mathfrak{s}([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$ we have in (2-3) an equality.

Proof. The proof of (i) can be found on several places (for example [Sh2]). Proofs of (ii) and (iii) are very similar. Complete proof of (ii) can be found in [T7]. Applying the generalized Zelevinsky involution ([Au], [B] or [SnSt]), one gets (iii). □

The Steinberg representation of a connected reductive group G over F is defined in [C1]. We shall denote this representation by St_G. The trivial representation of G will be denoted by 1_G, while the trivial representation of the trivial group will be denoted simply by 1. Now a simple computation of the modular characters of the minimal parabolic subgroups imply the following:

2.5. Proposition.
(i) $St_{Sp(n,F)} = \delta([\nu 1_{F^\times}, \nu^n 1_{F^\times}], 1), \quad St_{SO(2n+1,F)} = \delta([\nu^{1/2} 1_{F^\times}, \nu^{n-1/2} 1_{F^\times}], 1)$.
(ii) $1_{Sp(n,F)} = \mathfrak{s}([\nu 1_{F^\times}, \nu^n 1_{F^\times}], 1), \quad 1_{SO(2n+1,F)} = \mathfrak{s}([\nu^{1/2} 1_{F^\times}, \nu^{n-1/2} 1_{F^\times}], 1)$. □

A.V. Zelevinsky defined involution $\pi \mapsto \pi^t$ on representations of general linear groups over F ([Z]). A generalization of this involution on irreducible representations of reductive groups is constructed in [Au], [B] and [SnSt]. This generalization is called generalized Zelevinsky involution (we shall use in this paper generalization from [Au]).

3. Some general arguments for reducibility and irreducibility of representations

In this section G will denote a connected reductive group over F. We fix a maximal split torus A in G and a minimal parabolic subgroup $P_{\min}$ in G containing A. Let Σ be the set of all (reduced) roots of G relative to A (see [C2] for more details regarding notation that we use in this section). The minimal parabolic subgroup $P_{\min}$ determines the basis Δ of Σ. For $\Theta \subseteq \Delta$ let P_Θ be the corresponding standard parabolic subgroup. The unipotent radical of P_Θ is denoted by N_Θ. Denote by A_Θ the connected component of $\cap_{\beta\in\Theta}\mathrm{Ker}(\beta)$ and denote by M_Θ the centralizer of A_Θ in G. Then $P_\Theta = M_\Theta N_\Theta$ is a Levi decomposition of P_Θ.

All parabolic subgroups that we consider in this section will be assumed to be standard with respect to $P_{\min}$ (i.e., to contain $P_{\min}$). All Levi decompositions of parabolic subgroups will be assumed to be of the type described above.

For a parabolic subgroup $P = MN$ and an admissible representation π of G, the Jacquet module of π with respect to P will be denoted by $r_M^G(\pi)$ (with one exception, this notation will be used only in this section). We can lift r_M^G to a homomorphism $\mathfrak{R}(G) \to \mathfrak{R}(M)$, which we denote also by r_M^G. Note that this homomorphism is positive: if $\pi \geq 0$, then $r_M^G(\pi) \geq 0$. This implies that r_M^G is monotone: if $\pi_1 \leq \pi_2$, then $r_M^G(\pi_1) \leq r_M^G(\pi_2)$. For an admissible representation σ of M, we denote by $\mathrm{Ind}_P^G(\sigma)$ the parabolically induced representation of G, induced by σ.

The following simple lemmas explain how we shall conclude reducibility and irreducibility of parabolically induced representations in the most cases.

3.1. Lemma. *Let π, π' and Π be admissible representations of G of finite length. Suppose:*

(i) $\pi \leq \Pi$ *and* $\pi' \leq \Pi$;
(ii) *there exist parabolic subgroups $P_1 = M_1N_1$ and $P_2 = M_2N_2$ of G so that*

$$r_{M_1}^G(\pi) \not\leq r_{M_1}^G(\pi') \quad \textit{and} \quad r_{M_2}^G(\pi) + r_{M_2}^G(\pi') \not\leq r_{M_2}^G(\Pi).$$

Then π is reducible (and has a common irreducible subquotient with π').

Proof. Note that (ii) implies $\pi \not\leq \pi'$ and $\pi + \pi' \not\leq \Pi$. Using (i) one now gets directly the lemma. $\square$

The addition among representations in the above formulas is addition among semi simplifications in the Grothendieck group.

3.2. Remark. We shall usually apply the above lemma in the following setting: let $P_0 = M_0N_0, P' = M'N'$, $P'' = M''N''$ be parabolic subgroups of G, let σ_0, σ', σ'' be irreducible admissible representations of M_0, M', M'' respectively, and suppose that:
(i) $\text{Ind}_{P_0}^G(\sigma_0) \leq \text{Ind}_{P''}^G(\sigma'')$ and $\text{Ind}_{P'}^G(\sigma') \leq \text{Ind}_{P''}^G(\sigma'')$;
(ii) there exist parabolic subgroups $P_1 = M_1N_1$ and $P_2 = M_2N_2$ of G such that $r_{M_1}^G(\text{Ind}_{P_0}^G(\sigma_0)) \not\leq r_{M_1}^G(\text{Ind}_{P'}^G(\sigma'))$ and $r_{M_2}^G(\text{Ind}_{P_0}^G(\sigma_0))+r_{M_2}^G(\text{Ind}_{P'}^G(\sigma')) \not\leq r_{M_2}^G(\text{Ind}_{P''}^G(\sigma''))$;
then $\text{Ind}_{P_0}^G(\sigma_0)$ reduces (and has a common irreducible subquotient with $\text{Ind}_{P'}^G(\sigma')$).

Note that for admissible representations π_1, π_2 of G of finite length, $\pi_1 \leq \pi_2$ if and only if for any irreducible admissible representation σ of G, the multiplicity of σ in π_1 is less than or equal to the multiplicity of σ in π_2. Therefore to show $\pi_1 \not\leq \pi_2$ it is enough to find irreducible σ such that its multiplicity in π_1 is greater than the multiplicity in π_2. In particular, it is enough to find irreducible σ which is a subquotient of π_1 but not of π_2 (if such σ exists). Clearly, to show $\pi_1 \not\leq \pi_2$ it is enough to find some parabolic subgroup $P = MN$ of G such that $r_M^G(\pi_1) \not\leq r_M^G(\pi_2)$ since the Jacquet functors are monotone (we already used that in the proof of Lemma 3.1).

Denote by $\mathcal{P}$ the set of all standard parabolic subgroups of G and set $\mathfrak{R}_+(G) = \{x \in \mathfrak{R}(G); x \geq 0\}$.

3.3. Definition. Let $P_0 = M_0N_0$ be a parabolic subgroup of G, let σ_0 be an irreducible admissible representation of G, let X be a non-empty subset of $\mathcal{P}$ and let ℓ be an integer ≥ 2. A function $\phi = (\phi_1, \dots, \phi_\ell) : X \to (\mathfrak{R}_+(G))^\ell$ will be called coherent X-decomposition of order ℓ of Jacquet modules of $\text{Ind}_{P_0}^G(\sigma_0)$ if

(i) $\sum_{i=1}^{\ell} \phi_i(P) = r_M^G(\text{Ind}_{P_0}^G(\sigma_0))$ for all $P \in X$;
(ii) $r_{M'}^{M''}(\phi_i(P'')) = \phi_i(P')$, when $P', P'' \in X$, $P' \subseteq P''$ and $1 \leq i \leq \ell$;
(iii) $\phi_i(P) = 0$ if and only if $\phi_j(P) = 0$, for all $P \in X$ and $i, j \in \{1, \dots, \ell\}$.

We call a coherent X-decomposition of order ℓ of Jacquet modules of $\text{Ind}_{P_0}^G(\sigma_0)$ non-trivial, if $\phi_1(P) \neq 0$ for some $P \in X$. A coherent $\mathcal{P}$-decomposition of order ℓ of Jacquet modules of $\text{Ind}_{P_0}^G(\sigma_0)$ will be called full coherent decomposition of order ℓ of Jacquet modules of $\text{Ind}_{P_0}^G(\sigma_0)$. Coherent X-decomposition of order 2 of Jacquet modules of $\text{Ind}_{P_0}^G(\sigma_0)$ will be simply called coherent X-decomposition of Jacquet modules of $\text{Ind}_{P_0}^G(\sigma_0)$.

From the above definition it is clear that each full coherent decomposition ϕ of Jacquet modules of $\text{Ind}_{P_0}^G(\sigma_0)$ is non-trivial. Further, it is completely determined with $\phi(G)$, what is a decomposition of $\text{Ind}_{P_0}^G(\sigma_0)$ into a sum of two strictly positive elements of $\mathfrak{R}(G)$. From the proof of the following lemma we can conclude that the converse is also true, each decomposition of $\text{Ind}_{P_0}^G(\sigma_0)$ into a sum of two strictly positive elements of $\mathfrak{R}(G)$ determines a non-trivial full coherent decomposition of Jacquet modules of $\text{Ind}_{P_0}^G(\sigma_0)$.

3.4. Lemma. *Suppose that σ_0 is an irreducible admissible representation of M_0. If $Ind_{P_0}^G(\sigma_0)$ reduces, then there exists a full coherent decomposition of Jacquet modules of $Ind_{P_0}^G(\sigma_0)$. This decomposition is non-trivial.*

Proof. Suppose that $\mathrm{Ind}_{P_0}^G(\sigma_0)$ reduces. Chose a non-trivial proper subrepresentation π of $\mathrm{Ind}_{P_0}^G(\sigma_0)$. Define $\phi : \mathcal{P} \to \mathfrak{R}_+(G) \times \mathfrak{R}_+(G)$ by the formula

$$\phi(P) = (\mathrm{s.s.}(r_M^G(\pi)), \mathrm{s.s.}(r_M^G(\mathrm{Ind}_{P_0}^G(\sigma_0)/\pi))),$$

where $P = MN$. Note that the property (i) of Definition 3.3 holds since Jacquet functors are exact. The property (ii) follows from the transitivity of Jacquet modules ((c) of Proposition 1.9 of [BZ]). It remains to see that ϕ satisfies also the third property of the definition. Let $P = MN$ be a parabolic subgroup of G, let τ be an irreducible subquotient of $\mathrm{Ind}_{P_0}^G(\sigma_0)$ and suppose that $r_M^G(\mathrm{Ind}_{P_0}^G(\sigma_0)) \neq 0$. To prove (iii), it is enough to show that $r_M^G(\tau) \neq 0$.

Choose a parabolic subgroup $P_0' \subseteq P_0$ of G such that there exists an irreducible cuspidal representation σ_0' of M_0' satisfying $\sigma_0 \hookrightarrow \mathrm{Ind}_{P_0' \cap M_0}^{M_0}(\sigma_0')$. Then $\mathrm{Ind}_{P_0}^G(\sigma_0) \hookrightarrow \mathrm{Ind}_{P_0'}^G(\sigma_0')$. Chose a parabolic subgroup $P' \subseteq P$ of G which satisfies $r_{M'}^G(\mathrm{Ind}_{P_0'}^G(\sigma_0')) \neq 0$ and which is minimal among all parabolic subgroups which satisfy this. Then $r_{M'}^G(\mathrm{Ind}_{P_0'}^G(\sigma_0')) \neq 0$ is a cuspidal representation (otherwise, we could choose smaller P' which satisfies the above requirements). Let σ' be some irreducible quotient of $r_{M'}^G(\mathrm{Ind}_{P_0'}^G(\sigma_0'))$. Then Frobenius reciprocity implies that there exists a non-trivial intertwining of $\mathrm{Ind}_{P_0'}^G(\sigma_0')$ into $\mathrm{Ind}_{P'}^G(\sigma')$. Theorem 2.9 of [BZ] implies that P' and P_0' are associate parabolic subgroups. Now Lemma 2.12.4 of [Si] implies that $r_{M'}^G(\tau) \neq 0$. Since $r_{M'}^G(\tau) = r_{M'}^M(r_M^G(\tau))$, we obtain $r_M^G(\tau) \neq 0$. This finishes the proof. □

3.5. Remark. From the above proof we see that the following fact holds. Let σ_0 be an irreducible admissible representation of M_0 and let τ be a non-zero subquotient of $\mathrm{Ind}_{P_0}^G(\sigma_0)$. If $r_M^G(\mathrm{Ind}_{P_0}^G(\sigma_0)) \neq 0$ for some parabolic subgroup $P = MN$, then $r_M^G(\tau) \neq 0$.

We could easily prove also that if $\mathrm{Ind}_{P_0}^G(\sigma_0)$ has length $\geq k$, then there exists a full coherent decomposition of order k of Jacquet modules of $\mathrm{Ind}_{P_0}^G(\sigma_0)$.

Let $Y \subset X \subset \mathcal{P}$. Suppose that ϕ is a coherent X-decomposition of Jacquet modules of $\mathrm{Ind}_{P_0}^G(\sigma_0)$ and suppose that $r_P^G(\mathrm{Ind}_{P_0}^G(\sigma_0)) \neq 0$ for some $P \in Y$. Then the restriction $\phi|Y$ is a non-trivial coherent Y-decomposition of Jacquet modules of $\mathrm{Ind}_{P_0}^G(\sigma_0)$. Therefore we have the following:

3.6. Lemma. *Let $P_0 = M_0N_0$ be a parabolic subgroup of G and let σ_0 be an irreducible admissible representation of G. Assume that $X \subset \mathcal{P}$ and that there exists $P \in X$ such that $r_P^G(Ind_{P_0}^G(\sigma_0)) \neq 0$. Suppose that it does not exist a coherent X-decomposition of Jacquet modules of $Ind_{P_0}^G(\sigma_0)$. Then $Ind_{P_0}^G(\sigma_0)$ is irreducible.* □

Suppose that P, P' and P'' are proper parabolic subgroups of G such that $P \subsetneqq P'$, $P \subsetneqq P''$ and $P' \neq P''$. Coherent $\{P, P', P''\}$-decompositions play important role in proving irreducibility of parabolically induced representations. We can call coherent $\{P, P', P''\}$-decompositions of Jacquet modules with P, P', P'' as above, coherent decompositions of V-type.

The following lemma enables one to check sometimes in a simple way the condition of nonexistence from the above lemma. This lemma will enable us to prove irreducibility in a number of cases.

3.7. Lemma. *Let $P_o = M_0N_0$ be a parabolic subgroup of G and let σ_0 be an irreducible admissible representation of M_0. Let P', P'', P''' be parabolic subgroups of G such that $P' \subseteq P''$, $P' \subseteq P'''$ and $r^G_{M'}(Ind^G_{P_0}(\sigma_0)) \neq 0$. Suppose that there exists an irreducible subquotient τ'' of $r^G_{M''}(Ind^G_{P_0}(\sigma_0))$ such that for any irreducible subquotient τ''' of $r^G_{M'''}(Ind^G_{P_0}(\sigma_0))$ we have*

$$r^{M''}_{M'}(\tau'') + r^{M'''}_{M'}(\tau''') \not\leq r^G_{M'}(Ind^G_{P_0}(\sigma_0)).$$

Then $Ind^G_P(\sigma)$ is irreducible.

Sometimes is convenient to write the condition $r^{M''}_{M'}(\tau'') + r^{M'''}_{M'}(\tau''') \not\leq r^G_{M'}(\mathrm{Ind}^G_{P_0}(\sigma_0))$ in the form

$$r^{M'''}_{M'}(\tau''') \not\leq r^{M''}_{M'}(r^G_{M''}(\mathrm{Ind}^G_{P_0}(\sigma_0)) - \tau'').$$

Proof. We shall show that there does not exist a coherent $\{P', P'', P'''\}$-decomposition of Jacquet modules of $\mathrm{Ind}^G_{P_0}(\sigma_0)$. Suppose that some such decomposition ϕ exists. Without lost of generality we can assume that $\tau'' \leq \phi_1(P'')$. Now $r^G_{M'}(\mathrm{Ind}^G_{P_0}(\sigma_0)) = \phi_1(P') + \phi_2(P') = r^{M''}_{M'}(\phi_1(P'')) + r^{M'''}_{M'}(\phi_2(P''')) \geq r^{M''}_{M'}(\tau'') + r^{M'''}_{M'}(\phi_2(P'''))$. Since $\phi_2(P''') \neq 0$, this contradicts to (ii). $\square$

In the case of induction by unitarizable irreducible representations, the following lemma lists some useful facts.

3.8. Lemma. *Let $P_0 = M_0N_0$ be a parabolic subgroup of G and let σ_0 be an irreducible unitarizable admissible representation of M.*

(a) If the multiplicity of σ_0 in $r^G_{M_0}(Ind^G_{P_0}(\sigma_0))$ is one, then $Ind^G_{P_0}(\sigma_0)$ is irreducible.

(b) If the multiplicity of σ_0 in $r^G_{M_0}(Ind^G_{P_0}(\sigma_0))$ is two, then $Ind^G_{P_0}(\sigma_0)$ is either irreducible or a direct sum of two irreducible non-isomorphic representations.

(c) Let P_0' be a parabolic subgroups of G such that $P_0' \subseteq P_0$. Suppose that there exists an irreducible subquotient τ_0 of $r^G_{M_0}(Ind^G_{P_0}(\sigma_0))$ of multiplicity one. Let σ_0' be an irreducible admissible representation of M_0'. Suppose that the following conditions hold:

(i) *$Ind^G_{P_0}(\sigma_0) \hookrightarrow Ind^G_{P_0'}(\sigma_0')$*

(ii) *$\sigma_0' \not\leq r^{M_0}_{M_0'}(r^G_{M_0}(Ind^G_{P_0}(\sigma_0)) - \tau_0)$ (i.e. the multiplicity of σ_0' in $r^{M_0}_{M_0'}(r^G_{M_0}(Ind^G_{P_0}(\sigma_0)) - \tau_0)$ is 0; note that $r^G_{M_0}(Ind^G_{P_0}(\sigma_0)) - \tau_0 \geq 0$).*

Then $Ind^G_{P_0}(\sigma_0)$ is irreducible.

(d) Let P' and P'' be parabolic subgroups of G such that $P' \subseteq P_0$ and $P' \subseteq P''$. Suppose that there exists an irreducible subquotient τ'' of $r^G_{M''}(Ind^G_{P_0}(\sigma_0))$ of multiplicity one. Let

τ_0 be an irreducible subquotient of $r^G_{M_0}(Ind^G_{P_0}(\sigma_0))$ and let σ' be an irreducible admissible representations of M'. Suppose that the following conditions hold:

(i) $Ind^G_{P_0}(\sigma_0) \hookrightarrow Ind^G_{P'}(\sigma')$

(ii) *If τ_0' is an irreducible subquotient of $r^G_{M_0}(Ind^G_{P_0}(\sigma_0))$ which is not isomorphic to τ_0, then σ' is not a subquotient of $r^{M_0}_{M'}(\tau_0')$.*

(iii) *There exists an irreducible subquotient ρ' of $r^{M_0}_{M'}(\tau_0)$ such that the multiplicities of ρ' in $r^{M''}_{M'}(\tau'')$ and $r^G_{M'}(Ind^G_{P_0}(\sigma_0))$ are the same.*

Then $Ind^G_{P_0}(\sigma_0)$ is irreducible.

Proof. Write $\mathrm{Ind}^G_{P_0}(\sigma_0) = \oplus_{i=1}^k m_i\pi_i$ into a direct sum of irreducible representations such that $\pi_i \not\cong \pi_j$ if $i \neq j$. Then $d = \dim_{\mathbb{C}} \mathrm{End}_G\ (\mathrm{Ind}^G_{P_0}(\sigma_0)) = \sum_{i=1}^k m_i^2$. Further Frobenius reciprocity implies $\dim_{\mathbb{C}} \mathrm{Hom}_{M_0}\ (r^G_{M_0}(\mathrm{Ind}^G_{P_0}(\sigma_0)), \sigma_0) = \sum_{i=1}^k m_i^2$. Clearly, if the multiplicity of σ_0 in $r^G_{M_0}(\mathrm{Ind}^G_{P_0}(\sigma_0))$ is one (resp. 2), then $d \leq 1$ (resp. $d \leq 2$). This proves (a) and (b).

(c) There exists an irreducible subquotient π of $\mathrm{Ind}^G_{P_0}(\sigma_0)$ such that $\tau_0 \leq r^G_{M_0}(\pi)$. The multiplicity of π in $\mathrm{Ind}^G_{P_0}(\sigma_0)$ is one. Suppose that $\mathrm{Ind}^G_{P_0}(\sigma_0)$ is reducible. Let π' be some irreducible subquotient of $\mathrm{Ind}^G_{P_0}(\sigma_0)$ which is not isomorphic to π. Then τ_0 is not a subquotient of $r^G_{M_0}(\pi')$. Since $\mathrm{Ind}^G_{P_0}(\sigma_0)$ is completely reducible, π' is a subrepresentation of $\mathrm{Ind}^G_{P_0}(\sigma_0)$. Thus $\pi' \hookrightarrow \mathrm{Ind}^G_{P_0'}(\sigma_0')$ by (i). Frobenius reciprocity (F-R) implies that σ_0' is a quotient of $r^G_{M_0'}(\pi') = r^{M_0}_{M_0'}(r^G_{M_0}(\pi'))$. Now (ii) implies that τ_0 is a subquotient of $r^G_{M_0}(\pi')$. This contradicts to our choice of π'. The contradiction completes the proof of (c).

(d) Chose an irreducible subquotient π of $\mathrm{Ind}^G_{P_0}(\sigma_0)$ such that $\tau'' \leq r^G_{M_0}(\pi)$. Then the multiplicity of π in $\mathrm{Ind}^G_{P_0}(\sigma_0)$ is one. Suppose that $\mathrm{Ind}^G_{P_0}(\sigma_0)$ reduces. Let π' be some irreducible subquotient of $\mathrm{Ind}^G_{P_0}(\sigma_0)/\pi$. Then as in the proof of (c) we see that σ' is a quotient of $r^G_{M'}(\pi')$. Since $r^G_{M'}(\pi') = r^{M_0}_{M'}(r^G_{M_0}(\pi'))$, we conclude that τ_0 is a subquotient of $r^G_{M_0}(\pi')$. Now

$$r^G_{M'}(\mathrm{Ind}^G_{P_0}(\sigma_0)) \geq r^G_{M'}(\pi) + r^G_{M'}(\pi') = r^{M''}_{M'}(r^G_{M''}(\pi)) + r^{M_0}_{M'}(r^G_{M_0}(\pi')) \geq r^{M''}_{M'}(\tau'') + r^{M_0}_{M'}(\tau_0).$$

From here we see that the multiplicity of ρ' in $r^{M''}_{M'}(\tau'')$ is at least for one less than the multiplicity in $\mathrm{Ind}^G_{P_0}(\sigma_0)$, what contradicts to (iii). This contradiction completes the proof. □

Note that (c) is a special case of (d).

4. On unitary induction of GL-type

We shall describe first the types of cuspidal reducibilities with which we shall work in this paper. Suppose that ρ is an irreducible cuspidal representation of $GL(p, F)$, and σ a similar representation of S_q. Write $\rho = \nu^\beta \rho^u$, where ρ^u is unitarizable and $\beta \in \mathbb{R}$. If $\rho \rtimes \sigma$ reduces, then $\rho^u \cong (\rho^u)\tilde{}$. The Langlands conjectures about the cuspidal representations suggest that the following should hold:

($\mathcal{R}_{(1/2)\mathbb{Z}}$) there exists $\alpha_0 \in (1/2)\,\mathbb{Z}$ such that$\nu^\alpha \rho^u \rtimes \sigma$ is irreducible for $\alpha \in \mathbb{R}\backslash\{\pm\alpha_0\}$

(see [T7] for more explanations regarding this property). Note that in the above formulation we do not claim that there must be reducibility at $\pm\alpha_0$. If the above condition holds for a pair ρ and σ, then we shall say that they have reducibility in $(1/2)\,\mathbb{Z}$, or $(1/2)\,\mathbb{Z}$-reducibility. If for a pair ρ and σ one can find α_0 already in $\{0, \pm 1/2, \pm 1\}$ such that the above condition holds for that pair, then we shall say that ρ and σ have generic cuspidal reducibility (see [T7]). F. Shahidi has proved that if σ is generic, then ρ and σ have generic cuspidal reducibility ([Sh2]). He has informed us that his conjecture 9.4 from [Sd1] would imply that $\mathcal{R}_{(1/2)\mathbb{Z}}$ holds in general (for char $F = 0$). C. Mœglin has a conjectural description of α_0 (from $\mathcal{R}_{(1/2)\mathbb{Z}}$) in terms of Langlands correspondences.

Let us note that both Steinberg representations and degenerate principal series representations show up in the setting of generic cuspidal reducibilities. In understanding of reducibility of parabolically induced representations, the first classes of representations to be studied are degenerate principal series representations and representations parabolically induced by (twists of) Steinberg representations. This is the reason that our paper mainly deals with parabolically induced representations related to the generic cuspidal reducibilities (it is important to note that with respect to the irreducibility, a lot of the work done in the setting of generic cuspidal reducibilities applies also to the setting of non-generic cuspidal reducibilities). Our method applies also, without any significant modification, to the setting of non-generic cuspidal $(1/2)\,\mathbb{Z}$-reducibilities. It seems that before the summer of 1996 there were not known examples of reducibilities which are not generic ([Mg],[Rd3]). The last section of the paper is devoted to the setting of non-generic cuspidal reducibilities.

In whole this section we shall assume that ρ is an irreducible unitarizable cuspidal representation of $GL(p, F)$ and σ an irreducible cuspidal representation of S_q, while n will be a positive integer and m will be a non-negative integer.

Irreducibility result that we prove in this section will follow from Lemma 3.8, while reducibility results will follow from Remark 3.2.

4.1. Proposition. *Assume that $\nu^{1/2+k}\rho \rtimes \sigma$ is irreducible for any $k \in \mathbb{Z}$. Then the representation $\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$ is irreducible.*

Proof. Using (1-3) we compute

$$M^*(\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho])) = (m \otimes 1) \circ (\sim \otimes m^*) \circ \kappa \circ m^*(\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]))$$

$$= (m \otimes 1) \circ (\sim \otimes m^*) \circ \kappa (\sum_{k=-m-1}^{m+1} \delta([\nu^{k+1/2}\rho, \nu^{m+1/2}\rho]) \otimes \delta([\nu^{-m-1/2}\rho, \nu^{k-1/2}\rho]))$$

$$= (m \otimes 1) \circ (\sim \otimes m^*) (\sum_{k=-m-1}^{m+1} \delta([\nu^{-m-1/2}\rho, \nu^{k-1/2}\rho]) \otimes \delta([\nu^{k+1/2}\rho, \nu^{m+1/2}\rho]))$$

$$= (m \otimes 1)(\sum_{k=-m-1}^{m+1} \delta([\nu^{-k+1/2}\tilde{\rho}, \nu^{m+1/2}\tilde{\rho}]) \otimes$$

$$(\sum_{l=k}^{m+1} \delta([\nu^{l+1/2}\rho, \nu^{m+1/2}\rho]) \otimes \delta([\nu^{k+1/2}\rho, \nu^{l-1/2}\rho])))$$

$$= \sum_{k=-m-1}^{m+1} \sum_{l=k}^{m+1} \delta([\nu^{-k+1/2}\tilde{\rho}, \nu^{m+1/2}\tilde{\rho}]) \times \delta([\nu^{l+1/2}\rho, \nu^{m+1/2}\rho]) \otimes \delta([\nu^{k+1/2}\rho, \nu^{l-1/2}\rho]).$$

Theorem 2.3 implies $\mu^*(\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma) = M^*(\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho])) \rtimes \mu^*(\sigma) = M^*(\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho])) \rtimes (1 \otimes \sigma)$. Now we can see easily semi simplification of the Jacquet module of $\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$ for any standard parabolic subgroup. For this proof we need only

$$\text{(4-1)} \qquad \text{s.s.}(s_{GL}(\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma)) = \sum_{k=-m-1}^{m+1} \delta([\nu^{-k+1/2}\tilde{\rho}, \nu^{m+1/2}\tilde{\rho}]) \times \delta([\nu^{k+1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma.$$

Note that all representations in the above sum are irreducible (see Theorem 1.1).

Suppose that ρ is not selfcontragredient, i.e. $\rho \not\cong \tilde{\rho}$. Then GL-supports of representations in the above sum are all different. From this we conclude that the multiplicity of $\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$ in (4-1) is one. Now (a) of Lemma 3.8 implies irreducibility.

We shall assume that ρ is selfcontragredient in the rest of the proof. In proof of irreducibility we shall apply (c) of Lemma 3.8. Since $\rho = \tilde{\rho}$, we can write (4-1) in the following way

$$\text{(4-2)} \qquad \text{s.s.}(s_{GL}(\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma)) = \delta([\nu^{1/2}\rho, \nu^{m+1/2)}\rho]) \times \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma + 2\sum_{k=0}^{m} \delta([\nu^{-1/2-k}\rho, \nu^{m+1/2)}\rho]) \times \delta([\nu^{3/2+k}\rho, \nu^{m+1/2}\rho]) \otimes \sigma.$$

Denote $\tau_0 = \delta([\nu^{1/2}\rho, \nu^{m+1/2)}\rho]) \times \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$. Obviously, the multiplicity of τ_0 in (4-2) is one (one sees this easily from the fact that all elements of the sum in the last row have different GL-supports than τ_0).

From (1-2) we get $\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma \hookrightarrow \nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{-m-1/2}\rho \rtimes \sigma$. We shall now use repeatedly the fact that $\nu^{\ell}\rho \times \nu^{\ell'}\rho$ is irreducible for $\ell, \ell' \in \mathbb{R}$ if $|\ell - \ell'| \neq 1$ (see Theorem 1.1), Proposition 2.1 and (1-1), to show the following isomorphisms:

$$\begin{aligned}
&\nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \nu^{m-3/2}\rho \times \cdots \times \nu^{-m+3/2}\rho \times \nu^{-m+1/2}\rho \times \nu^{-m-1/2}\rho \rtimes \sigma \\
&\quad = (\nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{-m+3/2}\rho \times \nu^{-m+1/2}\rho) \rtimes (\nu^{-m-1/2}\rho \rtimes \sigma) \\
&\quad \cong (\nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{-m+3/2}\rho \times \nu^{-m+1/2}\rho) \rtimes (\nu^{m+1/2}\rho \rtimes \sigma) \\
&\quad \cong (\nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{-m+3/2}\rho \times (\nu^{-m+1/2}\rho \times \nu^{m+1/2}\rho)) \rtimes \sigma \\
&\quad \cong (\nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{-m+3/2}\rho \times (\nu^{m+1/2}\rho \times \nu^{-m+1/2}\rho)) \rtimes \sigma \\
&\quad \cong (\nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times (\nu^{-m+3/2}\rho \times \nu^{m+1/2}\rho) \times \nu^{-m+1/2}\rho) \rtimes \sigma \\
&\quad \cong (\nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{m+1/2}\rho \times \nu^{-m+3/2}\rho \times \nu^{-m+1/2}\rho) \rtimes \sigma \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cong \nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{1/2}\rho \times \nu^{m+1/2}\rho \times \nu^{-1/2}\rho \times \nu^{-3/2}\rho \times \cdots \times \nu^{-m+1/2}\rho \rtimes \sigma.
\end{aligned}$$

Repeating the above procedure with $\nu^{-m+1/2}\rho, \nu^{-m+3/2}\rho, \dots, \nu^{1/2}\rho$, we get

$$\nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{-m-1/2}\rho \rtimes \sigma$$
$$\cong \nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{1/2}\rho \times \nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{1/2}\rho \rtimes \sigma.$$

Therefore

$$\text{(4-3)}\quad \delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma \hookrightarrow \nu^{m+1/2}\rho \times \cdots \times \nu^{1/2}\rho \times \nu^{m+1/2}\rho \times \cdots \times \nu^{1/2}\rho \rtimes \sigma.$$

Denote $\sigma_0' = \nu^{m+1/2}\rho \otimes \cdots \otimes \nu^{1/2}\rho \otimes \nu^{m+1/2}\rho \otimes \cdots \otimes \nu^{1/2}\rho \otimes \sigma$.

According to (4-2), to prove that the condition (ii) in (c) of Lemma 3.8 holds, it is enough to prove that σ_0' is not a subquotient of $r_{(p)^{2m+2}}(\delta([\nu^{-1/2-k}\rho, \nu^{m+1/2)}\rho]) \times \delta([\nu^{3/2+k}\rho, \nu^{m+1/2}\rho])) \otimes \sigma$ for any $0 \leq k \leq m$. This follows easily from the fact that each $\delta([\nu^{-1/2-k}\rho, \nu^{m+1/2)}\rho]) \times \delta([\nu^{3/2+k}\rho, \nu^{m+1/2}\rho]), 0 \leq k \leq m$, has some $\nu^{\ell}\rho$ in the support with $\ell < 0$. We have proved that conditions in (c) of Lemma 3.8 hold. The proof is now complete. □

4.2. Proposition. *Suppose that $\nu^k\rho \rtimes \sigma$ is irreducible for any $k \in \mathbb{Z}$. Then $\delta([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma$ is irreducible.*

Proof. If ρ is not selfcontragredient, then one gets as in the proof of the above proposition that $\delta([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma$ is irreducible. We shall assume in further that ρ is selfcontragredient. We shall prove irreducibility in this case using (d) of Lemma 3.8. From Theorem 2.3 and (1-3) we get in a similar way as in the proof of the preceding proposition

$$\text{(4-4)}\quad \text{s.s.}(s_{(2np)}(\delta([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma))$$
$$= \sum_{k=-n}^{n} \delta([\nu^{-k+1}\rho, \nu^n\rho]) \times \delta([\nu^{k+1}\rho, \nu^n\rho]) \otimes \nu^k\rho \rtimes \sigma$$
$$= \delta([\nu\rho, \nu^n\rho]) \times \delta([\nu\rho, \nu^n\rho]) \otimes \rho \rtimes \sigma + 2\sum_{k=1}^{n} \delta([\nu^{-k+1}\rho, \nu^n\rho]) \times \delta([\nu^{k+1}\rho, \nu^n\rho]) \otimes \nu^k\rho \rtimes \sigma,$$

$$\text{(4-5)}\quad \text{s.s.}(s_{GL}(\delta([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma)) = 2\sum_{k=0}^{n} \delta([\nu^{-k}\rho, \nu^n\rho]) \times \delta([\nu^{k+1}\rho, \nu^n\rho]) \otimes \sigma$$

In the formula (4-4), we see that all the elements in the sum in the second row are irreducible. Denote $\tau'' = \delta([\nu\rho, \nu^n\rho]) \times \delta([\nu\rho, \nu^n\rho]) \otimes \rho \rtimes \sigma$. Then τ'' has multiplicity one in (4-4). Further all representations in the sum of the right hand side of (4-5) are irreducible. Denote the first representation $\delta([\rho, \nu^n\rho]) \times \delta([\nu\rho, \nu^n\rho]) \otimes \sigma$ in that sum by τ_0.

Now in a similar way as in the last proof we obtain

$$\delta([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma \hookrightarrow \nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu^{-n+1}\rho \times \nu^{-n}\rho \rtimes \sigma$$
$$\cong \nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu^{-n+1}\rho \times \nu^n\rho \rtimes \sigma$$
$$\cong \nu^n\rho \times \cdots \times \nu\rho \times \rho \times \nu^n\rho \times \nu^{-1}\rho \times \cdots \times \nu^{-n+1}\rho \rtimes \sigma \cong \dots$$
$$\cdots \cong \nu^n\rho \times \cdots \times \nu\rho \times \rho \times \nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu\rho \rtimes \sigma.$$

Denote $\sigma' = \nu^n\rho \otimes \cdots \otimes \nu\rho \otimes \rho \otimes \nu^n\rho \otimes \cdots \otimes \nu\rho \otimes \sigma$.

To prove condition (ii) in (d) of Lemma 3.8, it is enough to show that σ' is not a subquotient of $r_{(p)^{2n+1}}(\delta([\nu^{-k}\rho, \nu^n\rho]) \times \delta([\nu^{k+1}\rho, \nu^n\rho])) \otimes \sigma$ for $1 \leq k \leq n$ (see (4-5)). This follows from the fact that each representation $\delta([\nu^{-k}\rho, \nu^n\rho]) \times \delta([\nu^{k+1}\rho, \nu^n\rho]) \otimes \sigma$, $1 \leq k \leq n$ has in GL-support some $\nu^\ell\rho$ with $\ell < 0$ (for example $\nu^{-k}\rho$; for the above argumentation see the connection between Jacquet modules and supports described in the first section).

From (1-3) we see that

$$\tau_0 \cong \delta([\nu\rho, \nu^n\rho]) \times \delta([\rho, \nu^n\rho]) \otimes \rho \hookrightarrow \nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu\rho \times \nu^n\rho \times \cdots \times \nu\rho \times \rho \rtimes \sigma.$$

Therefore $\rho' = \nu^n\rho \otimes \nu^{n-1}\rho \otimes \cdots \otimes \nu\rho \otimes \nu^n\rho \otimes \cdots \otimes \nu\rho \otimes \rho \otimes \sigma$ is a quotient of $s_{(p)^{2n+1}}(\tau_0)$.
It remains to prove (iii). From (4-5) we see that it is enough to show that ρ' can not be a subquotient of $r_{(p)^{2n+1}}(\delta([\nu^{-k+1}\rho, \nu^n\rho]) \times \delta([\nu^{k+1}\rho, \nu^n\rho])) \otimes s_{(p)}(\nu^k\rho \rtimes \sigma)$ for $1 \leq k \leq n$. Since s.s.$(s_{(p)}(\nu^k\rho \rtimes \sigma)) = \nu^k\rho \otimes \sigma + \nu^{-k}\rho \otimes \sigma$ by Theorem 2.3, we see that this is true (note that ρ' has $\rho \otimes \sigma$ at the end of the tensor product). This completes the proof. □

4.3. Proposition. *Suppose that* $\nu^{1/2}\rho \rtimes \sigma$ *reduces. Then* $\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$ *reduces into a sum of two inequivalent irreducible representations.*

Proof. The proposition follows from Theorem 4.2 of [T8]. For the sake of completeness, we shall sketch the proof. We shall use Lemma 3.1 in the form of Remark 3.2 to prove reducibility. From (1-2) and (ii) of Proposition 2.4 we get easily embeddings

$$\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma \hookrightarrow \nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{-m-1/2}\rho \rtimes \sigma, \tag{4-6}$$

$$\begin{aligned} &\delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma) \\ &\qquad \hookrightarrow \nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \ldots \nu^{1/2}\rho \times \nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \ldots \nu^{1/2}\rho \rtimes \sigma. \end{aligned} \tag{4-7}$$

Note that $\nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{-m-1/2}\rho \rtimes \sigma = \nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \ldots \nu^{1/2}\rho \times \nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \ldots \nu^{1/2}\rho \rtimes \sigma$ in $R(S)$ (use Proposition 2.2 and commutativity of R). Now embeddings (4-6) and (4-7) give corresponding inequalities in R(S).

Using Theorem 2.3 we can see that the multiplicity of $\delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho])^2 \otimes \sigma$ in each of $s_{GL}(\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma)$, $s_{GL}(\delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma))$ and $s_{GL}(\nu^{-m-1/2}\rho \times \nu^{-m+1/2}\rho \times \nu^{-m+3/2}\rho \times \cdots \times \nu^{m+1/2}\rho \rtimes \sigma)$ is one. This implies

$$\begin{aligned} &s_{GL}(\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma) + s_{GL}(\delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma)) \\ &\qquad \not\leq s_{GL}(\nu^{-m-1/2}\rho \times \nu^{-m+1/2}\rho \times \nu^{-m+3/2}\rho \times \cdots \times \nu^{m+1/2}\rho \rtimes \sigma). \end{aligned}$$

From Theorem 2.3 we get that the multiplicities of $\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$ in $s_{GL}(\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma)$ and $s_{GL}(\delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma))$ are two and one respectively. This implies

$$s_{GL}(\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma) \not\leq s_{GL}(\delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma)).$$

Now we can conclude reducibility from Remark 3.2.

We already mentioned the fact that the multiplicity of $\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$ in $s_{GL}(\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma)$ is two. Thus $\delta([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$ splits into two irreducible inequivalent representations by (b) of Lemma 3.8, since we have already proved reducibility. This finishes the proof. □

4.4. Proposition. *Suppose that $\nu\rho \rtimes \sigma$ or $\rho \rtimes \sigma$ reduces. Then $\delta([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma$ is a sum of two inequivalent irreducible representations.*

Proof. The proposition follows from Theorems 5.4 and 6.4 of [T8]. We sketch very briefly the proof here since it is very similar to the previous one. It is based also on the principle exposed in Lemma 3.1 and Remark 3.2. Suppose that $\nu\rho \rtimes \sigma$ reduces. One considers $\delta([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma$, $\delta([\rho, \nu^n\rho]) \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)$ and $\nu^{-n}\rho \times \nu^{-n+1}\rho \times \cdots \times \nu^n\rho \rtimes \sigma$. The multiplicities of $\delta([\rho, \nu^n\rho]) \times \delta([\nu\rho, \nu^n\rho]) \otimes \sigma$ in the Jacquet modules are now two. One proves the reducibility as above. Now suppose that $\rho \rtimes \sigma$ reduces. Write $\rho \rtimes \sigma = \tau_1 \oplus \tau_2$ as a sum of two irreducible representations. Consider $\delta([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma$, $\delta([\nu\rho, \nu^n\rho])^2 \rtimes \tau_1$ and $\nu^{-n}\rho \times \nu^{-n+1}\rho \times \cdots \times \nu^n\rho \rtimes \sigma$. The multiplicities of $\delta([\rho, \nu^n\rho]) \times \delta([\nu\rho, \nu^n\rho]) \otimes \sigma$ in the Jacquet modules are now 2, 1, 2, respectively. Again one proves the reducibility as in the previous proposition. □

4.5. Remark. Using the generalized Zelevinsky involution, Propositions 4.1-4.4 imply the dual result: with the same assumptions on ρ, σ, n and m as in the beginning of Theorem 4.1, we have:
(i) If $\nu\rho \rtimes \sigma$ or $\rho \rtimes \sigma$ reduces, then $\mathfrak{s}([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma$ is a sum of two inequivalent irreducible representations.
(ii) If $\nu^k\rho \rtimes \sigma$ is irreducible for any $k \in \mathbb{Z}$, then $\mathfrak{s}([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma$ is irreducible.
(iii) If $\nu^{1/2}\rho \rtimes \sigma$ reduces, then $\mathfrak{s}([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$ reduces into a sum of two inequivalent irreducible representations.
(iv) If $\nu^{1/2+k}\rho \rtimes \sigma$ is irreducible for any $k \in \mathbb{Z}$, then $\mathfrak{s}([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$ is irreducible.

5. On irreducibility of $\nu^\beta\rho \rtimes \delta(\nu^\beta\rho, \sigma)$ and $\nu^\beta\rho \rtimes L(\nu^\beta\rho, \sigma)$ $(\beta \in (1/2)\mathbb{Z},\ \beta \geq 1)$

In this section, and the following one, we shall prove irreducibility of a parabolically induced representations for which there exist coherent $\{\mathcal{P}\backslash\{S_n\}\}$-decompositions of Jacquet modules ($\mathcal{P}$ denotes the set of all standard parabolic subgroups in S_n). These two cases are the only cases of non-unitarizable irreducibilities considered in this paper, which can not be concluded proving non-existence of coherent $\{\mathcal{P}\backslash\{S_n\}\}$-decompositions of Jacquet modules. The existing coherent decompositions of Jacquet modules (in these two cases) play indirectly a role in proving irreducibility. The ideas used in the proofs are similar to those ones used in the third section, but slightly more sophisticated.

5.1. Proposition. *Let ρ be an irreducible unitarizable cuspidal representation of $GL(p, F)$ and let σ be an irreducible cuspidal representation of S_q. Suppose that $\beta > 1/2$ is in $(1/2)\,\mathbb{Z}$ and that $\nu^\beta\rho \rtimes \sigma$ reduces. Then $\nu\rho^\beta \rtimes \delta(\nu^\beta\rho, \sigma)$ and $\nu^\beta\rho \rtimes L(\nu^\beta\rho, \sigma)$ are irreducible.*

Proof. It is enough to prove that $\nu^\beta\rho \rtimes \delta(\nu^\beta\rho, \sigma)$ is irreducible (the irreducibility of the other representation follows using the generalized Zelevinsky involution). Suppose that the induced representation reduces. Note that

$$s_{GL}(\nu^\beta\rho \rtimes \delta(\nu^\beta\rho, \sigma)) = \nu^\beta\rho \times \nu^\beta\rho \otimes \sigma + \nu^{-\beta}\rho \times \nu^\beta\rho \otimes \sigma \tag{5-1}$$

by Theorem 2.3. Since (5-1) has length two, and since we have supposed reducibility of $\nu^\beta\rho \rtimes \delta(\nu^\beta\rho, \sigma)$, there exists a subquotient π of $\nu^\beta\rho \rtimes \delta(\nu^\beta\rho, \sigma)$ which satisfies

$$s_{GL}(\pi) = \nu^\beta\rho \times \nu^\beta\rho \otimes \sigma.$$

Now evidently

$$\delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \rtimes \pi \leq \delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \times \nu^{-\beta} \times \nu^\beta \rtimes \sigma, \tag{5-2}$$

$$\delta([\nu^{-\beta}\rho, \nu^{\beta}\rho]) \rtimes \sigma \leq \delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \times \nu^{-\beta} \times \nu^\beta \rtimes \sigma. \tag{5-3}$$

From Theorem 2.3 and (1-3) we get

$$\text{s.s.}(s_{GL}(\delta([\nu^{-\beta}\rho, \nu^{\beta}\rho]) \rtimes \sigma)) = \sum_{k=-\beta-1}^{\beta} \delta([\nu^{-k}\rho, \nu^{\beta}\rho]) \times \delta([\nu^{k+1}\rho, \nu^{\beta}\rho]) \otimes \sigma, \tag{5-4}$$

$$\begin{aligned} &\text{s.s.}(s_{GL}(\nu^\beta\rho \times \nu^{-\beta}\rho \times \delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \rtimes \sigma)) \\ &\quad = (\nu^\beta\rho + \nu^{-\beta}\rho) \times (\nu^\beta\rho + \nu^{-\beta}\rho) \times \sum_{k=-\beta}^{\beta-1} \delta([\nu^{-k}\rho, \nu^{\beta-1}\rho]) \times \delta([\nu^{k+1}\rho, \nu^{\beta-1}\rho]) \otimes \sigma, \end{aligned} \tag{5-5}$$

$$\begin{aligned} &\text{s.s.}(s_{GL}(\delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \rtimes \pi)) \\ &\quad = \nu^\beta\rho \times \nu^\beta\rho \times \sum_{k=-\beta}^{\beta-1} \delta([\nu^{-k}\rho, \nu^{\beta-1}\rho]) \times \delta([\nu^{k+1}\rho, \nu^{\beta-1}\rho]) \otimes \sigma. \end{aligned} \tag{5-6}$$

Suppose that τ is a subquotient of $\delta([\nu^{-\beta}\rho, \nu^{\beta}\rho]) \rtimes \sigma$. Since the later representation is unitarizable, Frobenius reciprocity implies

$$\delta([\nu^{-\beta}\rho, \nu^{\beta}\rho]) \otimes \sigma \leq s_{GL}(\tau). \tag{5-7}$$

Note that $\delta([\nu^{-\beta}\rho, \nu^{\beta}\rho]) \otimes \sigma$ is not a subquotient of (5-6) (observe that $\nu^{-\beta}\rho$ can not appear in the support of $\nu^\beta\rho \times \nu^\beta\rho \times \sum_{k=-\beta}^{\beta-1} \delta([\nu^{-k}\rho, \nu^{\beta-1}\rho]) \times \delta([\nu^{k+1}\rho, \nu^{\beta-1}\rho])$, which is on the right hand side of (5-6)). Therefore, if τ is a subquotient of $\delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \rtimes \pi$, then

$$\delta([\nu^{-\beta}\rho, \nu^{\beta}\rho]) \otimes \sigma \not\leq s_{GL}(\tau). \tag{5-8}$$

We shall discuss now two separate cases, although the principles of our thinking in both cases is the same.

Suppose that $\beta \in 1/2 + \mathbb{Z}$. Then the formulas (5-4), (5-5) and (5-6) imply that the multiplicity of $\delta([\nu^{1/2}\rho, \nu^{\beta}\rho]) \times \delta([\nu^{1/2}\rho, \nu^{\beta}\rho]) \otimes \sigma$ in each of $s_{GL}(\delta([\nu^{-\beta}\rho, \nu^{\beta}\rho]) \rtimes \sigma)$, $s_{GL}(\nu^\beta\rho \times \nu^{-\beta}\rho \times \delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \rtimes \sigma)$ and $s_{GL}(\delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \rtimes \pi)$ is 1. This, and inequalities (5-2) and (5-3) imply that there must exist a common irreducible subquotient τ of $\delta([\nu^{-\beta}\rho, \nu^{\beta}\rho]) \rtimes \sigma$ and $\delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \rtimes \pi$. Now (5-7) and (5-8) hold for the same τ. This is a contradiction which completes the proof in this case.

Suppose now that $\beta \in \mathbb{Z}$. Similarly as before, we see from the formulas (5-4), (5-5) and (5-6) that the multiplicity of $\delta([\rho, \nu^{\beta}\rho]) \times \delta([\nu\rho, \nu^{\beta}\rho]) \otimes \sigma$ in each of $s_{GL}(\delta([\nu^{-\beta}\rho, \nu^{\beta}\rho]) \rtimes \sigma)$, $s_{GL}(\nu^\beta\rho \times \nu^{-\beta}\rho \times \delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \rtimes \sigma)$ and $s_{GL}(\delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \rtimes \pi)$ is 2. Now we get a contradiction in the same way as in the first case (using (5-2) and (5-3); again must exist τ as above). This ends the proof. □

6. On irreducibility of $\delta([\rho,\nu\rho])\rtimes\sigma$ and $L(\rho,\nu\rho)\rtimes\sigma$

Let ρ be an irreducible selfcontragredient cuspidal representation of $GL(p,F)$ (selfcontragredient means that $\rho\cong\tilde\rho$). An irreducible cuspidal representation of S_q will be denoted by σ. In this section we shall assume that $\rho\rtimes\sigma$ and $\nu\rho\rtimes\sigma$ are irreducible. The following lemma follows directly from Proposition 4.2 and results in [Go] about R-groups of $Sp(n,F)$ and $SO(2n+1,F)$.

6.1. Lemma. *Suppose that char $F=0$. Then the representation $\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\sigma$ is irreducible.* □

6.2. Lemma. *The multiplicity of $\delta([\rho,\nu\rho])\times\delta([\rho,\nu\rho])\otimes\sigma$ in $\mu^*(\rho\times\rho\times\nu\rho\times\nu\rho\rtimes\sigma)$ is 4. It has the same multiplicity in $\mu^*(\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\sigma)$.*

Proof. Observe that

$$\text{s.s.}(s_{(4p)}((\rho\times\rho\times\nu\rho\times\nu\rho\rtimes\sigma)))=4\sum_{(\varepsilon_1,\varepsilon_2)\in\{\pm1\}^2}\rho\times\rho\times\nu^{\varepsilon_1}\rho\times\nu^{\varepsilon_2}\rho\otimes\sigma.$$

Use Theorem 1.1 to see the first claim of the lemma. The other claim follows from

$$\text{s.s.}(s_{(4p)}(\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\sigma))=4\rho\times\delta([\nu^{-1}\rho,\nu\rho])\otimes\sigma+4\rho\times\nu\rho\times\delta([\rho,\nu\rho])\otimes\sigma$$

and Theorem 1.1. □

6.3. Proposition. *Suppose that char $F=0$. If $\rho\rtimes\sigma$, $\nu\rho\rtimes\sigma$ are irreducible, then the representation $\delta([\rho,\nu\rho])\rtimes\sigma$ is irreducible.*

Proof. Suppose that we have a reduction. Write

$$\text{(6-1)}\quad \mu^*(\delta([\rho,\nu\rho])\rtimes\sigma)=1\otimes\delta([\rho,\nu\rho])\rtimes\sigma+ [\nu\rho\otimes\rho\rtimes\sigma+\rho\otimes\nu\rho\rtimes\sigma]+[2\delta([\rho,\nu\rho])+L((\rho,\nu\rho))+\delta([\nu^{-1}\rho,\rho])]\otimes\sigma].$$

This implies that there exists an irreducible subquotient π such that $s_{(p)}(\pi)=\nu\rho\otimes\rho\rtimes\sigma$. One sees directly that $\text{s.s.}(s_{(2p)}(\pi))=2\delta([\rho,\nu\rho])\otimes\sigma$. Now consider $\delta([\rho,\nu\rho])\rtimes\pi$. One gets that

$$\text{s.s.}(s_{(4p)}(\delta([\rho,\nu\rho])\rtimes\pi))=2\delta([\rho,\nu\rho])^2\otimes\sigma + 2\rho\times\nu\rho\times\delta([\rho,\nu\rho])\otimes\sigma+2\delta([\nu^{-1}\rho,\rho])\times\delta([\rho,\nu\rho])\otimes\sigma.$$

Since $4\delta([\rho,\nu\rho])^2\otimes\sigma\le\mu^*(\delta([\rho,\nu\rho])\rtimes\pi)$, we have by the preceding two lemmas $\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\sigma\le\delta([\rho,\nu\rho])\rtimes\pi$. This implies $s_{(4p)}(\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\sigma)\le s_{(4p)}(\delta([\rho,\nu\rho])\rtimes\pi)$, and furthermore,

$$4\rho\times\delta([\nu^{-1}\rho,\nu\rho])\otimes\sigma+4\rho\times\nu\rho\times\delta([\rho,\nu\rho])\otimes\sigma \le 2\delta([\rho,\nu\rho])^2\otimes\sigma+2\rho\times\nu\rho\times\delta([\rho,\nu\rho])\otimes\sigma+2\delta([\nu^{-1}\rho,\rho])\times\delta([\rho,\nu\rho])\otimes\sigma.$$

Looking at $\rho\times\delta([\nu^{-1}\rho,\nu\rho])\otimes\sigma$ we see that this cannot be the case. This completes the proof. □

Now the generalized Zelevinsky involution implies the following

6.4. Corollary. *Assume char $F=0$. If $\rho\rtimes\sigma$ and $\nu\rho\rtimes\sigma$ are irreducible, then the representation $L(\rho,\nu\rho)\rtimes\sigma$ is irreducible.* □

7. Reducibility points of some generalized principal series and generalized degenerate principal series representations (cuspidal reducibility at 1)

Reducibility and irreducibility results in the next three sections will be obtained on the basis of principles of Lemma 3.1 (and related Remark 3.2), and Lemma 3.7.

Since $\pi \rtimes \sigma \cong \tilde{\pi} \rtimes \sigma$ in $R(S)$ by Proposition 2.2, we shall consider only the case of $\alpha \geq 0$ in the theorems in this and the next section. From this case one can easily describe the case of $\alpha < 0$.

7.1. Theorem. *Suppose that ρ and ρ_0 are irreducible unitarizable cuspidal representations of $GL(p, F)$ and $GL(p_0, F)$ respectively. Let σ be an irreducible cuspidal representation of S_q. Assume that $\nu\rho \rtimes \sigma$ reduces. Let n be a positive integer and $\alpha \in \mathbb{R}, \alpha \geq 0$.*
(i) Suppose $\rho \ncong \rho_0$. Then $\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)$ reduces if and only if $\nu^\alpha\rho_0 \rtimes \sigma$ reduces. If $\rho_0 \rtimes \sigma$ reduces, then $\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)$ is a sum of two inequivalent irreducible tempered representations. If $\alpha > 0$ and $\nu^\alpha\rho_0 \rtimes \sigma$ reduces, then $\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)$ contains a unique square integrable subquotient, which we denote by $\delta(\nu^\alpha\rho_0, [\nu\rho, \nu^n\rho], \sigma)$. We then have $\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma) = \delta(\nu^\alpha\rho_0, [\nu\rho, \nu^n\rho], \sigma) + L(\nu^\alpha\rho_0, \delta([\nu\rho, \nu^n\rho], \sigma))$ in the Grothendieck group.
(ii) Suppose $\rho \cong \rho_0$ and suppose that $\nu^\alpha\rho \rtimes \sigma$ is irreducible whenever $\alpha \neq 1$ ($\alpha \geq 0$). Then $\nu^\alpha\rho \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)$ reduces if and only if $\alpha \in \{0, n+1\}$. The representation $\rho \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)$ is a sum of two inequivalent irreducible tempered representations. We have $\nu^{n+1}\rho \rtimes \delta([\nu\rho, \nu^n\rho], \sigma) = \delta([\nu\rho, \nu^{n+1}\rho], \sigma) + L(\nu^{n+1}\rho, \delta([\nu\rho, \nu^n\rho], \sigma))$ in the Grothendieck group.
(iii) If $\alpha > 0$ and $\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)$ is irreducible, then $\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma) \cong L(\nu^\alpha\rho_0, \delta([\nu\rho, \nu^n\rho], \sigma))$.

Proof. Theorem 2.3 and (2-1) imply

$$\mu^*(\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)) = \big((\nu^{-\alpha}\tilde{\rho}_0 \otimes 1 + \nu^\alpha\rho_0 \otimes 1) + 1 \otimes \nu^\alpha\rho_0\big) \rtimes \big(\sum_{k=0}^{n} \delta([\nu^{k+1}\rho, \nu^n\rho]) \otimes \delta([\nu\rho, \nu^k\rho], \sigma)\big)$$

We read directly from the above formula the Jacquet module of GL-type

$$\text{(7-1)}\quad \text{s.s.}(s_{GL}(\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma))) = \nu^{-\alpha}\tilde{\rho}_0 \times \delta([\nu\rho, \nu^n\rho]) \otimes \sigma + \nu^\alpha\rho_0 \times \delta([\nu\rho, \nu^n\rho]) \otimes \sigma.$$

We also see that

$$\text{(7-2)}\quad s_{(np)}(\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)) \geq \delta([\nu\rho, \nu^n\rho]) \otimes \nu^\alpha\rho_0 \rtimes \sigma,$$

$$\text{(7-3)}\quad s_{((n-1)p)}(\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)) \geq \delta([\nu^2\rho, \nu^n\rho]) \otimes \nu^\alpha\rho_0 \rtimes \delta(\nu\rho, \sigma).$$

Suppose $\rho \ncong \rho_0$.

From (7-1), Theorem 1.1 and Remark 3.5 we obtain that $\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)$ is a multiplicity one representation of length ≤ 2 for $\alpha > 0$. If $\alpha = 0$, then the above formula for

$\mu^*(\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma))$ implies that the multiplicity of $\rho_0 \otimes \delta([\nu\rho, \nu^n\rho], \sigma)$ in $s_{(p_0)}(\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma))$ is ≤ 2. Now (b) of Lemma 3.8 implies that we have a multiplicity one representation of length ≤ 2 also for $\alpha = 0$.

Suppose that $\nu^\alpha\rho_0 \rtimes \sigma$ reduces for some $\alpha > 0$. Looking at the Jacquet modules of GL-type, we can easily conclude that $\delta([\nu\rho, \nu^n\rho]) \rtimes \delta(\nu^\alpha\rho_0, \sigma)$ and $\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)$ have exactly one irreducible factor π in common and that $s_{GL}(\pi) = \nu^\alpha\rho_0 \times \delta([\nu\rho, \nu^n\rho]) \otimes \sigma$. Note that we are in the regular situation, i.e. all Jacquet modules of the full induced representation $\nu^\alpha\rho_0 \times \nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu\rho \rtimes \sigma$ ($= \nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu\rho \times \nu^\alpha\rho_0 \rtimes \sigma$ in $R(S)$) are multiplicity one representations. Because of this, it is very easy to analyze such situations (see for example [Ro1]). The Casselman square integrability criterion (Theorem 4.4.6 of [C2], see also the sixth section of [T5]) implies that π is square integrable. We denote π by $\delta(\nu^\alpha\rho_0, [\nu\rho, \nu^n\rho], \sigma)$. Now, clearly we have $\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma) = \delta(\nu^\alpha\rho_0, [\nu\rho, \nu^n\rho], \sigma) + L(\nu^\alpha\rho_0, \delta([\nu\rho, \nu^n\rho], \sigma))$ in $R(S)$. For more details regarding such regular situations, one can consult [T5].

Now suppose that $\rho_0 \rtimes \sigma$ reduces. Write $\rho_0 \rtimes \sigma = \tau_1 \oplus \tau_2$ as a sum of irreducible representations. Now the multiplicities of $\rho_0 \times \delta([\nu\rho, \nu^n\rho]) \otimes \sigma$ in $s_{GL}(\delta([\nu\rho, \nu^n\rho]) \rtimes \tau_1)$, $s_{GL}(\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma))$ and $s_{GL}(\rho_0 \times \nu\rho \times \nu^2\rho \times \cdots \times \nu^n\rho \rtimes \sigma)$ are 1, 2 and 2 respectively. Using Remark 3.2 we can conclude now the reducibility.

Now suppose that $\nu^\alpha\rho_0 \rtimes \sigma$ does not reduce. We shall apply Lemma 3.7 here. Denote $\tau'' = \delta([\nu\rho, \nu^n\rho]) \otimes \nu^\alpha\rho_0 \rtimes \sigma$, $P'' = P_{(np)}$, $P''' = P_{(np+p_0)}$ and $P' = P_{(p,p,\ldots,p,p_0)}$ where p appears n times in the last index. Further denote $\vartheta_+ = \nu^n\rho \otimes \nu^{n-1}\rho \otimes \cdots \otimes \nu\rho \otimes \nu^\alpha\rho_0 \otimes \sigma$ and $\vartheta_- = \nu^n\rho \otimes \nu^{n-1}\rho \otimes \cdots \otimes \nu\rho \otimes \nu^{-\alpha}\tilde{\rho}_0 \otimes \sigma$. Then one sees directly from (1-5) and Theorem 2.3 that

$$\vartheta_+ + \vartheta_- \leq (r_{(p)^n} \otimes s_{(p_0)})(\tau''). \tag{7-4}$$

Suppose $\nu^{-\alpha}\tilde{\rho}_0 \not\cong \nu^\alpha\rho_0$. Now multiplicities of ϑ_+ in

$$\begin{aligned} &r_{(p,p,\ldots,p,p_0)}(\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho]) \otimes \sigma)\\ &r_{(p,p,\ldots,p,p_0)}(\nu^{-\alpha}\tilde{\rho}_0 \rtimes \delta([\nu\rho, \nu^n\rho]) \otimes \sigma)\\ &s_{(p,p,\ldots,p,p_0)}(\nu^\alpha\rho_0 \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)) \end{aligned} \tag{7-5}$$

are 1, 0, 1 respectively, while the multiplicities of ϑ_- are 0, 1, 1 respectively. One uses (1-3), (1-5), and the structure of Hopf algebra on R to get this (more precisely, only the Hopf axiom is necessary). From (7-2), (7-4) and above multiplicities, we can conclude that the conditions of Lemma 3.7 hold. Therefore we have irreducibility in this case. Consider the remaining case: $\nu^{-\alpha}\tilde{\rho}_0 \cong \nu^\alpha\rho_0$, i.e. $\alpha = 0$ and $\rho_0 \cong \tilde{\rho}_0$. Then $\vartheta_+ \cong \vartheta_-$. The multiplicities of ϑ_+ in (7-5) are now 1, 1, 2 respectively. From this (and (7-2) and (7-4)), we conclude again irreducibility using Lemma 3.7..

Now suppose that $\rho_0 \cong \rho$.

Take $\alpha \geq 0$, $\alpha \notin \{0, 1, n+1\}$. Then one concludes the irreducibility from Lemma 3.7 in the same way as before taking $\tau'' = \delta([\nu\rho, \nu^n\rho]) \otimes \nu^\alpha\rho \rtimes \sigma$. We now consider the case $\alpha = 1$. Note that $\nu\rho \rtimes \delta(\nu\rho, \sigma)$ is irreducible by Proposition 3.1. One now gets irreducibility for $n > 1$ from Lemma 3.7 in a similar way as before, taking $\tau'' = \delta([\nu^2\rho, \nu^n\rho]) \otimes \nu\rho \rtimes \delta(\nu\rho, \sigma)$ and using (7-3).

For $\alpha = 0$, using an argument similar to that which we used before, we get that $\rho \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)$ is a multiplicity one representation of length ≤ 2. The multiplicities of $\delta([\rho, \nu^n\rho]) \otimes \sigma$ in $s_{GL}(\rho \rtimes \delta([\nu\rho, \nu^n\rho], \sigma))$, $s_{GL}(\delta([\rho, \nu^n\rho]) \rtimes \sigma)$, and $s_{GL}(\rho \times \nu\rho \times \cdots \times \nu^n\rho \rtimes \sigma)$ are all equal to two (use Theorems 2.3 and 1.1). Further one can easily obtain $s_{GL}(\rho \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)) \not\leq s_{GL}(\delta([\rho, \nu^n\rho]) \rtimes \sigma)$ again using Theorems 2.3 and 1.1. Remark 3.2 now implies the reducibility.

Now take $\alpha = n+1$. One gets easily that $\delta([\nu\rho, \nu^{n+1}\rho], \sigma) < \nu^{n+1}\rho \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)$ (use (ii) of Proposition 2.2 to get $\leq$, and Theorem 2.3 to get $\neq$ on the level of Jacquet modules of GL-type, what together implies the above strict inequality). Thus $\nu^{n+1}\rho \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)$ is reducible. We only need to check that the length is two. Note that the length of $s_{GL}(\nu^{n+1}\rho \rtimes \delta([\nu\rho, \nu^n\rho], \sigma))$ is three (Theorem 1.1). Take an irreducible subquotient π of $\nu^{n+1}\rho \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)$ such that $\delta([\nu\rho, \nu^n\rho]) \otimes \nu^{n+1}\rho \rtimes \sigma \leq \mu^*(\pi)$. One can get easily that the length of $s_{GL}(\pi)$ is ≥ 2. The argument is of similar type as in the proof of Lemma 3.7, although slightly more complicated. One shows here that there exist two different subquotients $\pi_1 \otimes \sigma$ and $\pi_2 \otimes \sigma$ of $s_{GL}(\nu^{n+1}\rho \rtimes \delta([\nu\rho, \nu^n\rho], \sigma))$ such that

$$r_{(p)^{n+1}}(\pi_i) \otimes \sigma + (r_{(p)^n} \otimes s_{(p)})(\delta([\nu\rho, \nu^n\rho]) \otimes \nu^{n+1}\rho \rtimes \sigma) \not\leq s_{(p)^{n+1}}(\nu^{n+1}\rho \rtimes \delta([\nu\rho, \nu^n\rho], \sigma))$$

for $i = 1, 2$. Now Remark 3.5 implies that the length of $\nu^{n+1}\rho \rtimes \delta([\nu\rho, \nu^n\rho], \sigma)$ is ≤ 2. This finishes the proof. $\square$

In the following theorem we shall compute Langlands parameters of irreducible subquotients of parabolically induced representations. We shall do it usually in one of the following two ways (suppose that $\mathrm{Ind}_{P_0}^G(\sigma_0)$ is some representation that we shall consider). In the simpler case, we shall construct a non-trivial intertwining $\mathrm{Ind}_P^G(\tau) \to \mathrm{Ind}_{P_0}^G(\sigma_0)$ using (1-2) and (iii) of Proposition 2.2, and $\mathrm{Ind}_P^G(\tau)$ will give a Langlands parameter. In the other case we shall have a surjective intertwining, say $\psi : \mathrm{Ind}_P^G(\pi) \twoheadrightarrow \mathrm{Ind}_{P_0}^G(\sigma_0)$ (again obtained with help of (1-2) and (iii) of Proposition 2.2) and $\mathrm{Ind}_{P'}^G(\tau') \hookrightarrow \mathrm{Ind}_P^G(\pi)$, where $\mathrm{Ind}_{P'}^G(\tau')$ will give a Langlands parameter if ψ is non-trivial on $\mathrm{Ind}_{P'}^G(\tau')$. To see this non-triviality, it will be enough to prove that

$$r_{M''}^G(\mathrm{Ind}_{P_0}^G(\sigma_0)) \not\leq r_{M''}^G(\mathrm{Ind}_P^G(\pi)) - r_{M''}^G(\mathrm{Ind}_{P'}^G(\tau')) \tag{7-6}$$

for some parabolic subgroup $P'' = M''N''$ of G.

This was only a very brief description of the ideas.

7.2. Theorem. *Let ρ, ρ_0, n and α be as in Theorem 7.1.*
(i) Suppose $\rho \not\cong \rho_0$. Then $\nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma)$ reduces if and only if $\nu^\alpha\rho_0 \rtimes \sigma$ reduces. If $\nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma)$ reduces for some $\alpha > 0$, then we have in the Grothendieck group

$$\nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma) = L(\nu^\alpha\rho_0, \nu\rho, \nu^2\rho, \ldots, \nu^n\rho, \sigma) + L(\nu\rho, \nu^2\rho, \ldots, \nu^n\rho, \delta(\nu^\alpha\rho_0, \sigma)).$$

If $\alpha = 0$, decompose $\rho_0 \rtimes \sigma = \overset{k}{\underset{i=1}{\oplus}} \tau_i$ into a sum of irreducible representations ($k \in \{1, 2\}$). Then $\rho_0 \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma) = \overset{k}{\underset{i=1}{\oplus}} L(\nu\rho, \nu^2\rho, \ldots, \nu^n\rho, \tau_i)$.

(ii) Suppose that $\rho_0 \cong \rho$ and suppose that $\nu^\alpha\rho \rtimes \sigma$ is irreducible for $\alpha \neq 1$ (we assume $\alpha \geq 0$). Then $\nu^\alpha\rho \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma)$ reduces if and only if $\alpha \in \{0, n+1\}$. We have

$$\nu^{n+1}\rho \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma) = L(\nu\rho,\nu^2\rho,\dots,\nu^{n+1}\rho,\sigma) + L(\nu\rho,\dots,\nu^{n-1}\rho,\delta([\nu^n\rho,\nu^{n+1}\rho]),\sigma),$$

$$\rho \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma) = L(\nu\rho,\nu^2\rho,\dots,\nu^{n+1}\rho,\rho\rtimes\sigma) \oplus L(\delta([\rho,\nu\rho]),\nu^2\rho,\dots,\nu^{n+1}\rho,\sigma).$$

The first equality holds in the Grothendieck group only.
(iii) If $\alpha > 0$ and $\nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma)$ is irreducible, then $\nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma) = L(\nu^\alpha\rho_0,\nu\rho,\nu^2\rho,\dots,\nu^n\rho,\sigma)$.

Proof. The reducibility points and lengths follow from Theorem 7.1, using the generalized Zelevinsky involution. We only need to prove the description of irreducible subquotients (in fact, we shall also prove the reducibilities claimed in the theorem, since we shall find in these cases always Langlands parameters of two non-isomorphic subquotients).

Suppose that $\rho \not\cong \rho_0$. Then we have an epimorphism

$$\nu^\alpha\rho_0 \times \nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu\rho \rtimes \sigma \to \nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma). \tag{7-7}$$

Since $\nu^\alpha\rho_0 \times \nu^k\rho \cong \nu^k\rho \times \nu^\alpha\rho_0$, we get that $L(\nu^\alpha\rho_0,\nu^n\rho,\nu^{n-1}\rho,\dots,\nu\rho,\sigma) \leq \nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma)$ for $\alpha > 0$. If $\alpha = 0$ and $\rho_0 \rtimes \sigma$ is irreducible, we get in a similar way $L(\nu^n\rho,\nu^{n-1}\rho,\dots,\nu\rho,\rho_0\rtimes\sigma) \leq \rho_0 \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma)$. We have equality here.

Suppose that $\rho_0 \rtimes \sigma$ reduces and write $\rho_0 \rtimes \sigma = \tau_1 \oplus \tau_2$. Then the restriction of (7-7) gives intertwinings $\varphi_i : \nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu\rho \rtimes \tau_i \to \rho \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma)$. Suppose that some $\varphi_i = 0$. Since $\varphi_1 \oplus \varphi_2$ is an epimorphism (7-7), we have $\rho_0 \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma) \leq \nu^n\rho \times \cdots \times \nu\rho \rtimes \tau_{3-i}$. Looking at the GL-type Jacquet module, we see that this cannot happen. Thus $L(\nu^n\rho,\nu^{n-1}\rho,\dots,\nu\rho,\tau_i) \leq \rho_0 \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma)$ for $i = 1,2$.

Now suppose that $\alpha > 0$ and that $\nu^\alpha\rho_0 \rtimes \sigma$ reduces. Restricting (7-7), we get an intertwining $\varphi : \nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu\rho \times \delta(\nu^\alpha\rho_0,\sigma) \to \nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma)$. Suppose $\varphi = 0$. Then there is an epimorphism $\nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu\rho \rtimes L(\nu^\alpha\rho_0,\sigma) \to \nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma)$. Looking at the GL-type Jacquet modules, we see that this is impossible. Thus $L(\nu^n\rho,\nu^{n-1}\rho,\dots,\nu\rho,\delta(\nu^\alpha\rho_0,\sigma)) \leq \nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma)$.

Now suppose that $\rho_0 \cong \rho$. First consider the case $\alpha = n+1$. Clearly

$$L(\nu^{n+1}\rho,\nu^n\rho,\nu^{n-1}\rho,\dots,\nu\rho,\sigma) \leq \nu^{n+1}\rho \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma).$$

Now consider the restriction of (7-7) to

$$\delta([\nu^n\rho,\nu^{n+1}\rho]) \times \nu^{n-1}\rho \times \nu^{n-2}\rho \times \cdots \times \nu\rho \rtimes \sigma \to \nu^{n+1}\rho \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma).$$

Suppose that it is zero. Then there exists an epimorphism

$$L(\nu^n\rho,\nu^{n+1}\rho) \times \nu^{n-1}\rho \times \nu^{n-2}\rho \times \cdots \times \nu\rho \rtimes \sigma \to \nu^{n+1}\rho \rtimes \mathfrak{s}([\nu\rho,\nu^n\rho],\sigma).$$

This implies

$$\begin{aligned}&(L(\nu^{-n-1}\rho,\nu^{-n}\rho) + \nu^{-n-1}\rho \times \nu^n\rho + L(\nu^n\rho,\nu^{n+1}\rho))\times \\ &\qquad(\nu^{-n+1}\rho + \nu^{n-1}\rho) \times (\nu^{-n+2}\rho + \nu^{n-2}\rho) \times \cdots \times (\nu\rho + \nu^{-1}\rho) \otimes \sigma \\ &\qquad\qquad \geq (\nu^{-n-1}\rho + \nu^{n+1}\rho) \times \mathfrak{s}([\nu^{-n}\rho,\nu^{-1}\rho]) \otimes \sigma.\end{aligned}$$

Further, we must have

$$L(\nu^{-n-1}\rho, \nu^{-n}\rho) \times \nu^{-n+1}\rho \times \cdots \times \nu^{-1}\rho \otimes \sigma \geq \nu^{-n-1}\rho \times \mathfrak{s}([\nu^{-n}\rho, \nu^{-1}\rho]) \otimes \sigma.$$

But this can not hold by Theorem 1.1. Thus $L(\delta([\nu^n\rho, \nu^{n+1}\rho]), \nu^{n-1}\rho, \cdots, \nu\rho, \sigma) \leq \nu^{n+1}\rho \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma)$.

Now consider the case $\alpha = 0$. There is an epimorphism

$$\nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu^2\rho \times \rho \times \nu\rho \rtimes \sigma \to \rho \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma). \tag{7-8}$$

Consider the restriction $\varphi : \nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu^2\rho \times L(\rho, \nu\rho) \rtimes \sigma \to \rho \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma)$. Suppose $\varphi = 0$. Then $\rho \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma) \leq \nu^n\rho \times \cdots \times \nu^2\rho \times \delta([\rho, \nu\rho]) \rtimes \sigma$. Thus

$$\begin{aligned} &2\rho \times \mathfrak{s}([\nu^{-n}\rho, \nu^{-1}\rho]) \otimes \sigma \\ &\qquad \leq (\nu^{-n}\rho + \nu^n\rho) \times \cdots \times (\nu^{-2}\rho + \nu^2\rho) \times (\delta([\nu^{-1}\rho, \rho]) + \rho \times \nu\rho + \delta([\rho, \nu\rho])) \otimes \sigma. \end{aligned}$$

This implies $2\rho \times \mathfrak{s}([\nu^{-n}\rho, \nu^{-1}\rho]) \leq \nu^{-n}\rho \times \cdots \times \nu^{-2}\rho \times \delta([\nu^{-1}\rho, \rho])$. This cannot hold. Thus $L(\nu^n\rho, \nu^{n-1}\rho, \ldots, \nu^2\rho, \nu\rho, \rho \rtimes \sigma) \leq \rho \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma)$. Now consider the natural epimorphism $\psi : \nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu^2\rho \times ((\rho \times \nu\rho)/L(\rho, \nu\rho)) \rtimes \sigma \twoheadrightarrow (\rho \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma))/\mathrm{Im}\varphi$. (note that $(\rho \times \nu\rho)/L(\rho, \nu\rho) \cong \delta([\rho, \nu\rho])$). Suppose $\psi = 0$. Then φ must be an epimorphism. Therefore $\rho \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma) \leq \nu^n\rho \times \cdots \times \nu^2\rho \times L(\rho, \nu\rho) \rtimes \sigma$. On the level of GL-type Jacquet modules, we get

$$\begin{aligned} &2\rho \times \mathfrak{s}([\nu^{-n}\rho, \nu^{-1}\rho]) \otimes \sigma \leq \\ &\qquad (\nu^{-n}\rho + \nu^n\rho) \times \cdots \times (\nu^{-2}\rho + \nu^2\rho) \times (L(\nu^{-1}\rho, \rho) + \nu^{-1}\rho \times \rho + L(\rho, \nu\rho)) \otimes \sigma. \end{aligned}$$

Thus $2\rho \times \mathfrak{s}([\nu^{-n}\rho, \nu^{-1}\rho]) \leq \nu^{-n}\rho \times \cdots \times \nu^{-2}\rho \times (L(\nu^{-1}\rho, \rho) + \nu^{-1}\rho \times \rho)$. Theorem 1.1 implies that this is not possible. Thus $L(\nu^n\rho, \nu^{n-1}\rho, \ldots, \nu^2\rho, \delta([\rho, \nu\rho]), \sigma) \leq \rho \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma)$.

If $\alpha \notin \{0, 1, \ldots, n, n+1\}$, then we get directly as in the first part of the proof $\nu^\alpha\rho \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma) = L(\nu^\alpha\rho, \nu^n\rho, \nu^{n-1}\rho, \ldots, \nu\rho, \sigma)$. For $\alpha = n$ the statement is obvious. Now suppose $\alpha = k \in \{1, \ldots, n-1\}$. Then we have epimorphisms

$$\nu^k\rho \times \mathfrak{s}([\nu^k\rho, \nu^n\rho]) \times \mathfrak{s}([\nu\rho, \nu^{k-1}\rho]) \rtimes \sigma \twoheadrightarrow \nu^k\rho \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho]) \rtimes \sigma \twoheadrightarrow \nu^k\rho \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma).$$

Using that $\nu^k\rho \times \mathfrak{s}([\nu^k\rho, \nu^n\rho]) \cong \mathfrak{s}([\nu^k\rho, \nu^n\rho]) \times \nu^k\rho$, we get that there exists an epimorphism

$$\begin{aligned} &\nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu^{k+1}\rho \times \nu^k\rho \times \nu^k\rho \times \nu^{k-1}\rho \times \nu^{k-2}\rho \times \cdots \times \nu\rho \rtimes \sigma \\ &\qquad\qquad \twoheadrightarrow \nu^k\rho \rtimes \mathfrak{s}([\nu\rho, \nu^n\rho], \sigma). \end{aligned}$$

This completes the proof. □

8. Reducibility points of some generalized principal series and generalized degenerate principal series representations (cuspidal reducibility at 1/2)

Suppose that ρ is an irreducible unitarizable cuspidal representations of $GL(p,F)$. Let σ be an irreducible cuspidal representation of S_q. Now suppose that $\nu^{1/2}\rho \rtimes \sigma$ reduces. We know from Proposition 4.3 that $\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]) \rtimes \sigma$ is a direct sum of two irreducible representations. Since $s_{GL}(\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]) \rtimes \sigma) = 2\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]) \otimes \sigma + \nu^{1/2}\rho \times \nu^{1/2}\rho \otimes \sigma$, Frobenius reciprocity (F-R) implies that the irreducible subrepresentations, say τ_1 and τ_2, satisfy $s_{GL}(\tau_1) = \delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]) \otimes \sigma + \nu^{1/2}\rho \times \nu^{1/2}\rho \otimes \sigma$ and $s_{GL}(\tau_2) = \delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]) \otimes \sigma$ (see [T8] for much more general construction of this type). We denote τ_1 by $\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\sigma)$ and τ_2 by $\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_-,\sigma)$. Note that $\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\sigma)$ can be characterized as the irreducible subquotient of $\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]) \rtimes \sigma$ whose Jacquet module of GL-type is reducible.

8.1. Theorem. *Suppose that ρ and ρ_0 are irreducible unitarizable cuspidal representations of $GL(p,F)$ and $GL(p_0,F)$ respectively. Let σ be an irreducible cuspidal representation of S_q. Assume that $\nu^{1/2}\rho \rtimes \sigma$ reduces. Let m be a non-negative integer and let $\alpha \in \mathbb{R}, \alpha \geq 0$.*

(i) Suppose $\rho \not\cong \rho_0$. Then $\nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu^{1/2}\rho,\nu^{1/2+m}\rho],\sigma)$ reduces if and only if $\nu^\alpha\rho_0 \rtimes \sigma$ reduces. If $\nu^\alpha\rho_0 \rtimes \sigma$ reduces for some $\alpha > 0$, then in the Grothendieck group, we have

$$\nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu^{1/2}\rho,\nu^{1/2+m}\rho],\sigma) = L(\nu^\alpha\rho_0,\nu^{1/2}\rho,\nu^{3/2}\rho,\dots,\nu^{1/2+m}\rho,\sigma) + L(\nu^{1/2}\rho,\nu^{3/2}\rho,\dots,\nu^{1/2+m}\rho,\delta(\nu^\alpha\rho_0,\sigma)).$$

If $\alpha = 0$, write $\rho_0 \rtimes \sigma = \bigoplus_{i=1}^{k}\tau_i$ as a sum of irreducible representations ($k \in \{1,2\}$). Then

$$\rho_0 \rtimes \mathfrak{s}([\nu^{1/2}\rho,\nu^{1/2+m}\rho],\sigma) = \bigoplus_{i=1}^{k} L(\nu^{1/2}\rho,\nu^{3/2}\rho,\dots,\nu^{1/2+m}\rho,\tau_i).$$

(ii) Suppose that $\rho_0 \cong \rho$ and suppose that $\nu^\alpha\rho \rtimes \sigma$ is irreducible for $\alpha \neq 1/2$ (we assume $\alpha \geq 0$). Then $\nu^\alpha\rho \rtimes \mathfrak{s}([\nu^{1/2}\rho,\nu^{1/2+m}\rho],\sigma)$ reduces if and only if $\alpha \in \{1/2, m+3/2\}$. In the Grothendieck group, we have

$$\nu^{m+3/2}\rho \rtimes \mathfrak{s}([\nu^{1/2}\rho,\nu^{1/2+m}\rho],\sigma) = \mathfrak{s}([\nu^{1/2}\rho,\nu^{3/2+m}\rho],\sigma) + L(\nu^{1/2}\rho,\nu^{3/2}\rho,\dots,\nu^{m-1/2}\rho,\delta([\nu^{m+1/2}\rho,\nu^{m+3/2}\rho]),\sigma),$$

$$\nu^{1/2}\rho \rtimes \mathfrak{s}([\nu^{1/2}\rho,\nu^{1/2+m}\rho],\sigma) = L(\nu^{1/2}\rho,\nu^{1/2}\rho,\nu^{3/2}\rho,\nu^{5/2}\rho,\dots,\nu^{1/2+m}\rho,\sigma) + L(\nu^{3/2}\rho,\nu^{5/2}\rho,\dots,\nu^{1/2+m}\rho,\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_-,\sigma)).$$

(iii) If $\alpha > 0$ and $\nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu^{1/2}\rho,\nu^{1/2+m}\rho],\sigma)$ is irreducible, then

$$\nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu^{1/2}\rho,\nu^{1/2+m}\rho],\sigma) = L(\nu^\alpha\rho_0,\nu^{1/2}\rho,\nu^{3/2}\rho,\dots,\nu^{1/2+m}\rho,\sigma).$$

Proof. Theorem 2.3 and (2-2) imply $\mu^*(\nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma)) =$

$$\big((\nu^{-\alpha}\tilde{\rho}_0\otimes 1+\nu^\alpha\rho_0\otimes 1)+1\otimes\nu^\alpha\rho_0\big) = \rtimes \sum_{k=-1}^{m} \mathfrak{s}([\nu^{-m-1/2}\rho, \nu^{-k-3/2}\rho])\otimes\mathfrak{s}([\nu^{1/2}\rho, \nu^{1/2+k}\rho], \sigma).$$

In particular,

$$\text{(8-1)}\qquad \text{s.s.}(s_{GL}(\nu^\alpha\rho_0 \rtimes \mathfrak{s}([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma))) = \nu^{-\alpha}\tilde{\rho}_0 \times \mathfrak{s}([\nu^{-m-1/2}\rho, \nu^{-1/2}\rho]) \otimes \sigma + \nu^\alpha\rho_0 \times \mathfrak{s}([\nu^{-m-1/2}\rho, \nu^{-1/2}\rho]) \otimes \sigma.$$

The proof of (i) is just a simple modification of the proof of (i) of Theorem 7.2.

Suppose $\rho_0 \cong \rho$. The proof of irreducibility for $\alpha \notin \{1/2, m+3/2\}$ is analogous to the proof in the preceding theorem. For $\alpha = m+3/2$, one gets that $\mathfrak{s}([\nu^{1/2}\rho, \nu^{m+3/2}\rho], \sigma)$ and $L(\nu^{1/2}\rho, \dots, \nu^{m-1/2}\rho, \delta([\nu^{m+1/2}\rho, \nu^{m+3/2}\rho], \sigma))$ are $\leq \nu^{m+1/2}\rho\rtimes\mathfrak{s}([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma)$ using a similar argument to that used earlier. Similarly, one gets that the length is ≤ 2. For a complete proof of (ii), it remains to consider the case $\alpha = 1/2$. First, we have an epimorphism

$$\nu^{1/2}\rho \times \mathfrak{s}([\nu^{1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma \twoheadrightarrow \nu^{1/2}\rho \rtimes \mathfrak{s}([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma),$$

and further

$$\nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{3/2}\rho \times \nu^{1/2}\rho \times \nu^{1/2}\rho \rtimes \sigma \twoheadrightarrow \nu^{1/2}\rho \rtimes \mathfrak{s}([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma).$$

Thus $L(\nu^{1/2}\rho, \nu^{1/2}\rho, \nu^{3/2}\rho, \nu^{5/2}\rho, \dots, \nu^{m+1/2}\rho, \sigma) \leq \nu^{1/2}\rho \rtimes \mathfrak{s}([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma)$. Furthermore, we have an epimorphism

$$\nu^{-1/2}\rho \times \nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{3/2}\rho \times \nu^{1/2}\rho \rtimes \sigma \twoheadrightarrow \nu^{-1/2}\rho \rtimes \mathfrak{s}([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma).$$

Therefore, we have an epimorphism

$$\nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{3/2}\rho \times \nu^{-1/2}\rho \times \nu^{1/2}\rho \rtimes \sigma \twoheadrightarrow \nu^{-1/2}\rho \rtimes \mathfrak{s}([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma).$$

Consider the restriction φ of the above epimorphism to

$$\nu^{m+1/2}\rho \times \nu^{m-1/2}\rho \times \cdots \times \nu^{3/2}\rho \times L(\nu^{-1/2}\rho, \nu^{1/2}\rho) \rtimes \sigma.$$

Suppose that φ is an epimorphism. Then

$$(\nu^{1/2}\rho + \nu^{-1/2}\rho) \times \mathfrak{s}([\nu^{-m-1/2}\rho, \nu^{-1/2}\rho]) \otimes \sigma \leq (\nu^{-m-1/2}\rho + \nu^{m+1/2}\rho) \times (\nu^{-m+1/2}\rho + \nu^{m-1/2}\rho) \times \cdots \times (\nu^{-3/2}\rho + \nu^{3/2}\rho) \times (2L(\nu^{-1/2}\rho, \nu^{1/2}\rho) + \nu^{-1/2}\rho \times \nu^{-1/2}\rho) \otimes \sigma.$$

This cannot hold (use Theorem 1.1 to see that $Z([\nu^{-m-1/2}\rho, \nu^{-1/2}\rho], \nu^{1/2}\rho) \otimes \sigma$ is a subquotient of the left hand side but not of the right hand side; note that $L(\nu^{-1/2}\rho, \nu^{1/2}\rho) =$

$\mathfrak{s}([\nu^{-1/2}\rho,\nu^{1/2}\rho]))$. Thus φ is not an epimorphism. Therefore we have a non-trivial intertwining (moreover, an epimorphism)

$$\psi:\nu^{m+1/2}\rho\times\nu^{m-1/2}\rho\times\cdots\times\nu^{3/2}\rho\times\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho])\rtimes\sigma \\ \to(\nu^{-1/2}\rho\rtimes\mathfrak{s}([\nu^{1/2}\rho,\nu^{m+1/2}\rho],\sigma))/\mathrm{Im}\varphi.$$

Recall that $\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho])\rtimes\sigma=\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\sigma)\oplus\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_-,\sigma)$. Suppose that ψ is nontrivial on $\nu^{m+1/2}\rho\times\nu^{m-1/2}\rho\times\cdots\times\nu^{3/2}\rho\rtimes\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\sigma)$.

At this point we need an information about Jacquet modules of the Langlands quotient $L(\nu^{m+1/2}\rho,\cdots,\nu^{3/2}\rho,\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\tilde{\sigma}))$. Since there exists an epimorphism

$$\nu^{m+1/2}\rho\times\cdots\times\nu^{3/2}\rho\rtimes\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\tilde{\sigma}) \\ \twoheadrightarrow L(\nu^{m+1/2}\rho,\cdots,\nu^{3/2}\rho,\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\tilde{\sigma})),$$

there exists an embedding

$$L(\nu^{m+1/2}\rho,\dots,\nu^{3/2}\rho,\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\sigma))\hookrightarrow \\ \nu^{-m-1/2}\rho\times\nu^{-m+1/2}\rho\times\cdots\times\nu^{-3/2}\rho\rtimes\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\sigma) \tag{8-2}$$

We have used above the formula for the contragredient in the Langlands classification (see [T5]) and the fact that $\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\tilde{\sigma})\tilde{}\cong\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\sigma)$. The last isomorphism follows from the fact that

$$\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\tilde{\sigma})\tilde{}\hookrightarrow\delta([\nu^{-1/2}\tilde{\rho},\nu^{1/2}\tilde{\rho}])\rtimes\sigma\cong\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho])\rtimes\sigma$$

and the fact that $s_{GL}(\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\tilde{\sigma})\tilde{})$ is reducible, what one can conclude from Corollary 4.2.5 of [C2] (see [T8] for arguments of such type in a more complex situations). Frobenius reciprocity (F-R) and existence of non-trivial intertwining (8-2) imply that $\nu^{-m-1/2}\rho\otimes\nu^{-m+1/2}\rho\otimes\cdots\otimes\nu^{-3/2}\rho\otimes\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\sigma)$ is a quotient of a suitable Jacquet module of $L(\nu^{m+1/2}\rho,\dots,\nu^{3/2}\rho,\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\sigma))$. Further from $s_{GL}(\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\sigma))$ we see that $\nu^{-m-1/2}\rho\otimes\nu^{-m+1/2}\rho\otimes\cdots\otimes\nu^{-3/2}\rho\otimes\nu^{1/2}\rho\otimes\nu^{1/2}\rho\otimes\sigma$ is also a subquotient of a suitable Jacquet module of the same representation.

Since we have supposed that ψ is non-trivial on $\nu^{m+1/2}\rho\times\nu^{m-1/2}\rho\times\cdots\times\nu^{3/2}\rho\rtimes\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_+,\sigma)$, $\nu^{-m-1/2}\rho\otimes\nu^{-m+1/2}\rho\otimes\cdots\otimes\nu^{-3/2}\rho\otimes\nu^{1/2}\rho\otimes\nu^{1/2}\rho\otimes\sigma$ must be a subquotient of a suitable Jacquet module of $\nu^{-1/2}\rho\rtimes\mathfrak{s}([\nu^{1/2}\rho,\nu^{m+1/2}\rho],\sigma)$. From (8-1), we see that this cannot be the case (use the Hopf algebra structure on R, (1-4) and (1-6)). This contradiction proves

$$L(\nu^{3/2}\rho,\nu^{5/2}\rho,\dots,\nu^{m+1/2}\rho,\delta([\nu^{-1/2}\rho,\nu^{1/2}\rho]_-,\sigma))\le\nu^{1/2}\rho\rtimes\mathfrak{s}([\nu^{1/2}\rho,\nu^{m+1/2}\rho],\sigma).$$

For a complete proof, we need to prove that the length of $\nu^{1/2}\rho\rtimes\mathfrak{s}([\nu^{1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ is two. For this, it is enough to prove that the length is ≤ 2. From (8-1) (and Remark 3.5 and Theorem 1.1), we see that the length is ≤ 3. To prove that the length is ≤ 2, it is enough to show that there does not exist a subquotient π of $\nu^{1/2}\rho \rtimes \mathfrak{s}([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma)$ with

$$s_{GL}(\pi) = \nu^{-1/2}\rho \times \mathfrak{s}([\nu^{-1/2}\rho, \nu^{-m-1/2}\rho]) \otimes \sigma.$$

Suppose that such a π exists. First, the multiplicity of $\mathfrak{s}([\nu^{-1/2}\rho, \nu^{-m-1/2}\rho])^2 \otimes \sigma$ in

$$s_{GL}(\nu^{-m-1/2}\rho \times \nu^{-m+1/2}\rho \times \cdots \times \nu^{m+1/2}\rho \rtimes \sigma)$$

is one (use Theorems 2.3 and 1.1). Let τ denote the irreducible subquotient which contains $\mathfrak{s}([\nu^{-1/2}\rho, \nu^{-m-1/2}\rho])^2 \otimes \sigma$ as a subquotient in its Jacquet module. Then τ has multiplicity one in $\nu^{-m-1/2}\rho \times \nu^{-m+1/2}\rho \times \cdots \times \nu^{m+1/2}\rho \rtimes \sigma$. Note that

$$\mathfrak{s}([\nu^{-1/2}\rho, \nu^{-m-1/2}\rho])^2 \otimes \sigma \leq s_{GL}(\mathfrak{s}([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma)$$

(use Theorem 2.3). Thus $\tau \leq \mathfrak{s}([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$. Frobenius reciprocity implies

$$\mathfrak{s}([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \leq s_{GL}(\tau).$$

Consider $\vartheta = \mathfrak{s}([\nu^{-m-1/2}\rho, \nu^{-3/2}\rho]) \rtimes \pi$. Then $\mathfrak{s}([\nu^{-m-1/2}\rho, \nu^{-1/2}\rho])^2 \otimes \sigma \leq s_{GL}(\vartheta)$. Thus $\tau \leq \vartheta$. But one gets directly from Theorem 2.3 that $\mathfrak{s}([\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \not\leq s_{GL}(\vartheta)$. This is a contradiction. The proof is now complete. $\square$

Similarly, we get

8.2. Theorem. *Let ρ, ρ_0, σ, m and α be as in Theorem 8.1.*
(i) Suppose $\rho \not\cong \rho_0$. Then $\nu^\alpha\rho_0 \rtimes \delta([\nu^{1/2}\rho, \nu^{1/2+m}\rho], \sigma)$ reduces if and only if $\nu^\alpha\rho_0 \rtimes \sigma$ reduces. If $\nu^\alpha\rho_0 \rtimes \sigma$ reduces for some $\alpha > 0$, then $\nu^\alpha\rho_0 \rtimes \delta([\nu^{1/2}\rho, \nu^{1/2+m}\rho], \sigma)$ contains a unique irreducible square integrable subquotient. We denote that subquotient by $\delta(\nu^\alpha\rho_0, [\nu^{1/2}\rho, \nu^{1/2+m}\rho], \sigma)$. In the Grothendieck group, we have

$$\nu^\alpha\rho_0 \rtimes \delta([\nu^{1/2}\rho, \nu^{1/2+m}\rho], \sigma) = \delta(\nu^\alpha\rho_0, [\nu^{1/2}\rho, \nu^{1/2+m}\rho], \sigma) + L(\nu^\alpha\rho_0, \delta([\nu^{1/2}\rho, \nu^{1/2+m}\rho], \sigma)).$$

If $\rho_0 \rtimes \sigma$ reduces, then $\rho_0 \rtimes \delta([\nu^{1/2}\rho, \nu^{1/2+m}\rho], \sigma)$ is a direct sum of two inequivalent irreducible tempered representations.
(ii) Suppose $\rho \cong \rho_0$ and suppose that $\nu^\alpha\rho \rtimes \sigma$ is irreducible for $\alpha \neq 1/2$ (we assume $\alpha \geq 0$). Then $\nu^\alpha\rho \rtimes \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma)$ reduces if and only if $\alpha \in \{1/2, m+3/2\}$. In the Grothendieck group, we have

$$\nu^{m+3/2}\rho \rtimes \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma) = \delta([\nu^{1/2}\rho, \nu^{m+3/2}\rho], \sigma) + L(\nu^{m+3/2}\rho, \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma)).$$

If $\alpha = 1/2$ and $m > 0$, then there exists a unique irreducible square integrable subquotient in $\nu^{-1/2}\rho \rtimes \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma)$. We denote that subquotient by $\delta([\nu^{-1/2}\rho, \nu^{m+1/2}\rho]_+, \sigma)$. Then, in the Grothendieck group we have

$$\nu^{1/2}\rho \rtimes \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma) = L(\nu^{1/2}\rho, \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma)) + \delta([\nu^{-1/2}\rho, \nu^{m+1/2}\rho]_+, \sigma)$$

for any $m \geq 0$.
(iii) If $\alpha > 0$ and $\nu^\alpha\rho_0 \rtimes \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma)$ is irreducible, then

$$\nu^\alpha\rho_0 \rtimes \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma) = L(\nu^\alpha\rho_0, \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma)). \quad \square$$

9. On non-unitary induction of GL-type

For an irreducible cuspidal representation ρ of $GL(p,F)$ and a positive integer m set

$$\delta(\rho,m)=\delta([\nu^{-(m-1)/2}\rho,\nu^{(m-1)/2}\rho]).$$

It is easy to see that $\nu^\alpha\delta(\rho,m)\cong\delta(\nu^\alpha\rho,m)$ for $\alpha\in\mathbb{R}$.

9.1. Theorem. *Let ρ be an irreducible unitarizable cuspidal representation of $GL(p,F)$ and let σ be an irreducible cuspidal representation of S_q. Let m be a positive integer and $\alpha\in\mathbb{R}$.*
(i) If $\rho\not\cong\tilde\rho$, then $\nu^\alpha\delta(\rho,m)\rtimes\sigma$ is irreducible for any $\alpha\in\mathbb{R}$.
(ii) Suppose that $\nu^{1/2}\rho\rtimes\sigma$ reduces and that $\nu^\alpha\rho\rtimes\sigma$ is irreducible for any $\alpha\in\mathbb{R},|\alpha|\neq 1/2$. Assume that char $F=0$. Then $\nu^\alpha\delta(\rho,m)\rtimes\sigma$ reduces if and only if

$$\nu^\alpha\delta(\rho,m)\in\{\delta([\nu^{-m+1/2}\rho,\nu^{-1/2}\rho]),\delta([\nu^{-m+3/2}\rho,\nu^{1/2}\rho]),\delta([\nu^{-m+5/2}\rho,\nu^{3/2}\rho]),\\ \dots,\delta([\nu^{-1/2}\rho,\nu^{m-3/2}\rho]),\delta([\nu^{1/2}\rho,\nu^{m-1/2}\rho])\}.$$

In other words, we have reducibility if and only if $\alpha\in\{-m/2,-m/2+1,-m/2+2,\dots,m/2\}$.
(iii) Suppose that $\rho\rtimes\sigma$ reduces and that $\nu^\alpha\rho\rtimes\sigma$ is irreducible for any $\alpha\in\mathbb{R},\alpha\neq 0$. Then $\nu^\alpha\delta(\rho,m)\rtimes\sigma$ reduces if and only if

$$\nu^\alpha\delta(\rho,m)\in\{\delta([\nu^{-m+1}\rho,\rho]),\delta([\nu^{-m+2}\rho,\nu\rho]),\delta([\nu^{-m+3}\rho,\nu^2\rho]),\dots,\delta([\rho,\nu^{m-1}\rho])\},$$

i.e., if and only if $\alpha\in\{(-m+1)/2,(-m+1)/2+1,(-m+1)/2+2,\dots,(m-1)/2\}$.
(iv) Suppose that $\nu\rho\rtimes\sigma$ reduces and that $\nu^\alpha\rho\rtimes\sigma$ is irreducible for any $\alpha\in\mathbb{R},|\alpha|\neq 1$. Assume that $m\geq 2$. Then $\nu^\alpha\delta(\rho,m)\rtimes\sigma$ reduces if and only if

$$\nu^\alpha\delta(\rho,m)\in\{\delta([\nu^{-m}\rho,\nu^{-1}\rho]),\delta([\nu^{-m+1}\rho,\rho]),\delta([\nu^{-m+2}\rho,\nu\rho]),\dots,\delta([\nu\rho,\nu^{m}\rho])\},$$

i.e., if and only if $\alpha\in\{(-m-1)/2,(-m-1)/2+1,(-m-1)/2+2,\dots,(m+1)/2\}$.

Proof. Let $n\in\mathbb{Z},n\geq 0$. To shorten notation, in the proof we shall work with the representation $\nu^\beta\delta([\rho,\nu^n\rho])\rtimes\sigma\cong\delta([\nu^\beta\rho,\nu^{\beta+n}\rho])\rtimes\sigma$, where $\beta\in\mathbb{R}$ and $n\in\mathbb{Z},n\geq 0$. Clearly, $m=n+1$ and $\beta=(-m+1)/2+\alpha=-n/2+\alpha$. From Theorem 2.3 and (1-3) we get

$$\text{(9-1)}\quad \text{s.s.}(s_{(p)}(\delta([\nu^\beta\rho,\nu^{\beta+n}\rho])\rtimes\sigma))=\\ \nu^{\beta+n}\rho\otimes\delta([\nu^\beta\rho,\nu^{\beta+n-1}\rho])\rtimes\sigma+\nu^{-\beta}\tilde\rho\otimes\delta([\nu^{\beta+1}\rho,\nu^{\beta+n}\rho])\rtimes\sigma,$$

$$\text{(9-2)}\quad \text{s.s.}(s_{GL}(\delta([\nu^\beta\rho,\nu^{\beta+n}\rho])\rtimes\sigma)\\ =\sum_{k=0}^{n+1}\delta([\nu^{-\beta-n+k}\tilde\rho,\nu^{-\beta}\tilde\rho])\times\delta([\nu^{\beta+n-k+1}\rho,\nu^{\beta+n}\rho])\otimes\sigma.$$

In particular, considering members in the sum corresponding to $k = n-1$ and $k = n$ we get

$$\begin{aligned}(9\text{-}3)\quad & s_{GL}(\delta([\nu^{\beta}\rho, \nu^{\beta+n}\rho]) \rtimes \sigma) \\ & \geq \delta([\nu^{-\beta-1}\tilde{\rho}, \nu^{-\beta}\tilde{\rho}]) \times \delta([\nu^{\beta+2}\rho, \nu^{\beta+n}\rho]) \otimes \sigma + \nu^{-\beta}\tilde{\rho} \times \delta([\nu^{\beta+1}\rho, \nu^{\beta+n}\rho]) \otimes \sigma.\end{aligned}$$

We shall prove (i) - (iv) now.
(i) Suppose $\rho \not\cong \tilde{\rho}$. Then $\nu^{\beta+n}\rho \not\cong \nu^{-\beta}\tilde{\rho}$ for any $\beta \in \mathbb{R}$. We prove (i) by induction with respect to n. Assume $n \geq 1$. We shall show irreducibility using Lemma 3.7. Denote $\tau'' = \nu^{-\beta}\tilde{\rho} \times \delta([\nu^{\beta+1}\rho, \nu^{\beta+n}\rho]) \otimes \sigma, P'' = P_{((n+1)p)}, P''' = P_{(p)}$ and $P' = P_{(p)^{n+1}}$. Clearly, τ'' is irreducible. From (9-3) we know $\tau'' \leq s_{GL}(\delta([\nu^{\beta}\rho, \nu^{\beta+n}\rho]) \rtimes \sigma)$. By the inductive assumption, both representations on the right hand side of (9-1) are irreducible. We shall show now that conditions of Lemma 3.7 are fulfilled. First take $\tau''' = \nu^{\beta+n}\rho \otimes \delta([\nu^{\beta}\rho, \nu^{\beta+n-1}\rho]) \rtimes \sigma$. Suppose

$$(9\text{-}4)\qquad (1 \otimes s_{(p)^n})(\tau''') + (r_{(p)^{n+1}} \otimes 1)(\tau'') \leq s_{(p)^{n+1}}(\delta([\nu^{\beta}\rho, \nu^{\beta+n}\rho]) \rtimes \sigma).$$

Then (9-1) implies $(r_{(p)^{n+1}} \otimes 1)(\tau'') \leq \nu^{-\beta}\tilde{\rho} \otimes s_{(p)^n}(\delta([\nu^{\beta+1}\rho, \nu^{\beta+n}\rho]) \rtimes \sigma)$. We can see easily that this can not hold (use the Hopf algebra structure of R, (1-3) and (1-5)). Thus (9-4) can not hold. Analogously we see that (9-4) can not hold if we take $\tau''' = \nu^{-\beta}\tilde{\rho} \otimes \delta([\nu^{\beta+1}\rho, \nu^{\beta+n}\rho]) \rtimes \sigma$. Therefore the conditions of Lemma 3.7 are satisfied, and $\delta([\nu^{\beta}\rho, \nu^{\beta+n}\rho]) \rtimes \sigma$ is irreducible by that lemma.
(ii) We shall prove reducibilities first. We can easily conclude from (ii) of Proposition 2.4 that $\delta([\nu^{1/2}\rho, \nu^{v+1/2}\rho]) \rtimes \sigma$ reduces for $v \in \mathbb{Z}, v \geq 0$. In this situation, one subquotient is the square integrable representation $\delta([\nu^{1/2}\rho, \nu^{v+1/2}\rho], \sigma)$. Using Remark 3.2 we shall now prove the reducibility of $\delta([\nu^{-u-1/2}\rho, \nu^{v+1/2}\rho]) \rtimes \sigma$, where $u, v \in \mathbb{Z}, u, v \geq 0$. It is enough to prove it in the case $u \leq v$ (Proposition 2.2). One first shows using Theorems 2.3 and 1.1 that the multiplicity of $\delta([\nu^{1/2}\rho, \nu^{u+1/2}\rho]) \times \delta([\nu^{1/2}\rho, \nu^{v+1/2}\rho]) \otimes \sigma$ in the following three representations

$$s_{GL}(\nu^{-u-1/2}\rho \times \nu^{-u+1/2}\rho \times \nu^{-u+3/2}\rho \times \cdots \times \nu^{v-1/2}\rho \times \nu^{v+1/2}\rho \rtimes \sigma),$$
$$s_{GL}(\delta([\nu^{-u-1/2}\rho, \nu^{v+1/2}\rho]) \rtimes \sigma) \text{ and } s_{GL}(\delta([\nu^{1/2}\rho, \nu^{u+1/2}\rho]) \rtimes \delta([\nu^{1/2}\rho, \nu^{v+1/2}\rho], \sigma))$$

is 1, and

$$s_{GL}(\delta([\nu^{-u-1/2}\rho, \nu^{v+1/2}\rho]) \rtimes \sigma) \not\leq s_{GL}(\delta([\nu^{1/2}\rho, \nu^{u+1/2}\rho]) \rtimes \delta([\nu^{1/2}\rho, \nu^{v+1/2}\rho], \sigma)).$$

Now Remark 3.2 implies the reducibility of $\delta([\nu^{-u-1/2}\rho, \nu^{v+1/2}\rho]) \rtimes \sigma$.

We shall now prove by induction the irreducibilities that we claim in (ii) (recall $\rho \cong \tilde{\rho}$). It is enough to consider only the case $\beta + n/2 \geq 0$ (otherwise, one passes to the contragredient). By Proposition 4.2, it is enough to consider only the case of $\beta + n/2 > 0$. Note that under these assumptions we always have $\nu^{\beta+n}\rho \not\cong \nu^{-\beta}\rho$. Suppose that $n \geq 1$ and that

$$\delta([\nu^{\beta}\rho, \nu^{\beta+n}\rho]) \not\in \{\delta([\nu^{-m+1/2}\rho, \nu^{-1/2}\rho]), \delta([\nu^{-m+3/2}\rho, \nu^{1/2}\rho]), \ldots, \delta([\nu^{1/2}\rho, \nu^{m-1/2}\rho])\}.$$

By the inductive assumption, both representations on the right hand side of (9-1) are irreducible.

First, consider the case $\beta \neq 0$. We shall conclude the irreducibility using Lemma 3.7. Denote $\tau'' = \nu^{-\beta}\rho \times \delta([\nu^{\beta+1}\rho, \nu^{\beta+n}\rho]) \otimes \sigma$ and take P', P'', P''' as before. Since $\beta \neq 0$ and $\beta + n/2 > 0$, τ'' is irreducible. Now one checks that conditions of Lemma 3.7 hold in the same way as in the previous application of that lemma. Thus $\delta([\nu^{\beta}\rho, \nu^{\beta+n}\rho]) \rtimes \sigma$ is irreducible.

Consider the case $\beta = 0$ now. By Proposition 6.3, it is enough to consider only the case $n \geq 2$. Denote $\tau'' = \delta([\nu^{-\beta-1}\rho, \nu^{-\beta}\rho]) \times \delta([\nu^{\beta+2}\rho, \nu^{\beta+n}\rho]) \otimes \sigma = \delta([\nu^{-1}\rho, \rho]) \times \delta([\nu^{2}\rho, \nu^{n}\rho]) \otimes \sigma$ and take P', P'', P''' as before. Note that τ'' is irreducible, and $\tau'' \leq s_{GL}(\delta([\nu^{\beta}\rho, \nu^{\beta+n}\rho]) \rtimes \sigma) = s_{GL}(\delta([\rho, \nu^{n}\rho]) \rtimes \sigma)$ by (9-3). In the same way as before we get from Lemma 3.7 that $\delta([\nu^{\beta}\rho, \nu^{\beta+n}\rho]) \rtimes \sigma = \delta([\rho, \nu^{n}\rho]) \rtimes \sigma$ is irreducible. This completes the proof of (ii).

The proofs of (iii) and (iv) proceed along similar lines, using formulas (9-1) and (9-2) (here we do not even have a delicate point as in the proof of (ii), i.e. we do not need to use Proposition 6.3). Therefore, we shall not write these proofs here. □

The previous theorem holds in the same form for Zelevinsky segment representations $\mathfrak{s}(\rho, m) = \mathfrak{s}([\nu^{-(m-1)/2}\rho, \nu^{(m-1)/2}\rho])$ (use the generalized Zelevinsky involution).

10. A simple example of cuspidal reducibilities

The group $GSp(n, F)$ is a semi-direct product of $Sp(n, F)$ and $\{\text{q-diag}(I_n, \lambda I_n), \lambda \in F^\times\}$. Further, the map q-diag$(I_n, \lambda I_n)\, h \mapsto \lambda$, where $\lambda \in F^\times$ and $h \in Sp(n, F)$, defines an epimorphism of $GSp(n, F)$ onto $F^\times$. Using this epimorphism, we shall identify characters of $GSp(n, F)$ with characters of $F^\times$.

Using $\lambda \mapsto \lambda I_n$, we identify $F^\times$ with the center of $GL(n, F)$. If π is an irreducible admissible representation of $GL(n, F)$, then the central character of π is denoted by ω_π. Using the homomorphism $\det : GL(n, F) \to F^\times$, the characters of $GL(n, F)$ are identified with characters of $F^\times$.

10.1. Proposition. *Let ρ be an irreducible selfcontragredient cuspidal representation of the group $GL(p, F)$ and let σ be an irreducible cuspidal representation of $GSp(q, F)$. Suppose that $\sigma \not\cong \omega_\rho \sigma$. Let σ_0 be any irreducible $Sp(q, F)$-subrepresentation of the restriction $\sigma|Sp(q, F)$. Then $\rho \rtimes \sigma_0$ reduces into a sum of two inequivalent irreducible representations. Further, $\nu^\alpha \rho \rtimes \sigma_0$ is irreducible for any $\alpha \in \mathbb{R}^\times$.*

Proof. One introduces $\rtimes$ for groups GSp in a similar way to that done here for symplectic groups (see [T5] or [T6]). The assumptions imply that $\rho \otimes \sigma$ is regular. By (i) of Proposition 3.1 in [T7], the representation $\nu^\alpha \rho \rtimes \sigma$ of $GSp(p+q, F)$ is irreducible for any $\alpha \in \mathbb{R}$ (one can see that easily from Frobenius reciprocity if $\alpha = 0$, and from Proposition 7.1.3 of [C2] if $\alpha \neq 0$). Note that for the restrictions we have $(\nu^\alpha \rho \rtimes \sigma)|Sp(p+q, F) \cong \nu^\alpha \rho \rtimes (\sigma|Sp(q, F))$ as representations of $Sp(p+q, F)$. Proposition 2.7, (iii), in [T3] implies that $\nu^\alpha \rho \rtimes \sigma_0$ is irreducible for $\alpha \in \mathbb{R}^\times$ (one can twist $\nu^\alpha \rho \rtimes \sigma$ with a suitable character to get a unitary central character). Further, $\rho \rtimes \sigma_0$ reduces. This follows from the p-adic Clifford theory ([GbKn], see also [T3]), since $\omega_\rho \neq 1_{F^\times}$ and $\omega_\rho(\rho \rtimes \sigma) \cong \rho \rtimes \sigma$. This reducibility also follows from the fact that the complementary series have finite length. □

10.2. Remarks. (i) If $\sigma|Sp(q,F)$ is a multiplicity one representation, then the condition $\sigma \not\cong \omega_\rho\sigma$ from the above proposition, can be expressed in a simple way just in terms of σ_0 (see Remark 2.6 of [T3], for example).
(ii) It would be interesting to know if $\sigma|Sp(q,F)$ is a multiplicity one representation when σ is an irreducible admissible representation of $GSp(q,F)$. This seems to be generally expected (for $q = 1$ it is well-known that we have always multiplicity one). Let us note that if one proves multiplicity one for irreducible tempered representations of $GSp(q,F)$'s, this would imply multiplicity one for all irreducible admissible representations of $GSp(q,F)$ (see Lemma 6.2 of [T5]).

The last proposition directly implies the following result of Shahidi (he assumes char $F = 0$; our proof requires char $F \neq 2$).

10.3. Corollary (Shahidi, [Sh2]). *If ρ is an irreducible selfcontragredient cuspidal representation of $GL(p,F)$ such that $\omega_\rho \neq 1_{F^\times}$, then $\rho \rtimes 1$ reduces into a sum of two inequivalent irreducible representations and $\nu^\alpha\rho \rtimes 1$ is irreducible for any $\alpha \in \mathbb{R}^\times$.* □

Shahidi's proof uses a method based on analysis of local Langlands L-functions. His method also works in a number of other situations.

11. Applications

In this section, we shall list some of the most interesting consequences of theorems of sections 7, 8 and 9.

Let χ be a character of $F^\times$. Recall that the representation $\chi \rtimes 1$ of $Sp(1,F) = SL(2,F)$ reduces if and only if χ is a character of order two or $\chi = \nu^{\pm 1}1_{F^\times}$. Further, the representation $\chi \rtimes 1$ of $SO(3,F)$ reduces if and only if $\chi^2 = \nu^{\pm 1}1_{F^\times}$. We have directly now:

11.1. Theorem. *Let χ be a character of $F^\times$ and let n be positive integer. Then $\chi \rtimes 1_{Sp(n,F)}$ reduces if and only if $\chi \rtimes St_{Sp(n,F)}$ reduces. We have reducibility if and only if $\chi^2 = 1_{F^\times}$ or $\chi = \nu^{\pm(n+1)}1_{F^\times}$. In the case of reducibility, we have a multiplicity one representation of length two (the Langlands parameters of the irreducible subquotients can be seen in Theorems 7.1 and 7.2).*

Proof. Theorems 7.1 and 7.2. □

11.2. Theorem. *Let χ be a character of $F^\times$. Then $\chi \rtimes 1_{SO(2n+1,F)}$ reduces if and only if $\chi \rtimes St_{SO(2n+1,F)}$ reduces. Write $\chi = \nu^\alpha\chi_0$, where χ_0 is a unitary character of $F^\times$ and $\alpha \in \mathbb{R}$. We have reducibility if and only if $\chi_0^2 = 1_{F^\times}$ and $\alpha = \pm 1/2$, or $\chi = \nu^{\pm(n+1/2)}1_{F^\times}$. In the case of reducibility, we have a multiplicity one representation of length two (the Langlands parameters of the irreducible subquotients can be seen in Theorems 8.1 and 8.2).*

Proof. Theorems 8.1 and 8.2. □

11.3. Theorem. *In this theorem, we only consider representations of $Sp(n,F)$. Let χ be a character of $F^\times$. Then $\chi 1_{GL(n,F)} \rtimes 1$ is reducible if and only if $\chi St_{GL(n,F)} \rtimes 1$ is reducible. Write $\chi = \nu^\alpha \chi_0$, where χ_0 is a unitary character of $F^\times$ and $\alpha \in \mathbb{R}$. We have reducibility if and only if*

$$\chi_0^2 = 1_{F^\times} \text{ and } \alpha \in \{(-n+1)/2, (-n+1)/2+1, (-n+1)/2+2, \dots, (n-1)/2\},$$

or $\chi = \nu^{\pm(n+1)/2} 1_{F^\times}$.

Proof. Theorem 9.1 and and the generalized Zelevinsky involution imply the theorem. □

The case of reducibilities of the degenerate principal series representation $\chi 1_{GL(n,F)} \rtimes 1$ of $Sp(n,F)$ covered by the last theorem was already settled by Kudla and Rallis in [KuRa]. They also described the irreducible subquotients. They assume char $F=0$. The unramified degenerate principal series case was settled before by [Gu].

11.4. Theorem. *Assume char $F=0$. In this theorem, we only consider representations of $SO(2n+1,F)$. Let χ be a character of $F^\times$. Then $\chi 1_{GL(n,F)} \rtimes 1$ reduces if and only if $\chi St_{GL(n,F)} \rtimes 1$ reduces. Write $\chi = \nu^\alpha \chi_0$, where χ_0 is a unitary character of $F^\times$ and $\alpha \in \mathbb{R}$. We have reducibility if and only if $\chi_0^2 = 1_{F^\times}$ and $\alpha \in \{-n/2, -n/2+1, -n/2+2, \dots, n/2\}$.*

Proof. Theorem 9.1. □

Part of the description of reducibilities of the degenerate principal series representations considered in the above theorems was already obtained by C. Jantzen in [J1] and [J2]. Following the investigation of this paper, he made in [J3] a great step forward in understanding of reducibility of (generalized) degenerate principal series (among others, he obtained all reducibility points and all irreducible subquotients when induction goes from a maximal parabolic subgroup; [J4] considers reducibility points for any parabolic).

Shahidi proved the following results:

11.5. Theorem (Shahidi, [Sh2]). *Assume char $F=0$. Let ρ be an irreducible unitarizable cuspidal representation of $GL(p,F)$ where $p \geq 2$. Suppose $\rho \cong \tilde{\rho}$ (then the representation $\nu^\alpha \rho \rtimes 1$ of S_p reduces for some $\alpha \in \mathbb{R}$; S_p is either $Sp(p,F)$ or $SO(2p+1)$). Then*
(i) $\nu^\alpha \rho \rtimes 1$ is irreducible for $\alpha \in \mathbb{R} \backslash \{0, \pm 1/2\}$.
(ii) $\rho \rtimes 1$ is irreducible if and only if $\nu^{\pm 1/2} \rho \rtimes 1$ is reducible.
(iii) The representation $\rho \rtimes 1$ of $Sp(p,F)$ reduces if and only if the representation $\rho \rtimes 1$ of $SO(2p+1,F)$ is irreducible.
(iv) If p is odd, then the representation $\rho \rtimes 1$ of $Sp(p,F)$ is reducible (recall that then $p \geq 3$).
(v) If $p=2$, then the representation $\rho \rtimes 1$ of $Sp(2,F)$ reduces if and only if $\omega_\rho \neq 1_{F^\times}$.

From this theorem and previous sections, the following results are immediate (recall that $\delta(\rho,m) = \delta([\nu^{-(m-1)/2}\rho, \nu^{(m-1)/2}\rho])$).

11.6. Theorem. *In this theorem, we only consider representations of the groups $Sp(n,F)$. Assume char $F = 0$. Let ρ be an irreducible unitarizable cuspidal representation of $GL(p,F)$ where $p > 1$ is odd (for $p = 1$ see Theorem 11.3). Let m be a positive integer and $\alpha \in \mathbb{R}$. Then $\nu^\alpha\delta(\rho,m) \rtimes 1$ reduces if and only if $\nu^\alpha\mathfrak{s}(\rho,m) \rtimes 1$ reduces. We have reducibility if and only if $\rho \cong \tilde{\rho}$ and $\alpha \in \{(-m+1)/2, (-m+1)/2+1, (-m+1)/2+2, \dots, (m-1)/2\}$.*

Proof. This is a direct consequence of Proposition 3.5 of [Sh2] (see also Theorem 11.5 above), and (iii) of Theorem 9.1. □

Let us note that selfcontragredient irreducible cuspidal representations are not very often (see [Ad]).

11.7. Corollary. *In this corollary, we only consider representations of $Sp(n,F)$ (char $F =$ 0). Let δ be an irreducible essentially square integrable representation of $GL(n,F)$ and assume that n is odd. Suppose that δ is not a twist of the Steinberg representation by a character of $F^\times$ (for this case see Theorem 11.3). Then there exist an irreducible unitarizable cuspidal representation ρ of $GL(p,F), p \geq 2$, a positive integer m and $\alpha \in \mathbb{R}$ so that $\delta \cong \nu^\alpha\delta(\rho,m)$ (note that p and m are odd). Write $m = 2k+1$. Then $\delta \rtimes 1 \cong \nu^\alpha\delta(\rho,2k+1) \rtimes 1$ reduces if and only if $\rho \cong \tilde{\rho}$ and $\alpha \in \{-k, -k+1, -k+2, \dots, k\}$.* □

11.8. Theorem. *In this theorem, we only consider representations of $SO(2n+1,F)$. Assume char $F = 0$. Let ρ be an irreducible unitarizable cuspidal representation of $GL(p,F)$ where p is odd. Let m be a positive integer and $\alpha \in \mathbb{R}$. Then $\nu^\alpha\delta(\rho,m) \rtimes 1$ reduces if and only if $\nu^\alpha\mathfrak{s}(\rho,m) \rtimes 1$ reduces. We have reducibility if and only if $\rho \cong \tilde{\rho}$ and $\alpha \in \{-m/2, -m/2+1, -m/2+2, \dots, m/2\}$.*

Proof. This follows from Proposition 3.10 of [Sh2] (see also above Theorem 11.5), and (ii) of Theorem 9.1. □

11.9. Corollary. *In this corollary, we only consider representations of $SO(2n+1,F)$ (char $F = 0$). Let δ be an irreducible essentially square integrable representation of $GL(n,F)$ and assume that n is odd. There exist an irreducible unitarizable cuspidal representation ρ of $GL(p,F)$, a positive integer m and $\alpha \in \mathbb{R}$ so that $\delta \cong \nu^\alpha\delta(\rho,m)$ (p and m are odd). Write $m = 2k+1$. Then $\delta \rtimes 1 \cong \nu^\alpha\delta(\rho,2k+1) \rtimes 1$ reduces if and only if $\rho \cong \tilde{\rho}$ and $\alpha \in \{-k-1/2, -k+1/2, -k+3/2, \dots, k+1/2\}$.* □

11.10. Theorem. *In this theorem, we only consider representations of $Sp(n,F)$'s. Assume char $F = 0$. Let ρ be an irreducible unitarizable cuspidal representation of $GL(2,F)$. Let m be a positive integer and $\alpha \in \mathbb{R}$. If $\rho \not\cong \tilde{\rho}$, then $\nu^\alpha\delta(\rho,m) \rtimes 1$ is irreducible for any $\alpha \in \mathbb{R}$. Suppose that $\rho \cong \tilde{\rho}$. Then:*
(i) If $\omega_\rho = 1_{F^\times}$, then $\nu^\alpha\delta(\rho,m) \rtimes 1$ reduces if and only if $\alpha \in \{-m/2, -m/2+1, -m/2+2, \dots, m/2\}$.
(ii) If $\omega_\rho \neq 1_{F^\times}$, then $\nu^\alpha\delta(\rho,m) \rtimes 1$ reduces if and only if $\alpha \in \{(-m+1)/2, (-m+1)/2+1, (-m+1)/2+2, \dots, (m-1)/2\}$. □

11.11. Theorem. *In this theorem, we only consider representations of the groups $SO(2n+1,F)$. Assume char $F = 0$. Let ρ be an irreducible unitarizable cuspidal representation of $GL(2,F)$. Let m be a positive integer and $\alpha \in \mathbb{R}$. If $\rho \not\cong \tilde{\rho}$, then $\nu^\alpha\delta(\rho,m) \rtimes 1$ is irreducible for any $\alpha \in \mathbb{R}$. Suppose that $\rho \cong \tilde{\rho}$. Then:*

(i) If $\omega_\rho \neq 1_{F^\times}$*, then* $\nu^\alpha \delta(\rho, m) \rtimes 1$ *reduces if and only if* $\alpha \in \{-m/2, -m/2+1, -m/2+2, \dots, m/2\}$.
(ii) If $\omega_\rho = 1_{F^\times}$*, then* $\nu^\alpha \delta(\rho, m) \rtimes 1$ *reduces if and only if* $\alpha \in \{(-m+1)/2, (-m+1)/2+1, (-m+1)/2+2, \dots, (m-1)/2\}$. □

11.12. Remark. The last two theorems also hold for Zelevinsky segment representations.

These were some applications. We can apply our general theorems to a number of other cases. We shall see only one application more. In the following example, we handle the case of the representations $\chi \mathrm{St}_{GL(n,F)} \rtimes \sigma$ and $\chi 1_{GL(n,F)} \rtimes \sigma$ of the group $Sp(n+1, F)$ where σ is an irreducible cuspidal representation of $Sp(1, F)$.

Let σ be an irreducible cuspidal representation of $Sp(1, F)$ $(= SL(2, F))$. Then there exists an irreducible cuspidal representation Σ of $GSp(1, F)$ $(=GL(2, F))$ so that σ is a subrepresentation of $\Sigma | Sp(1, F)$. Let χ be a character of $F^\times$. Here is a complete list of the points of reducibility of the representation $\chi \rtimes \sigma$ of $Sp(2, F)$:
(i) $\chi = 1_{F^\times}$;
(ii) χ is a character of order two which satisfies $\chi\Sigma \not\cong \Sigma$;
(iii) $\chi = \nu^{\pm 1}\chi_0$ where χ_0 is a character of order two which satisfies $\chi_0 \Sigma \cong \Sigma$.
The above reducibility result was proved by J.-L. Waldspurger, and also by F. Shahidi (Waldspurger's proof does not require char $F = 0$). The reducibility condition can be expressed purely in terms of σ (without Σ). One can find such an interpretation in the fifth section of [SaT].

We shall write a character χ of $F^\times$ as $\chi = \nu^\alpha \chi_0$, where χ_0 is unitary and $\alpha \in \mathbb{R}$. With notation as above, we have:

11.13. Theorem. *Let* $n \in \mathbb{Z}, n \geq 2$*. In this theorem, we only consider representations of* $Sp(n+1, F)$*. Then the representation* $\chi 1_{GL(n,F)} \rtimes \sigma$ *reduces if and only if* $\chi St_{GL(n,F)} \rtimes \sigma$ *is reducible. We have reducibility exactly when*

$$\alpha \in \{(-n+1)/2, (-n+1)/2+1, (-n+1)/2+2, \dots, (n-1)/2\} \quad \textit{and} \quad \chi_0^2 = 1_{F^\times}$$

or $\alpha \in \{\pm(n+1)/2\}$ *and* χ_0 *is a character of order two which satisfies* $\chi_0 \Sigma \cong \Sigma$. □

For a description of the condition $\chi_0 \Sigma \cong \Sigma$ in terms of σ, see the fifth section of [SaT].

In closing, let us say that using results about reducibilities of $\rho \rtimes 1$ when ρ is a cuspidal irreducible representation of a general linear group, proved by Shahidi in [Sh2], and using only Propositions 4.1-4.4, one can prove the results of Shahidi in [Sh2] about reducibilities of $\delta \rtimes 1$ when δ is a (non-cuspidal) irreducible square integrable representation. Shahidi's proof of this step is based on L-functions. Propositions 4.1-4.4 provide an alternative proof of this step.

12. GL-duality

Let π be an admissible representations of $GL(n, F)$. If we consider the representation $\pi \rtimes 1$ of $Sp(n, F)$ (resp. of $SO(2n+1, F)$), then we shall denote it by $\pi \rtimes_{Sp(n,F)} 1$ (resp. by $\pi \rtimes_{SO(2n+1,F)} 1$) in this section .

Shahidi proved the following duality (Theorem 1.2 of [Sh2]): if $\delta \not\cong 1_{F^\times}$ is a selfcontragredient irreducible square integrable representation of $GL(n,F)$, then $\delta \rtimes_{Sp(n,F)} 1$ reduces if and only if $\delta \rtimes_{SO(2n+1,F)} 1$ is irreducible (char $F=0$). Propositions 4.1 – 4.4 and Shahidi's results about cuspidal reducibilities (Theorem 11.5) imply this duality (for selfcontragredient non-cuspidal irreducible square integrable representations). Shahidi's duality can be extended to the non-unitary case (i.e. to essentially square integrable representations) in the following way:

12.1. Theorem. *Assume char $F=0$. Let δ be an irreducible square integrable representation of $GL(n,F)$ and $\alpha \in \mathbb{R}$. Write $\delta = \delta(\rho,m)$, where ρ is an irreducible unitarizable cuspidal representation of a general linear group and m is a positive integer.*
(i) Suppose that $\delta \not\cong St_{GL(n,F)}$. If

$$\rho \not\cong \tilde{\rho} \text{ or } \alpha \notin \{(-m)/2, (-m+1)/2, (-m+2)/2, (-m+3)/2, \dots, (m-1)/2, m/2\},$$

then both $\nu^\alpha\delta \rtimes_{Sp(n,F)} 1$ and $\nu^\alpha\delta \rtimes_{SO(2n+1,F)} 1$ are irreducible. If

$$\rho \cong \tilde{\rho} \text{ and } \alpha \in \{(-m)/2, (-m+1)/2, (-m+2)/2, (-m+3)/2, \dots, (m-1)/2, m/2\},$$

then $\nu^\alpha\delta \rtimes_{Sp(n,F)} 1$ reduces if and only if $\nu^\alpha\delta \rtimes_{SO(2n+1,F)} 1$ is irreducible (and conversely).
(ii) Suppose $n \geq 2$. If

$$\alpha \notin \{(-n-1)/2, (-n)/2, (-n+1)/2, (-n+2)/2, \dots, (n-1)/2, n/2, (n+1)/2\},$$

then both $\nu^\alpha St_{GL(n,F)} \rtimes_{Sp(n,F)} 1$ and $\nu^\alpha St_{GL(n,F)} \rtimes_{SO(2n+1,F)} 1$ are irreducible. If

$$\alpha \in \{(-n-1)/2, (-n)/2, (-n+1)/2, (-n+2)/2, \dots, (n-1)/2, n/2, (n+1)/2\},$$

then $\nu^\alpha St_{GL(n,F)} \rtimes_{Sp(n,F)} 1$ reduces if and only if $\nu^\alpha St_{GL(n,F)} \rtimes_{SO(2n+1,F)} 1$ is irreducible.
(iii) If $\alpha \in \{\pm 1/2, \pm 1\}$, then $\nu^\alpha 1_{F^\times} \rtimes_{Sp(1,F)} 1$ reduces if and only if $\nu^\alpha 1_{F^\times} \rtimes_{SO(3,F)} 1$ is irreducible. Both $\nu^\alpha 1_{F^\times} \rtimes_{Sp(1,F)} 1$ and $\nu^\alpha 1_{F^\times} \rtimes_{SO(3,F)} 1$ are irreducible for $\alpha \in \mathbb{R}\backslash\{\pm 1/2, \pm 1\}$. □

12.2. Remarks. (i) The particular case of $\alpha = 0$ and $\rho \cong \tilde{\rho}$ is Shahidi's duality.
(ii) The duality from the above theorem also holds for Zelevinsky segment representations. One needs to replace the irreducible square integrable representation $\delta(\rho,m)$ with the unitarizable Zelevinsky segment representation $\mathfrak{s}(\rho,m) = \mathfrak{s}([\nu^{-(m-1)/2}\rho, \nu^{(m-1)/2}\rho])$, and $\mathrm{St}_{GL(n,F)}$ with $1_{GL(n,F)}$ (a special case of this duality is a duality for degenerate principal series representations).

13. The case of non-generic cuspidal reducibilities

In this section we shall study reducibility of parabolically induced representations in the setting of non-generic cuspidal $(1/2)\mathbb{Z}$-reducibilities. In particular, we shall pay special attention to some new square integrable representations which are specific for the non-generic cuspidal reducibilities. Our method applies in this setting without essential changes. This is the reason why we shall supply only brief proofs in this section. We shall rely often on our paper [T7] (there one can find definition of regular representations, and more details regarding them).

We shall start this section with considering of cases similar to those ones considered in the seventh and the eighth sections. First we shall recall of a new regular irreducible square integrable representations related to the non-generic cuspidal reducibilities, which have some similarities to the square integrable representations introduced in Proposition 2.4 (they have also some significant differences). One of the main similarities with the square integrable representations introduced in Proposition 2.4, is that all their Jacquet modules are also irreducible.

Suppose that ρ is an irreducible unitarizable cuspidal representation of $GL(p, F)$ and σ a similar representation of S_q. Suppose that $\nu^\beta\rho \rtimes \sigma$ reduces for some $\beta > 0$. Take $k \in \mathbb{Z}$ which satisfies $0 < \beta - k \leq \beta$. Then the representation $\nu^{\beta-k} \times \nu^{\beta-k+1} \times \cdots \times \nu^\beta\rho \rtimes \sigma$ contains a unique irreducible subrepresentation, which we denote by

$$\delta([\nu^{\beta-k}\rho, \nu^\beta\rho], \sigma).$$

Then $\delta([\nu^{\beta-k}\rho, \nu^\beta\rho], \sigma)$ is a (regular) square integrable representation which satisfies

$$\mu^*(\delta([\nu^{\beta-k}\rho, \nu^\beta\rho], \sigma)) = \sum_{i=o}^{k+1} \mathfrak{s}([\nu^{\beta-k}\rho, \nu^{\beta-i}\rho]) \otimes \delta([\nu^{\beta-i+1}\rho, \nu^\beta\rho], \sigma)$$

(see the seventh section of [T7] for more details, and for proofs).

If $\beta > 1$, then we can take $k \geq 1$ such that $0 < \beta - k$. Now the square integrable representations $\delta([\nu^{\beta-k}\rho, \nu^\beta\rho], \sigma)$ are not covered by Proposition 2.4 (they are of different type then the square integrable representations considered there).

13.1. Proposition. *Let ρ and ρ_0 be irreducible unitarizable cuspidal representations of $GL(p, F)$ and $GL(p_0, F)$ respectively, let σ be an irreducible cuspidal representation of S_q and let $\beta \in (1/2)\mathbb{Z}$ be positive (i.e. > 0). Suppose that $\nu^\beta\rho \rtimes \sigma$ reduces, and that $\nu^\alpha\rho \rtimes \sigma$ is irreducible for any $\alpha \in \mathbb{R}\backslash\{\pm\beta\}$. Chose $k, l \in \mathbb{Z}$ such that $0 < \beta - k \leq \beta \leq \beta + l$. Let $\alpha \in \mathbb{R}$. Then:*
(i) If $\rho \not\cong \rho_0$, then $\nu^\alpha\rho_0 \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma)$ (resp. $\nu^\alpha\rho_0 \rtimes \delta([\nu^{\beta-k}\rho, \nu^\beta\rho], \sigma)$) reduces, if and only if $\nu^\alpha\rho_0 \rtimes \sigma$ reduces.
(ii) $\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma)$ reduces if and only if $\alpha \in \{\pm(\beta - 1), \pm(\beta + l + 1)\}$.
(iii) $\nu^\alpha\rho \rtimes \delta([\nu^{\beta-k}\rho, \nu^\beta\rho], \sigma)$ reduces if and only if $\alpha \in \{\pm(\beta - k - 1), \pm(\beta + 1)\}$.

Proof. The proposition is enough to prove in the case $\alpha \geq 0$. Write

$$\text{(13-1)} \quad \mu^*(\nu^\alpha\rho_0 \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma))$$
$$= (1 \otimes \nu^\alpha\rho_0 + \nu^\alpha\rho_0 \otimes 1 + \nu^{-\alpha}\tilde{\rho}_0 \otimes 1) \rtimes \left(\sum_{j=-1}^{l} \delta([\nu^{\beta+j+1}\rho, \nu^{\beta+l}\rho]) \otimes \delta([\nu^\beta\rho, \nu^{\beta+j}\rho], \sigma) \right).$$

We shall proceed now in the same way as in Proposition 7.1. From (13-1) we get

$$\text{(13-2)} \quad \text{s.s.}(s_{GL}(\nu^\alpha\rho_0 \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma)))$$
$$= \nu^{-\alpha}\tilde{\rho}_0 \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes \sigma + \nu^\alpha\rho_0 \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes \sigma,$$

$$s_{(lp)}(\nu^\alpha\rho_0 \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma)) \geq \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes \nu^\alpha\rho_0 \rtimes \sigma, \tag{13-3}$$

$$s_{((l-1)p)}(\nu^\alpha\rho_0 \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma)) \geq \delta([\nu^{\beta+1}\rho, \nu^{\beta+l}\rho]) \otimes \nu^\alpha\rho_0 \rtimes \delta(\nu^\beta\rho, \sigma). \tag{13-4}$$

We shall analyze first the case $\rho_0 \not\cong \rho$. The length of (13-2) is then 2.

Suppose that $\nu^\alpha\rho_0 \rtimes \sigma$ is irreducible. If $\nu^\alpha\rho_0 \not\cong \nu^{-\alpha}\tilde{\rho}_0$, then (13-3) and (13-2) imply the irreducibility of $\nu^\alpha\rho_0 \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma)$ (this follows from the transitivity of Jacquet modules; Lemma 3.7). Similarly one sees the irreducibility of $\nu^\alpha\rho_0 \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma)$ for $\rho_0 \cong \tilde{\rho}_0$ (one then needs to look at multiplicities, and use (13-2) and (13-3) again).

Assume now that $\nu^\alpha\rho_0 \rtimes \sigma$ reduces. If $\alpha > 0$, then we are in the regular situation. Now (13-2) and Theorem 7.4 of [T7] imply the reducibility of $\nu^\alpha\rho_0 \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma)$. Suppose $\alpha = 0$. Write $\rho_0 \rtimes \sigma = \tau_1 \oplus \tau_2$ where τ_i are irreducible. Multiplicity of $\rho_0 \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes \sigma$ in $s_{GL}(\delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \rtimes \tau_1)$, $s_{GL}(\rho_0 \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma))$ and $s_{GL}(\rho_0 \times \nu^\beta\rho \times \nu^{\beta+1}\rho \times \cdots \times \nu^{\beta+l}\rho \rtimes \sigma)$ is 1, 2 and 2 respectively. Remark 3.2 now implies the reducibility of $\nu^\alpha\rho_0 \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma)$.

We shall now prove (ii). We shall use formulas (13-2), (13-3) and (13-4) where we shall take ρ_0 to be ρ.

If $\alpha = \beta + l + 1$, then we are in regular situation. Now (13-2) and Theorem 6.3 of [T7] imply the reducibility. If $\beta \in \{1/2, 1\}$ and $\alpha \in \{\pm(\beta - 1)\}$, then (ii) of Theorems 7.1 and 8.1 imply the reducibility. Suppose $\beta > 1$. Then again we are in the regular situation, and (13-2) and Proposition 7.2 of [T7] imply the reducibility.

It remains to prove the irreducibility of $\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma)$ claimed in the proposition. Suppose $\alpha \notin \{\pm(\beta - 1), \pm(\beta + l + 1)\}$. This assumption implies that (13-2) has length 2. First consider case when $\alpha \notin \{0, \beta\}$. Then (13-2) and (13-3) imply the irreducibility (Lemma 3.7). Suppose $\alpha = \beta$. Now (13-2), (13-4) and Proposition 5.1 imply the irreducibility. If $\alpha = 0$, then one gets the irreducibility in the same way as in the case $\rho_0 \not\cong \rho$ (and $\alpha = 0$; one needs to consider multiplicities).

Up to now, we have proved the proposition for representations $\nu^\alpha\rho_0 \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma)$. Now we shall prove the proposition for representations $\nu^\alpha\rho_0 \rtimes \delta([\nu^{\beta-k}\rho, \nu^\beta\rho], \sigma)$. It is enough to consider $\beta \geq 3/2$.

First we have

$$\begin{aligned} &\mu^*(\nu^\alpha\rho_0 \rtimes \delta([\nu^{\beta-k}\rho, \nu^\beta\rho], \sigma)) \\ &= (1 \otimes \nu^\alpha\rho_0 + \nu^\alpha\rho_0 \otimes 1 + \nu^{-\alpha}\tilde{\rho}_0 \otimes 1) \rtimes \left(\sum_{i=0}^{k+1} \mathfrak{s}([\nu^{\beta-k}\rho, \nu^{\beta-i}\rho]) \otimes \delta([\nu^{\beta-i+1}\rho, \nu^\beta\rho], \sigma)\right). \end{aligned} \tag{13-5}$$

This implies

$$\begin{aligned} &\text{s.s.}(s_{GL}(\nu^\alpha\rho_0 \rtimes \delta([\nu^{\beta-k}\rho, \nu^\beta\rho], \sigma))) \\ &\qquad = \nu^{-\alpha}\tilde{\rho}_0 \times \mathfrak{s}([\nu^{\beta-k}\rho, \nu^\beta\rho]) \otimes \sigma + \nu^\alpha\rho_0 \times \mathfrak{s}([\nu^{\beta-k}\rho, \nu^\beta\rho]) \otimes \sigma, \end{aligned} \tag{13-6}$$

$$s_{(kp)}(\nu^\alpha\rho_0 \rtimes \delta([\nu^{\beta-k}\rho, \nu^\beta\rho], \sigma)) \geq \mathfrak{s}([\nu^{\beta-k}\rho, \nu^\beta\rho]) \otimes \nu^\alpha\rho_0 \rtimes \sigma, \tag{13-7}$$

(13-8) $$s_{((k-1)p)}(\nu^\alpha\rho_0 \rtimes \delta([\nu^{\beta-k}\rho,\nu^\beta\rho],\sigma)) \geq \mathfrak{s}([\nu^{\beta-k}\rho,\nu^{\beta-1}\rho]) \otimes \nu^\alpha\rho_0 \rtimes \delta(\nu^\beta\rho,\sigma).$$

The proof now goes similarly as in the previous case (we shall mainly do necessary modifications).

Consider first the case $\rho_0 \not\cong \rho$. The length of (13-6) is 2.

Suppose that $\nu^\alpha\rho_0 \rtimes \sigma$ is irreducible. If $\nu^\alpha\rho_0 \not\cong \nu^{-\alpha}\tilde{\rho}_0$, then (13-6) and (13-7) imply the irreducibility. One gets irreducibility for $\rho_0 \cong \tilde{\rho}_0$ considering multiplicities.

Suppose now that $\nu^\alpha\rho_0 \rtimes \sigma$ reduces. For $\alpha > 0$ we are in the regular situation. Then (13-6) and (i) of Theorem 7.4 in [T7] imply the reducibility. Suppose $\alpha = 0$. Write $\rho_0 \rtimes \sigma = \tau_1 \oplus \tau_2$ where τ_i are irreducible. Multiplicity of $\rho_0 \times \mathfrak{s}([\nu^{\beta-k}\rho,\nu^\beta\rho]) \otimes \sigma$ in $s_{GL}(\mathfrak{s}([\nu^{\beta-k}\rho,\nu^\beta\rho]) \rtimes \tau_1)$, $s_{GL}(\rho_0 \rtimes \delta([\nu^{\beta-k}\rho,\nu^\beta\rho],\sigma))$ and $s_{GL}(\rho_0 \times \nu^{\beta-k}\rho \times \nu^{\beta-k+1}\rho \times \cdots \times \nu^\beta\rho \rtimes \sigma)$ is 1, 2 and 2 respectively. Reducibility now follows from Remark 3.2.

We shall now prove (iii). We shall in the formulas (13-6), (13-7) and (13-8) now take ρ_0 to be ρ. For $\alpha = \beta + 1$ we are in the regular situation. Lemma 7.1 of [T7] and (13-6) imply the reducibility. We consider now the case $\alpha = \beta - k - 1$. If $\beta - k - 1 > 0$, then we are again in the regular situation, and Lemma 7.1 of [T7] and (13-6) again imply the reducibility. Suppose $\beta - k - 1 \leq 0$. Then $\beta - k - 1 \in \{0, -1/2\}$. First consider the case $\beta - k - 1 = 0$, i.e. $\beta = k + 1$. Observe that

$$\text{s.s.}(s_{GL}(\mathfrak{s}([\rho,\nu^\beta\rho]) \rtimes \sigma)) = \sum_{i=-1}^{\beta} \mathfrak{s}([\nu^{-\beta}\rho,\nu^{-i-1}\rho]) \times \mathfrak{s}([\rho,\nu^i\rho]) \otimes \sigma.$$

Now multiplicity of $\mathfrak{s}([\rho,\nu^\beta\rho]) \otimes \sigma$ in $s_{GL}(\mathfrak{s}([\rho,\nu^\beta\rho]) \rtimes \sigma)$, $s_{GL}(\rho \rtimes \delta([\nu\rho,\nu^\beta\rho],\sigma))$ and $s_{GL}(\rho \times \nu^{\beta-k}\rho \times \nu^{\beta-k+1}\rho \times \cdots \times \nu^\beta\rho \rtimes \sigma)$ is 1, 2, 2 respectively. This proves the reducibility of $\rho \rtimes \delta([\nu\rho,\nu^\beta\rho],\sigma)$. Consider now the case $\beta - k - 1 = -1/2$, i.e. $\beta = k + 1/2$. Multiplicity of $\mathfrak{s}([\nu^{-1/2}\rho,\nu^{k+1/2}\rho]) \otimes \sigma$ in $s_{GL}(\mathfrak{s}([\nu^{-1/2}\rho,\nu^{k+1/2}\rho]) \rtimes \sigma)$, $s_{GL}(\nu^{-1/2}\rho \rtimes \delta([\nu^{1/2}\rho,\nu^{k+1/2}\rho],\sigma))$ and $s_{GL}(\nu^{-1/2}\rho \times \mathfrak{s}([\nu^{1/2}\rho,\nu^{k+1/2}\rho]) \rtimes \sigma)$ is 1 in all cases. For the reducibility, it is enough to prove

(13-9) $$s_{GL}(\nu^{-1/2}\rho \rtimes \delta([\nu^{1/2}\rho,\nu^{k+1/2}\rho],\sigma)) \not\leq s_{GL}(\mathfrak{s}([\nu^{-1/2}\rho,\nu^{k+1/2}\rho]) \rtimes \sigma).$$

This follows from the fact that the multiplicity of $\nu^{1/2}\rho \rtimes \delta([\nu^{1/2}\rho,\nu^{k+1/2}\rho]) \otimes \sigma$ in the left-hand side of (13-9) is 1, and it is 0 in the right hand side.

It remains to prove the irreducibility claimed in (iii). Suppose $\alpha \notin \{\pm(\beta-k-1), \pm(\beta+1)\}$. Now (13-6) has length 2. If $\alpha \notin \{0,\beta\}$, then (13-6) and (13-7) imply the irreducibility. In the case $\alpha = \beta$, (13-6), (13-8) and Proposition 5.1 imply the irreducibility. In the case $\alpha = 0$, the irreducibility is obtained considering multiplicities. □

13.2. Theorem. *Let Δ be a segment in irreducible cuspidal representations of general linear groups, and let σ be an irreducible cuspidal representation of S_q. Suppose that $(\mathcal{R}_{(1/2)\mathbb{Z}})$ holds in general and suppose char $F = 0$. Then $\delta(\Delta) \rtimes \sigma$ reduces, if and only if $\rho \rtimes \sigma$ reduces for some $\rho \in \Delta$.*

Proof. Suppose that $\rho \rtimes \sigma$ is irreducible for all $\rho \in \Delta$. Now we shall show the irreducibility of $\delta(\Delta) \rtimes \sigma$.

First we consider the case when $\delta(\Delta)$ is unitary. Now proofs of Propositions 4.1 and 4.2 imply the irreducibility (in the proof of Proposition 4.1 we used only that $\rho' \rtimes \sigma$ is irreducible for $\rho' \in [\nu^{-m-1/2}\rho, \nu^{m+1/2}\rho]$; similarly we needed in the proof of Proposition 4.2 only that $\rho' \rtimes \sigma$ is irreducible for $\rho' \in [\nu^{-n}\rho, \nu^{n}\rho]$).

Suppose now that $\delta(\Delta)$ is not unitarizable. Write $\Delta = [\nu^{\beta}\rho, \nu^{\beta+n}\rho]$, where $\beta \in \mathbb{R}$, $n \in \mathbb{Z}, n \geq 0$, and ρ is a representation of $GL(p,F)$. The irreducibility is obvious for $n = 0$. Suppose $n \geq 1$, and suppose that we have proved irreducibility for lower n. Now we get the irreducibility in the same way as in the proof of Theorem 9.1 (using formulas (9-1) and (9-2), and Proposition 6.3).

Suppose that $\beta \in (1/2)\,\mathbb{Z}$, $\beta \geq 0$, and $\nu^{\beta}\rho \rtimes \sigma$ reduces. Let $k, l \in \mathbb{Z}, k \geq 0, l \geq 0$ such that $|\beta - k| \leq \beta + l$ (the last condition is equivalent to $k \leq 2\beta + l$). To complete the proof of the theorem, it is enough to prove that $\delta([\nu^{\beta-k}\rho, \nu^{\beta+l}\rho]) \rtimes \sigma$ reduces. If $k = 0$ and $\beta > 0$, then Theorem 6.3 of [T7] implies the reducibility. If $k = 0$ and $\beta = 0$, then (iii) of Theorem 9.1 implies the reducibility. Therefore, we shall suppose $k \geq 1$.

Write

$$\text{(13-10)} \qquad \text{s.s.}(s_{GL}(\delta([\nu^{\beta-k}\rho, \nu^{\beta-1}\rho]) \rtimes \delta([\nu^{\beta}\rho, \nu^{\beta+l}\rho], \sigma))) = \left(\sum_{i=0}^{k} \delta([\nu^{-\beta+1+i}\rho, \nu^{-\beta+k}\rho]) \times \delta([\nu^{\beta-i}\rho, \nu^{\beta-1}\rho]) \right) \times \delta([\nu^{\beta}\rho, \nu^{\beta+l}\rho]) \otimes \sigma.$$

$$\text{(13-11)} \qquad \text{s.s.}(s_{GL}(\delta([\nu^{\beta-k}\rho, \nu^{\beta+l}\rho]) \rtimes \sigma) = \sum_{i=0}^{k+l+1} \delta([\nu^{-\beta-l+i}\rho, \nu^{-\beta+k}\rho]) \times \delta([\nu^{\beta+l-i+1}\rho, \nu^{\beta+l}\rho]) \otimes \sigma.$$

$$\text{(13-12)} \qquad \text{s.s.}(s_{GL}(\delta([\nu^{\beta-k}\rho, \nu^{\beta-1}\rho]) \rtimes \delta([\nu^{\beta}\rho, \nu^{\beta+l}\rho]) \rtimes \sigma)) = \left(\sum_{i=0}^{k} \delta([\nu^{-\beta+1+i}\rho, \nu^{-\beta+k}\rho]) \times \delta([\nu^{\beta-i}\rho, \nu^{\beta-1}\rho]) \right) \times \left(\sum_{j=0}^{l+1} \delta([\nu^{-\beta-l+j}\rho, \nu^{-\beta}\rho]) \times \delta([\nu^{\beta+l-j+1}\rho, \nu^{\beta+l}\rho]) \right) \otimes \sigma.$$

Note that

$$\delta([\nu^{\beta-k}\rho, \nu^{\beta+l}\rho]) \rtimes \sigma \leq \delta([\nu^{\beta-k}\rho, \nu^{\beta-1}\rho]) \rtimes \delta([\nu^{\beta}\rho, \nu^{\beta+l}\rho]) \rtimes \sigma,$$

$$\delta([\nu^{\beta-k}\rho, \nu^{\beta-1}\rho]) \rtimes \delta([\nu^{\beta}\rho, \nu^{\beta+l}\rho], \sigma) \leq \delta([\nu^{\beta-k}\rho, \nu^{\beta-1}\rho]) \rtimes \delta([\nu^{\beta}\rho, \nu^{\beta+l}\rho]) \rtimes \sigma.$$

Suppose that $\beta - k > 0$. Then the multiplicity of $\delta([\nu^{\beta-k}\rho, \nu^{\beta+l}\rho]) \otimes \sigma$ in (13-10), (13-11) and (13-12) is 1 in all cases. Therefore, $\delta([\nu^{\beta-k}\rho, \nu^{\beta+l}\rho]) \rtimes \sigma$ and $\delta([\nu^{\beta-k}\rho, \nu^{\beta-1}\rho]) \rtimes$

$\delta([\nu^\beta\rho,\nu^{\beta+l}\rho],\sigma)$ have a common irreducible subquotient. Now to prove the reducibility of $\delta([\nu^{\beta-k}\rho,\nu^{\beta+l}\rho])\rtimes\sigma$, it is enough to prove

$$\delta([\nu^{\beta-k}\rho,\nu^{\beta+l}\rho])\rtimes\sigma\not\leq\delta([\nu^{\beta-k}\rho,\nu^{\beta-1}\rho])\rtimes\delta([\nu^\beta\rho,\nu^{\beta+l}\rho],\sigma). \tag{13-13}$$

This follows from the fact that $\delta([\nu^{-\beta-l}\rho,\nu^{-\beta+k}\rho])\otimes\sigma$ is a subquotient of (13-11), but not of (13-10).

Suppose now that $\beta-k\leq 0$. We shall proceed in a similar way as before. If $\beta\in(1/2)+\mathbb{Z}$ (resp. $\beta\in\mathbb{Z}$), then the multiplicity of $\delta([\nu^{1/2}\rho,\nu^{-\beta+k}\rho])\times\delta([\nu^{1/2}\rho,\nu^{\beta+l}\rho])\otimes\sigma$ (resp. $\delta([\nu\rho,\nu^{-\beta+k}\rho])\times\delta([\rho,\nu^{\beta+l}\rho])\otimes\sigma$) in (13-10), (13-11) and (13-12) is 1 (resp. 2) in all cases. To prove the reducibility, it is enough to show (13-13).

We have before introduced the condition $\beta-k\geq-\beta-l$ (see above). If $\beta-k>-\beta-l$, then $\delta([\nu^{-\beta-l}\rho,\nu^{-\beta+k}\rho])\otimes\sigma$ is a subquotient of (13-11), but not of (13-10). Thus (13-13) holds. If $\beta-k=-\beta-l$, then the multiplicity of $\delta([\nu^{-\beta-l}\rho,\nu^{-\beta+k}\rho])\otimes\sigma$ in (13-11) is 2, and in (13-10) it is 1. This proves again (13-13). The proof is now complete. □

We can also compute Langlands parameters of irreducible subquotients of representations studied in Proposition 13.1 in a similar way as we did in the seventh and the eighth sections. Instead of doing these computations, we shall compute reducibility points in one essentially new situation. Namely, we shall deal with the representation parabolically induced by a regular irreducible square integrable representation (which is related to a non-generic cuspidal reducibility), where Jacquet modules of the inducing representation are not irreducible always. The wider family of such square integrable representations was introduced in the seventh section of [T7]. We shall first briefly remind of this family.

Let ρ be an irreducible unitarizable cuspidal representation of $GL(p,F)$ and let σ be an irreducible cuspidal representation of S_q. Suppose that $\nu^\beta\rho\rtimes\sigma$ reduces for some $\beta>0$. Take $k,l\in\mathbb{Z}$ such that $0<\beta-k\leq\beta\leq\beta+l$. The representation

$$(\nu^{\beta+l}\times\nu^{\beta+l-1}\times\cdots\times\nu^{\beta+2}\rho\times\nu^{\beta+1}\rho)\times(\nu^{\beta-k}\times\nu^{\beta-k+1}\times\cdots\times\nu^{\beta-1}\rho\times\nu^\beta\rho)\rtimes\sigma$$

contains a unique irreducible subrepresentation, which we denote by

$$\delta([\nu^{\beta-k}\rho,\nu^{\beta+l}\rho],\sigma).$$

This subrepresentation is a regular square integrable representation and we have

$$\begin{aligned}\text{(13-14)}\quad & s_{GL}(\delta([\nu^{\beta-k}\rho,\nu^{\beta+l}\rho],\sigma))\\ &\qquad=L(\nu^{\beta-k}\rho,\nu^{\beta-k+1}\rho,\dots,\nu^{\beta-2}\rho,\nu^{\beta-1}\rho,\delta([\nu^\beta\rho,\nu^{\beta+l}\rho]))\otimes\sigma.\end{aligned}$$

We have considered already the above representations if $k=0$ or $l=0$ in Proposition 13.1. If $\beta>1$, then we can find $k\geq 1$ such that $0<\beta-k$. Take any $l\geq 1$. In that case there will exist Jacquet modules of $\delta([\nu^{\beta-k}\rho,\nu^{\beta+l}\rho],\sigma)$ which are reducible (already for some maximal parabolic subgroups). In the following lemma we shall deal with the reducibility of a parabolically induced representation related to one of representations $\delta([\nu^{\beta-k}\rho,\nu^{\beta+l}\rho],\sigma)$. In our case will be $k=l=1$ and $\beta=3$. Such example of a pair ρ and σ exists in the case of symplectic groups by C. Mœglin's results.

The example considered in the following lemma is enough to illustrate the method in the case of general $\delta([\nu^{\beta-k}\rho,\nu^{\beta+l}\rho],\sigma)$.

13.3. Lemma. *Take irreducible unitarizable cuspidal representations ρ and ρ_0 of groups $GL(p,F)$ and $GL(p_0,F)$ respectively, and take an irreducible cuspidal representation σ of S_q. Suppose that $\nu^3\rho \rtimes \sigma$ reduces, and that $\nu^\alpha\rho \rtimes \sigma$ is irreducible for any $\alpha \in \mathbb{R}\backslash\{\pm 3\}$. Let $\alpha \in \mathbb{R}$. Then:*
(i) If $\rho \not\cong \rho_0$, then $\nu^\alpha\rho_0 \rtimes \delta([\nu^2\rho,\nu^4\rho],\sigma)$ reduces, if and only if $\nu^\alpha\rho_0 \rtimes \sigma$ reduces.
(ii) $\nu^\alpha\rho \rtimes \delta([\nu^2\rho,\nu^4\rho],\sigma)$ reduces if and only if $\alpha \in \{\pm 1, \pm 3, \pm 5\}$.

Proof. It is not hard to see from (13-14) that

$$\mu^*(\delta([\nu^2\rho,\nu^4\rho],\sigma)) = 1 \otimes \delta([\nu^2\rho,\nu^4\rho],\sigma) + \nu^4\rho \otimes \delta([\nu^2\rho,\nu^3\rho],\sigma) + \nu^2\rho \otimes \delta([\nu^3\rho,\nu^4\rho],\sigma) + \nu^2\rho \times \nu^4\rho \otimes \delta(\nu^3\rho,\sigma) + L(\nu^2\rho,\delta([\nu^3\rho,\nu^4\rho])) \otimes \sigma.$$

using (13-14). Now from

$$\mu^*(\nu^\alpha\rho_0 \rtimes \delta([\nu^2\rho,\nu^4\rho],\sigma)) = (1 \otimes \nu^\alpha\rho_0 + \nu^\alpha\rho_0 \otimes 1 + \nu^{-\alpha}\tilde{\rho}_0 \otimes 1) \rtimes \mu^*(\nu^\alpha\rho_0 \rtimes \delta([\nu^2\rho,\nu^4\rho],\sigma))$$

we get

$$\text{(13-15)} \quad \text{s.s.}(s_{GL}(\nu^\alpha\rho_0 \rtimes \delta([\nu^2\rho,\nu^4\rho],\sigma))) = \nu^\alpha\rho_0 \times L(\nu^2\rho,\delta([\nu^3\rho,\nu^4\rho])) \otimes \sigma + \nu^{-\alpha}\tilde{\rho}_0 \times L(\nu^2\rho,\delta([\nu^3\rho,\nu^4\rho])) \otimes \sigma,$$

$$\text{(13-16)} \quad s_{(3p)}(\nu^\alpha\rho_0 \rtimes \delta([\nu^2\rho,\nu^4\rho],\sigma)) \geq L(\nu^2\rho,\delta([\nu^3\rho,\nu^4\rho])) \otimes \nu^\alpha\rho_0 \rtimes \sigma.$$

One proves now (i) from (13-15) and (13-16) in a similar way as we proved (i) of Proposition 13.1. If $\nu^\alpha\rho_0 \not\cong (\nu^\alpha\rho_0)\tilde{}$, and $\nu^\alpha\rho_0 \rtimes \sigma$ is irreducible, (13-15) and (13-16) imply the irreducibility of the induced representation. If $\rho_0 \cong (\rho_0)\tilde{}$ and $\rho_0 \rtimes \sigma$ is irreducible, we get the irreducibility from (13-15) and (13-16) considering multiplicities. If $\nu^\alpha\rho_0 \rtimes \sigma$ reduces and $\alpha \neq 0$, then Theorem 7.4 of [T7] implies the reducibility of the induced representation. Suppose that $\rho_0 \rtimes \sigma$ reduces. Write $\rho_0 \rtimes \sigma = \tau_1 \oplus \tau_2$ as a sum of irreducible subrepresentations. Using the representations $L(\nu^2\rho,\delta([\nu^3\rho,\nu^4\rho])) \rtimes \tau_1$ and $L(\nu^2\rho,\delta([\nu^3\rho,\nu^4\rho])) \times \rho_0 \rtimes \sigma$, one can easily show the reducibility of $\rho_0 \rtimes \delta([\nu^2\rho,\nu^4\rho],\sigma)$. For this one needs to compute the multiplicity of $\rho_0 \times L(\nu^2\rho,\delta([\nu^3\rho,\nu^4\rho])) \otimes \sigma$ in the corresponding Jacquet module of each of the last three representations. That multiplicities are 1, 2, 2 respectively. This implies the reducibility.

Suppose now $\alpha \in \mathbb{R}\backslash\{\pm 1, \pm 3, \pm 5\}$. If $\alpha \neq 0$, then the irreducibility of the representation $\nu^\alpha\rho \rtimes \delta([\nu^2\rho,\nu^4\rho],\sigma)$ follows directly from (13-15) and (13-16). If $\alpha = 0$, then the irreducibility follows considering multiplicities.

For $\alpha = 1$ or 5, the reducibility follows from Proposition 7.2 of [T7] (in the first case use the fact that $L(\nu\rho,\nu^2\rho,\delta([\nu^3\rho,\nu^4\rho]))$ is a subquotient of $\nu\rho \times L(\nu^2\rho,\delta([\nu^3\rho,\nu^4\rho]))$, and in the second case that $L(\nu^2\rho,\delta([\nu^3\rho,\nu^5\rho]))$ is a subquotient of $L(\nu^2\rho,\delta([\nu^3\rho,\nu^4\rho])) \times \nu^5\rho$; the last fact follows easily from the Bernstein-Zelevinsky theory).

It remains to show the reducibility of $\nu^3\rho \rtimes \delta([\nu^2\rho, \nu^4\rho], \sigma)$. First note

$$\nu^3\rho \rtimes \delta([\nu^2\rho, \nu^4\rho], \sigma) \leq \nu^3\rho \times \nu^2\rho \rtimes \delta([\nu^3\rho, \nu^4\rho], \sigma), \tag{13-17}$$

$$\mathfrak{s}([\nu^2, \nu^3\rho]) \rtimes \delta([\nu^3\rho, \nu^4\rho], \sigma) \leq \nu^3\rho \times \nu^2 \rtimes \delta([\nu^3\rho, \nu^4\rho], \sigma). \tag{13-18}$$

Further compute

$$\begin{aligned} \text{s.s.}(s_{GL}(\nu^3\rho \rtimes \delta([\nu^2\rho, \nu^4\rho], \sigma))) \\ = \nu^3\rho \times L(\nu^2\rho, \delta([\nu^3\rho, \nu^4\rho])) \otimes \sigma + \nu^{-3}\rho \times L(\nu^2\rho, \delta([\nu^3\rho, \nu^4\rho])) \otimes \sigma, \end{aligned} \tag{13-19}$$

$$\begin{aligned} \text{s.s.}(s_{GL}(\mathfrak{s}([\nu^2, \nu^3\rho]) \rtimes \delta([\nu^3\rho, \nu^4\rho], \sigma))) = \mathfrak{s}([\nu^2\rho \times \nu^3\rho]) \times \delta([\nu^3\rho, \nu^4\rho]) \otimes \sigma \\ + \nu^{-3}\rho \times \nu^2\rho \times \delta([\nu^3\rho, \nu^4\rho]) \otimes \sigma + \mathfrak{s}([\nu^{-3}\rho \times \nu^{-2}\rho]) \times \delta([\nu^3\rho, \nu^4\rho]) \otimes \sigma, \end{aligned} \tag{13-20}$$

$$\begin{aligned} \text{s.s.}(s_{GL}(\nu^3\rho \times \nu^2 \rtimes \delta([\nu^3\rho, \nu^4\rho], \sigma))) \\ = \nu^2\rho \times \nu^3\rho \times \delta([\nu^3\rho, \nu^4\rho]) \otimes \sigma + \nu^2\rho \times \nu^{-3}\rho \times \delta([\nu^3\rho, \nu^4\rho]) \otimes \sigma \\ + \nu^{-2}\rho \times \nu^3\rho \times \delta([\nu^3\rho, \nu^4\rho]) \otimes \sigma + \nu^{-2}\rho \times \nu^{-3}\rho \times \delta([\nu^3\rho, \nu^{-4}\rho]) \otimes \sigma. \end{aligned} \tag{13-21}$$

From the term $\nu^{-3}\rho \times L(\nu^2\rho, \delta([\nu^3\rho, \nu^4\rho])) \otimes \sigma$ in (13-19), and terms $\nu^2\rho \times \nu^{-3}\rho \times \delta([\nu^3\rho, \nu^4\rho]) \otimes \sigma$ in (13-20) and (13-21) we see that $\nu^3\rho \rtimes \delta([\nu^2\rho, \nu^4\rho], \sigma)$ and $\mathfrak{s}([\nu^2, \nu^3\rho]) \rtimes \delta([\nu^3\rho, \nu^4\rho], \sigma)$ must have a common irreducible subquotient. For the reducibility of $\nu^3\rho \rtimes \delta([\nu^2\rho, \nu^4\rho], \sigma)$ it is enough to show

$$\nu^3\rho \rtimes \delta([\nu^2\rho, \nu^4\rho], \sigma) \not\leq \mathfrak{s}([\nu^2, \nu^3\rho]) \rtimes \delta([\nu^3\rho, \nu^4\rho], \sigma). \tag{13-22}$$

From (13-19) and (13-20) we see that for this is enough to show

$$\nu^3\rho \times L(\nu^2\rho, \delta([\nu^3\rho, \nu^4\rho])) \not\leq \mathfrak{s}([\nu^2\rho \times \nu^3\rho]) \times \delta([\nu^3\rho, \nu^4\rho]). \tag{13-23}$$

The eleventh section of [Z] implies that right hand side of (13-23) is irreducible (more precisely, $L(\nu^3\rho, \nu^2\rho, \delta([\nu^3\rho, \nu^4\rho])) = \mathfrak{s}([\nu^2\rho, \nu^3\rho]) \times \delta([\nu^3\rho, \nu^4\rho]))$. Therefore, it is enough to prove that the left hand side is reducible. It is not hard to compute (using the eleventh section of [Z]) that

$$\nu^3\rho \times L(\nu^2\rho, \delta([\nu^3\rho, \nu^4\rho])) = L(\nu^3\rho, \nu^2\rho, \delta([\nu^3\rho, \nu^4\rho])) + L(\delta[\nu^2\rho, \nu^3\rho]), \delta([\nu^3\rho, \nu^4\rho])).$$

The proof of the lemma is now complete. □

13.4. Remark. *The above lemma suggests that it might be more convenient to denote* $\delta([\nu^{\beta-k}\rho, \nu^{\beta+l}\rho], \sigma)$ *by* $\delta(L(\nu^{\beta-k}\rho, \nu^{\beta-k+1}\rho, \dots, \nu^{\beta-2}\rho, \nu^{\beta-1}\rho, \delta([\nu^{\beta}\rho, \nu^{\beta+l}\rho])), \sigma)$ *for some purposes.*

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Department of Mathematics, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia
E-mail address: tadic@math.hr