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STRUCTURE ARISING FROM INDUCTION AND JACQUET MODULES OF REPRESENTATIONS OF CLASSICAL p-ADIC GROUPS

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Introduction

Jacquet modules are very useful in the study of parabolically induced representations of reductive groups over a p-adic field F (we shall assume char $F \neq 2$ in this paper). It is very hard to describe explicitly the structure of Jacquet modules of parabolically induced representations, particularly in the most interesting cases. There exists a description of factors of certain filtrations on them. That description was done by J. Bernstein and A. V. Zelevinsky in [BZ2], and by W. Casselman in [C]. In the case of general linear groups, the functor of parabolic induction and the Jacquet functor induce a structure of $\mathbb{Z}_+$-graded Hopf algebra on the sum R of Grothendieck groups of categories of smooth representations of $\mathrm{GL}(n,F)$'s of finite length, $n \geq 0$ ([Z1]). The multiplication $m : R \times R \to R$ is defined using parabolic induction, while the comultiplication $m^* : R \to R \otimes R$ is defined in terms of Jacquet modules. The most interesting part of the structure is the property that $m^* : R \to R \otimes R$ is a ring homomorphism (Hopf axiom). This enables one to compute composition series of parabolically induced representations in a very simple way. It is interesting to note that the existence of this strong structure did not have serious impact on the development of the representation theory of $\mathrm{GL}(n,F)$. One of the reasons for that may lay in the fact that for $\mathrm{GL}(n,F)$ there existed a very powerful theory of Gelfand-Kazhdan derivatives, and the main results of [Z1] were obtained using them. Nevertheless, A.V. Zelevinsky showed in [Z2] that some interesting parts of the representation theory of $\mathrm{GL}(n)$ over a finite field can be obtained as a structure theory of such Hopf algebra (defined in this setting). Besides that, one of the main tools for the study of representation theory of $\mathrm{GL}(n)$ over a central division F-algebra in [T1] is such Hopf algebra structure (in this situation are not available Gelfand-Kazhdan derivatives). It is natural to ask does some structure of this kind exist for other (simple split) classical p-adic groups. Since Levi factors of classical groups are isomorphic to direct products of general linear groups and smaller groups from the same series, one can expect for such structure to have some relation with R (if it exists).

In the third section we define the direct sum of Grothendieck groups $R(S)$, which corresponds either to the series $\mathrm{Sp}(n,F)$, $n \geq 0$, or $\mathrm{SO}(2n+1,F)$, $n \geq 0$, in a similar way as R was defined for general linear groups. The action $\rtimes$ of R on $R(S)$ is defined using the parabolic induction. In this way $R(S)$ becomes $\mathbb{Z}_+$-graded R-module. We can make $R(S)$ a $\mathbb{Z}_+$-graded comodule over R. The comodule structure map $\mu^* : R(S) \to R \otimes R(S)$ is again defined using the Jacquet modules, similarly as in the case of $\mathrm{GL}(n,F)$. It is not hard to see that $R(S)$ is not a Hopf module over R (see Remark 7.3). In this paper we

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TeX determine the structure of $R(S)$ over R (as we shall see, this structure is not far from the structure of Hopf module).

We shall now briefly describe this structure. First we need one definition. Let $\mathcal{H}$ be a Hopf algebra with comultiplication $m^* : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$. Suppose that $\mathcal{M}$ is a module and a comodule over $\mathcal{H}$. Let $\mu^* : \mathcal{H} \to \mathcal{H} \otimes \mathcal{M}$ be the comodule structure map. Suppose that

$$\Psi : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$$

is a ring homomorphism. Note that $\mathcal{H} \otimes \mathcal{H}$ acts in an obvious way on $\mathcal{H} \otimes \mathcal{M}$. Define $\mathcal{H}$-module structure on $\mathcal{H} \otimes \mathcal{M}$ by $h'.(h \otimes m) = \Psi(h')(h \otimes m)$. Then we shall say that $\mathcal{H} \otimes \mathcal{M}$ has a $\mathcal{H}$-module structure with respect to Ψ. We shall say that $\mathcal{M}$ is a Ψ-Hopf module if μ^* is a homomorphism of $\mathcal{H}$-modules, where we consider on $\mathcal{H}\otimes\mathcal{M}$ the $\mathcal{H}$-module structure with respect to Ψ (for $\Psi = m^*$, we get the usual Hopf module).

We return now to R and $R(S)$. The contragredient functor defines an automorphism $\sim\colon R \to R$ in a natural way. Let $s : R \otimes R \to R \otimes R$ be the homomorphism defined by $s(\sum_i x_i \otimes y_i) = \sum_i y_i \otimes x_i$. Consider the composition

$$M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^* : R \to R \otimes R$$

(here 1 denotes the identity mapping). Then:

Theorem. *$R(S)$ is a M^*-Hopf module over R.*

The natural action of $R \otimes R$ on $R \otimes R(S)$ will be also denoted by $\rtimes$. Let π be an irreducible smooth representation of $\mathrm{GL}(n, F)$ and let σ be a similar representation of $\mathrm{Sp}(m, F)$ or $\mathrm{SO}(2m + 1, F)$. Then the mail claim of the above theorem is that

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma).$$

The above formula connects the module and the comodule structures on $R(S)$. This is a combinatorial formula which enables one to obtain, in a simple manner, factors of filtrations of Jacquet modules of parabolically induced representations.

We expect that for other series of classical p-adic (not necessarily split) groups, we shall have also structures of Ψ-Hopf modules, with the same or a very similar Ψ to the above one. We also expect that the calculation which we do in this paper will be possible to use for other series of groups (particularly for those ones which have the same Weyl groups as symplectic groups).

At this point, let us explain a connection of this paper with an observation of J. Bernstein, P. Deligne and D. Kazhdan. In section 6.3 of [BDK], they noted that it would be interesting to study more thoroughly the combinatorial structure coming from composition of parabolic induction and Jacquet modules (they noted that such structures have some relation with Hopf algebras). For the simple split classical groups of type B_n and C_n we solved at least a part of the problem: such structure is a M^*-Hopf module over the Hopf algebra R. It seems that it is natural to look for an answer to the above problem in settings of groups of the same type.

Our interest for the investigation of the structure of $R(S)$ is motivated by possible applications to the representation theory of classical p-adic groups. In comparison with the well understood representation theory of general linear groups, representation theory for other classical groups is very poorly understood. Using the structure of $R(S)$, one can apply the representation theory of general linear groups in the representation theory of the other classical groups. An example of such use of this structure in the representation theory of the other classical groups, for the questions of reducibility of parabolically induced representations, can be found in [T6] and [J]. Our methods based on the structure of $R(S)$ provide a tool for settling such questions relatively simply. This structure can be also applied to the problem of construction of non-cuspidal square integrable representations of classical groups (see [T4] and [T5]).

Now we shall give more information about the content of this paper, section by section. In the first section, we recall of a general result of J. Bernstein and A. V. Zelevinsky in ([BZ2]), and also of W. Casselman ([C]). The second section recalls some general notation from the setting of the general linear groups. The main reference is [Z1]. The third section contains the basic facts regarding module and comodule structures for symplectic groups. The paper [T3] may be helpful for additional information regarding this structure. The fourth section is the technical heart of this paper. In this section the necessary calculations in the Weyl group are done. The complete structure for $\mathrm{Sp}(n, F)$ and $\mathrm{GSp}(n, F)$ is described in the fifth section. That section also contains the proof of the combinatorial formula. The sixth section describes what the structure is in the case of $\mathrm{SO}(2n+1, F)$. The relation with Hopf modules is studied in the seventh section.

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1. A general result on factors of some filtrations of Jacquet modules

In this section we shall recall of the Geometric Lemma of J.N. Bernstein and A.V. Zelevinsky from [BZ2]. The same result was obtained independently by W. Casselman in [C]. In this section we shall briefly present that result. For more details one should consult papers [BZ2] and [C]. Our presentation is based on Casselman's paper.

Let F be a non-archimedean local field. Let G denote the group of rational points of a connected reductive group defined over the field F. Fix a maximal split torus A in G. Let P be a minimal parabolic subgroup of G which contains A. Denote by Σ the set of (reduced) roots of G relative to A. The choice of $P_{\min}$ determines a basis Δ of Σ. It also determines a set of positive roots in Σ. The Weyl group W of Σ is a quotient of the normalizer of A in G by the centralizer of A in G. For $\Theta \subseteq \Delta$ let P_Θ be the standard parabolic subgroup of G determined by Θ. The unipotent radical of P_Θ is denoted by N_Θ. Let A_Θ be the connected component of $\cap_{\alpha\in\Theta}\mathrm{Ker}(\alpha)$. The centralizer of A_Θ in G is denoted by M_Θ. Then, $P_\Theta = M_\Theta N_\Theta$ is a Levi decomposition of P_Θ.

The group W acts on A by conjugation. For $\alpha \in \Delta$ set

$$W^\alpha = \{w \in W; w\alpha > 0\} \quad \text{and} \quad {}^\alpha W = \left\{w \in W; w^{-1}\alpha > 0\right\}.$$

Recall that $w \in W$ acts on a character χ by $(w\chi)(a) = \chi(w^{-1}a)$. Clearly $(W^\alpha)^{-1} =^\alpha W$.

For subsets $\Omega, \Omega_1, \Omega_2$ of the set of simple roots put

$$[W/W_\Omega] = \bigcap_{\alpha\in\Omega} W^\alpha, \quad [W_\Omega \backslash W] = \bigcap_{\alpha\in\Omega} {}^\alpha W, \quad [W_{\Omega_1}\backslash W/W_{\Omega_2}] = [W_{\Omega_1}\backslash W] \cap [W/W_{\Omega_2}],$$

as was done in [C]. Clearly $[W_{\Omega_1}\backslash W/W_{\Omega_2}]^{-1} = [W_{\Omega_2}\backslash W/W_{\Omega_1}]$.

Let P be a parabolic subgroup of G. Suppose that $P = MN$ is a Levi decomposition of P. Let σ be a smooth admissible representation of M. We denote by $\mathrm{Ind}_P^G(\sigma)$ the parabolically induced representation of G by σ from P. The induction that we consider is normalized; it induces the unitarizable representations to the unitarizable ones.

Let π be a smooth representation of G. Take a parabolic subgroup $P = MN$ of G. Set $V(N) = \mathrm{span}_{\mathbb{C}}\{\pi(n)v - v; n \in N, v \in V\}$. Define $V_N = V/V(N)$. Then there is a natural quotient action of M on V_N. Let δ_P be the modular function of P. We consider the action of M on V_N which is the quotient action twisted by $\delta_P^{-1/2}$. This representation of M is denoted by $r_{M,G}(\pi)$. The representation $r_{M,G}(\pi)$ is called the Jacquet module of π with respect to P.

Let $P = MN$ be a parabolic subgroup of G. Suppose that σ is a smooth representation of M. For $x \in G$, $x^{-1}\sigma$ denotes the representation of $x^{-1}Mx$ which is given by the formula $(x^{-1}\sigma)(x^{-1}mx) = \sigma(m)$, for $m \in M$. We can say also that $(x^{-1}\sigma)(m') = \sigma(xm'x^{-1})$, $m' \in x^{-1}Mx$.

We can state now the result of J.N. Bernstein, A.V. Zelevinsky (Geometric Lemma in [BZ2]) and W. Casselman (Proposition 6.3.3 of [C]). Let $\Theta, \Omega \subseteq \Delta$. Let σ be a smooth admissible representation of M_Θ. Then one can enumerate elements $w_1, w_2, \ldots, w_m$ of $[W_\Theta\backslash W/W_\Omega]$ in a such way that there exists a filtration

$$\{0\} = \tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_m = r_{M_\Omega,G}(\mathrm{Ind}_{P_\Theta}^G(\sigma))$$

of $r_{M_\Omega,G}(\mathrm{Ind}_{P_\Theta}^G(\sigma))$ such that for $1 \leq i \leq m$,

$$\tau_i/\tau_{i-1} \cong \mathrm{Ind}_{w^{-1}P_\Theta w\cap M_\Omega}^{M_\Omega}(w^{-1}(r_{M_{\Theta\cap w\Omega},M_\Theta}(\sigma))).$$

Note that $w^{-1}P_\Theta w \cap M_\Omega$ is a parabolic subgroup in M_Ω. One can take the subgroup $M_{w^{-1}\Theta\cap\Omega}$ for a Levi factor of that parabolic subgroup.

There is a canonical map from the objects of the category of all smooth representations of finite length of G, into the Grothendieck group of this category. This map is called semi-simplification, and we denote it by s.s.. There is a natural cone of positive elements in the above Grothendieck group (image of s.s.). Therefore, we have a natural partial order $\leq$ on such a group.

2. General linear group

Let F denote a non-archimedean local field. We shall briefly recall of some of the standard notation of the representation theory of p-adic general linear groups in this section.We shall mainly follow the notation introduced by J.N. Bernstein and A.V. Zelevinsky in [BZ2] and [Z1]. The proofs of the results which will be quoted in this section can be found there. All representations that we consider in this paper will be smooth and admissible.

The minimal parabolic subgroup of $\mathrm{GL}(n,F)$, which consists of all upper triangular matrices in $\mathrm{GL}(n,F)$, will be fixed.

Let π_1 be an admissible representation of $\mathrm{GL}(n_1,F)$, and let π_2 be an admissible representation of $\mathrm{GL}(n_2,F)$. Denote by $\pi_1\times\pi_2$ the representation of $\mathrm{GL}(n_1+n_2,F)$ which is parabolically induced by $\pi_1\otimes\pi_2$ from the standard parabolic subgroup whose Levi factor is naturally isomorphic to $\mathrm{GL}(n_1,F)\times\mathrm{GL}(n_2,F)$. A simple but very useful fact is that $\pi_1\times(\pi_2\times\pi_3)\cong(\pi_1\times\pi_2)\times\pi_3$ where π_i denotes an admissible representation of $\mathrm{GL}(n_i,F)$, for $i=1,2,3$.

The Grothendieck group of the category of smooth representations of $\mathrm{GL}(n,F)$ of finite length is denoted by R_n (the canonical mapping from the category to R_n was denoted by s.s.). Set $R=\underset{n\geq 0}{\oplus} R_n$. Equivalence classes of irreducible smooth representations of $\mathrm{GL}(n,F)$ form a $\mathbb{Z}$-basis of R_n. One defines a multiplication in R in a following way. Let $r_1=\sum_{i=1}^n a_i\pi_i,\quad r_2=\sum_{j=1}^m b_j\tau_j$, where π_i and τ_j are (classes of) irreducible smooth representations, and let $a_i,b_j\in\mathbb{Z}$. The product $r_1\times r_2$ is by definition $r_1\times r_2=\sum_{i=1}^n\sum_{j=1}^m a_ib_j\ \mathrm{s.s.}(\pi_i\times\tau_j)$. In this way R becomes a commutative associative graded ring. The induced mapping $R\otimes R\to R,\quad \sum_i\pi_i\otimes\tau_i\mapsto\sum_i\pi_i\times\tau_i$ is denoted by m.

Let π be a smooth representation of finite length of $\mathrm{GL}(n,F)$. Take an ordered partition $\alpha=(n_1,\dots,n_k)$ of n. There is a standard parabolic subgroup P_α of $\mathrm{GL}(n,F)$ whose Levi factor M_α is naturally isomorphic to $\mathrm{GL}(n_1,F)\times\ldots\times\mathrm{GL}(n_k,F)$ The Jacquet module of π with respect to P_α is denoted by $r_{\alpha,(n)}(\pi)$. One may consider in a natural way $\mathrm{s.s.}(r_{\alpha,(n)}(\pi))\in R_{n_1}\otimes\ldots\otimes R_{n_k}$. Set

$$m^*(\pi)=\sum_{k=0}^n \mathrm{s.s.}(r_{(k,n-k),(n)}(\pi))\in R\otimes R.$$

One extends m^* $\mathbb{Z}$-linearly to all of R.

Recall that $\mathcal{H}$ is a Hopf algebra if there are algebra and coalgebra structures on $\mathcal{H}$ (see [Sw] for these definitions), such that the comultiplication map $\mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$ is a ring homomorphism. We shall say that a Hopf algebra $\mathcal{H}$ is graded, if $\mathcal{H}$ is $\mathbb{Z}_+$-graded as an abelian group, and if the multiplication and the comultiplication maps are $\mathbb{Z}_+$-graded. We shall deal in this paper only with Hopf algebras over $\mathbb{Z}$.

With the multiplication m and the comultiplication m^*, R is a graded Hopf algebra ([Z1]).

For $g\in\mathrm{GL}(n,F)$ we denote by ${}^tg$ (resp. ${}^\tau g$) the transposed matrix of g (resp. the transposed matrix of g with respect to the second diagonal). If π is a representation of $\mathrm{GL}(n,F)$, then ${}^\tau\pi^{-1}$ denotes the representation $g\mapsto\pi({}^\tau g^{-1})$. We denote by $\tilde\pi$ the (smooth) contragredient representation of π. If π is irreducible, then by Theorem 2. of [GK] we have ${}^\tau\pi^{-1}\cong\tilde\pi$.

The center of $\mathrm{GL}(n,F)$ is identified with $F^\times$ in a standard way. Therefore, each character of the center will also be considered as a character of $F^\times$, and conversely, each character of $F^\times$ will also be considered as a character of the center.

3. Groups $\mathrm{Sp}(n,F)$ and $\mathrm{GSp}(n,F)$

In the rest of this paper we fix a local non-archimedean field of characteristic different from two.

Consider the $n \times n$ matrix
$$\begin{bmatrix} 00 & \dots & 01 \\ 00 & \dots & 10 \\ \vdots & & \\ 10 & \dots & 0 \end{bmatrix}.$$

Denote it by J_n. The identity matrix is denoted by I_n. For a $2n\times 2n$ matrix S with entries in F, set

$$^{\times}S = \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} {}^{t}S \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix},$$

as it was done in [F]. Note that $^{\times}(S_1S_2) = {}^{\times}S_2\, {}^{\times}S_1$.

The group $\mathrm{Sp}(n,F)$ (resp. $\mathrm{GSp}(n,F)$) consists of all $2n \times 2n$ matrices over F which satisfy $^{\times}SS = I_{2n}$ (resp $^{\times}SS \in F^{\times}I_{2n}$). We define $\mathrm{Sp}(0,F)$ to be the trivial group, and we define $\mathrm{GSp}(0,F)$ to be $F^{\times}$. We shall think of these two groups as groups of 0×0 matrices, formally.

For $S \in \mathrm{GSp}(n,F)$, we denote by $\psi(S)$ the element of the field F which satisfies $^{\times}SS = \psi(S)I_{2n}$. Clearly, $\psi : \mathrm{GSp}(n,F) \to F^{\times}$ is a homomorphism. The kernel of ψ is $\mathrm{Sp}(n,F)$. We have the following semidirect product decomposition

$$\mathrm{GSp}(n,F) = \mathrm{Sp}(n,F) \rtimes \left\{ \begin{bmatrix} I_n & 0 \\ 0 & \lambda I_n \end{bmatrix}; \lambda \in F^{\times} \right\}.$$

We identify characters of $F^{\times}$ with characters of $\mathrm{GSp}(n,F)$ using ψ.

We fix the maximal split torus A_0 in $\mathrm{Sp}(n,F)$ (resp. $\mathrm{GSp}(n,F)$) which consists of all diagonal matrices in $\mathrm{Sp}(n,F)$ (resp. $\mathrm{GSp}(n,F)$). The minimal parabolic subgroup $P_{\min}$, consisting of all upper triangular matrices in $\mathrm{Sp}(n,F)$ (resp. $\mathrm{GSp}(n,F)$), is fixed.

We define $a : (F^{\times})^n \to A_0$, $(x_1,\dots,x_n) \mapsto \mathrm{diag}(x_1,\dots,x_n,x_n^{-1},\dots,x_1^{-1})$. in the case of $\mathrm{Sp}(n,F)$. This is an isomorphism of $(F^{\times})^n$ onto A_0. In the case of $\mathrm{GSp}(n,F)$, we define $a : (F^{\times})^n \times F^{\times} \to A_0$,

$$(x_1,\dots,x_n,x) \mapsto \begin{bmatrix} \mathrm{diag}(x_1,\dots,x_n) & 0 \\ 0 & x\mathrm{diag}(x_n^{-1},\dots,x_1^{-1}) \end{bmatrix}.$$

This is now an isomorphism of $(F^{\times})^n \times F^{\times}$ onto A_0.

The Weyl groups defined by above tori in $\mathrm{Sp}(n,F)$ and $\mathrm{GSp}(n,F)$ are naturally isomorphic. These groups are denoted by W.

The simple roots in $\mathrm{Sp}(n,F)$ determined by $P_{\min}$, are

$$\alpha_i(a(x_1,\dots,x_n)) = x_ix_{i+1}^{-1}, \quad 1 \le i \le n-1, \quad \alpha_n(a(x_1,\dots,x_n)) = x_n^2.$$

The simple roots in $\mathrm{GSp}(n,F)$ are

$$\alpha_i(a(x_1,\dots,x_n,x)) = x_ix_{i+1}^{-1}, \quad 1 \le i \le n-1, \quad \alpha_n(a(x_1,\dots,x_n,x)) = x_n^2x^{-1}.$$

The sets of simple roots are denoted by Δ.

The standard parabolic subgroups of $\mathrm{Sp}(n, F)$ and $\mathrm{GSp}(n, F)$ are parameterized by subsets of Δ. We shall explain another parameterization of the standard parabolic subgroups.

If $X_1, \cdots, X_k$ are $p_i \times p_i$ matrices, then the quasi diagonal $(p_1 + \cdots + p_k) \times (p_1 + \cdots + p_k)$ matrix which has on the quasi diagonal matrices $X_1, \cdots, X_k$, will be denoted by

$$\text{q-diag}\ (X_1, \cdots, X_k).$$

Take an ordered partition $\alpha = (n_1, \dots, n_k)$ into positive integers of some non-negative integer $m \leq n$. If $m = 0$, then the only partition will be denoted by ϕ or (0). Let

$$M_\alpha = \left\{\text{q-diag}\,(g_1, \cdots, g_k, h, \ {}^\tau g_k^{-1}, \cdots, \ {}^\tau g_1^{-1}); g_i \in \mathrm{GL}(n_i, F), h \in \mathrm{Sp}(n-m, F)\right\}$$

Now, $P_\alpha = M_\alpha N_{\min}$ is a standard parabolic subgroup of $\mathrm{Sp}(n, F)$. It corresponds to the subset $\{\alpha_1, \dots, \alpha_n\} \backslash \{\alpha_{n_1}, \alpha_{n_1+n_2}, \dots, \alpha_{n_1+\ldots+n_k}\}$. The unipotent radical of P_α will be denoted by N_α.

One obtains the standard parabolic subgroups P_α of $\mathrm{GSp}(n, F)$ (resp. their Levi factors M_α) by multiplying the standard parabolic subgroups in $\mathrm{Sp}(n, F)$ (resp. multiplying their Levi factors) with the subgroup

$$\left\{\begin{bmatrix} I_n & 0 \\ 0 & \lambda I_n \end{bmatrix}; \lambda \in F^\times\right\}.$$

In the case of $\mathrm{GSp}(n, F)$, we have that M_α is the set of all

$$\text{q-diag}\,(g_1, \cdots, g_k, h, \ \psi(h)\ {}^\tau g_k^{-1}, \cdots, \ \psi(h)\ {}^\tau g_1^{-1})$$

where $g_i \in \mathrm{GL}(n_i, F), h \in \mathrm{GSp}(n-m, F)$. Note that in the case of $\mathrm{Sp}(n, F)$, M_α is naturally isomorphic to $\mathrm{GL}(n_1, F) \times \ldots \times \mathrm{GL}(n_k, F) \times \mathrm{Sp}(n-m, F)$. In the case of $\mathrm{GSp}(n, F)$, M_α is naturally isomorphic to $\mathrm{GL}(n_1, F) \times \ldots \times \mathrm{GL}(n_k, F) \times \mathrm{GSp}(n-m, F)$.

Two parabolic subgroups P_α and P_β are associate if and only if α and β are partitions of the same number, and if they are equal as unordered partitions.

Sometimes, we shall attach to an ordered partition of $m \leq n$ into $\alpha = (n_1, \dots, n_k)$ into non-negative integers, the parabolic subgroup $P_\alpha = M_\alpha N_\alpha$ by the same formulas. We shall get the same objects if we remove all zeros from α and apply the previous definition.

Let τ be an admissible representations of $\mathrm{Sp}(n, F)$ (resp. $\mathrm{GSp}(n, F)$). Let π be an admissible representation of $\mathrm{GL}(m, F)$. We denote by $\pi \rtimes \sigma$ the parabolically induced representation of $\mathrm{Sp}(n+m, F)$ (resp. $\mathrm{GSp}(n+m, F)$) from $P_{(n)}$ of $\pi \otimes \sigma$. Here $\pi \otimes \sigma$ maps

$$\begin{bmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & {}^\tau g^{-1} \end{bmatrix} \ (\text{resp.} \begin{bmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \psi(h){}^\tau g^{-1} \end{bmatrix})$$

to $\pi(g) \otimes \sigma(h)$. The following simple proposition can be proved directly. For more details one can also consult [T3].

Proposition 3.1. *(i) Suppose that π, π_1 and π_2 are admissible representations of groups $GL(n,F)$, $GL(n_1,F)$ and $GL(n_2,F)$ respectively. Let σ be an admissible representation of $Sp(n,F)$ or $GSp(m,F)$. Then $\pi_1 \rtimes (\pi_2 \rtimes \sigma) \cong (\pi_1 \times \pi_2) \rtimes \sigma$ and $(\pi \rtimes \sigma)^\sim \cong \tilde{\pi} \rtimes \tilde{\sigma}$.*

(ii) For an admissible representation π of $GL(n,F)$, for an admissible representation σ of $GSp(m,F)$, and for a character χ of $F^\times$ we have $\chi(\pi \rtimes \sigma) = \pi \rtimes (\chi\sigma)$. $\square$

The Grothendieck group of the category of all finite length smooth representations of $\mathrm{Sp}(n,F)$ (resp. $\mathrm{GSp}(n,F)$) is denoted by $R_n(S)$ (resp. $R_n(G)$). Set

$$R(S) = \underset{n\geq 0}{\oplus} R_n(S), \quad R(G) = \underset{n\geq 0}{\oplus} R_n(G).$$

We shall now introduce a multiplication $\rtimes : R \times R(S) \to R(S)$ (resp. $\rtimes : R \times R(G) \to R(G)$). For an irreducible smooth representation π from R and for an irreducible smooth representation σ from $R(S)$ (resp. $R(G)$), we put $\pi \rtimes \sigma = \text{ s.s.}(\pi \rtimes \sigma)$. We extend $\rtimes$ $\mathbb{Z}$-bilinearly to $R \times R(S)$ (resp. $R \times R(G)$).

In a natural way, one defines the contragredient involution $^\sim$ on R, $R(S)$ and $R(G)$.

Proposition 3.2. *(i) With the multiplication $\rtimes$, $R(S)$ is a $\mathbb{Z}_+$-graded module over R. One has $\pi \rtimes \sigma \cong \tilde{\pi} \rtimes \sigma$, for $\pi \in R, \sigma \in R(S)$.*

(ii) With the multiplication $\rtimes$, $R(G)$ is a $\mathbb{Z}_+$-graded module over R. We have $\pi \rtimes \sigma \cong \tilde{\pi} \rtimes \omega_\pi \sigma$, for $\sigma \in R(G)$ and for an irreducible representation π of $GL(n,F)$ whose central character is denoted by ω_π. $\square$

It is easy to prove the last proposition directly, using [BZ2] or [C]. One can also consult [T3] for more details.

We denote by $\mu : R \otimes R(S) \to R(S)$, (resp.$\mu : R \otimes R(G) \to R(G)$), the $\mathbb{Z}$-bilinear mapping which satisfies $\mu(\pi \otimes \sigma) = \text{ s.s.}(\pi \rtimes \sigma)$, where $\pi \in R, \sigma \in R(S)$ (resp. $\sigma \in R(G)$).

Let σ be a smooth representation of $\mathrm{Sp}(n,F)$ of finite-length. Let $\alpha = (n_1, \dots, n_k)$ be an ordered partition of a non-negative integer $m \leq n$. The Jacquet module of σ for P_α is denoted by $s_{\alpha,(0)}(\sigma)$. Since M_α is naturally isomorphic to

$$\mathrm{GL}(n_1,F) \times \mathrm{GL}(n_2,F) \times \ldots \times \mathrm{GL}(n_k,F) \times \mathrm{Sp}(n-m,F),$$

we may consider $\text{s.s.}(s_{\alpha,(0)}(\sigma)) \in R_{n_1} \otimes \ldots R_{n_k} \otimes R_{n-m}(S)$. We define a $\mathbb{Z}$-linear mapping $\mu^* : R(S) \to R \otimes R(S)$. It is defined on the basis of irreducible smooth representations by

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}(s_{(k),(0)}(\sigma)).$$

The mapping μ^* is $\mathbb{Z}_+$-graded. It is coassociative, i.e., the following diagram commutes

$$\begin{array}{ccc} R(S) & \xrightarrow{\mu^*} & R \otimes R(S) \\ \mu^* \downarrow & & \downarrow 1 \otimes \mu^* \\ R \otimes R(S) & \xrightarrow{m^* \otimes 1} & R \otimes R \otimes R(S). \end{array}$$

One defines $\mu^* : R(G) \to R \otimes R(G)$ by the same formula as for $R(S)$. Again, μ^* is $\mathbb{Z}_+$-graded. It is also coassociative.

4. Calculations in the root system, the case of C_n

Before we go to the computation of the Jacquet modules, we will do some preliminary computations in the Weyl group. First we shall describe the Weyl group in more detail.

The group of permutations of $\{1, 2, \ldots, n\}$ is denoted by $\mathrm{Sym}(n)$. We define an action of $p \in \mathrm{Sym}(n)$ on the standard maximal torus A_0 in $\mathrm{Sp}(n, F)$ by

$$pa(x_1, \ldots, x_n) = a(x_{p^{-1}(1)}, \ldots, x_{p^{-1}(n)}).$$

In the case of $\mathrm{GSp}(n, F)$, the action is given by

$$pa(x_1, \ldots, x_n, x) = a(x_{p^{-1}(1)}, \ldots, x_{p^{-1}(n)}, x).$$

We define an action of $\epsilon = (\epsilon_i) \in \{\pm 1\}^n$ on A_0 in the case of $\mathrm{Sp}(n, F)$ by

$$\epsilon a(x_1, \ldots, x_n) = a(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}).$$

In the case of $\mathrm{GSp}(n, F)$ we define

$$\epsilon a(x_1, \ldots, x_n, x) = a(x_1^{\epsilon_1} x^{(1-\epsilon_1)/2}, \ldots, x_n^{\epsilon_n} x^{(1-\epsilon_n)/2}, x).$$

We identify the Weyl group W with the group of transformations that W induces on A_0 acting by conjugation, and also we identify $\mathrm{Sym}(n)$ and $\{\pm 1\}^n$ with the groups of transformations of A_0. Then $\{\pm 1\}^n, \mathrm{Sym}(n) \subseteq W$. Also $W = \{\pm 1\}^n \rtimes Sym(n)$, and

$$p(\epsilon_1, \ldots, \epsilon_n)\, p^{-1} = \left(\epsilon_{p^{-1}(1)}, \ldots, \epsilon_{p^{-1}(n)}\right).$$

Therefore $[p(\epsilon_1, \ldots, \epsilon_n)]^{-1} = p^{-1}\left(\epsilon_{p^{-1}(1)}, \ldots, \epsilon_{p^{-1}(n)}\right)$ and

$$[(\epsilon_1, \ldots, \epsilon_n)p]^{-1} = \left(\epsilon_{p(1)}, \ldots, \epsilon_{p(n)}\right) p^{-1}$$

We consider the simple roots $\alpha_1, \ldots, \alpha_n$ determined by the standard minimal parabolic subgroup. An element $w \in W$ acts on a character χ by $(w\chi)(a) = \chi(w^{-1}a)$.

Now we consider the case of $\mathrm{Sp}(n, F)$. Introduce the characters a_i^* of A_0, given by

$$a_i^* : a(x_1, \ldots, x_n) \mapsto x_i\ , \qquad 1 \le i \le n.$$

Then $\epsilon a_i^* = (a_i^*)^{\epsilon_i}, \quad p a_i^* = a_{p(i)}^*$. Thus

$$(p\epsilon)\ (a_i^*) = (a_{p(i)}^*)^{\epsilon_i}.$$

Note that $\alpha_i = a_i^*(a_{i+1}^*)^{-1}, \quad 1 \le i \le n-1$, and $\alpha_n = (a_n^*)^2$. The positive roots are

$$a_i^*(a_j^*)^{-1},\ 1 \le i < j \le n, \quad a_i^* a_j^*,\ 1 \le i < j \le n, \quad (a_i^*)^2,\ 1 \le i \le n.$$

Recall that $W^{\alpha_i} = \{w \in W; w\alpha_i > 0\}$. From the above formulas, we have $(p\epsilon)(\alpha_n) = (a_{p(n)}^*)^{2\epsilon_n}$. This implies that $(p\epsilon)(\alpha_n) > 0$ if and only if $\epsilon_n = 1$. Thus

$$W^{\alpha_n} = \{p\epsilon \in W; \epsilon_n = 1\}.$$

Let $1 \le i \le n-1$. Then $(p\epsilon)(\alpha_i) = (p\epsilon)(a_i^*(a_{i+1}^*)^{-1}) = (a_p^*(i))^{\epsilon_i}(a_{p(i+1)}^*)^{-\epsilon_{i+1}}$. From the list of positive roots we obtain directly

Lemma 4.1. *(a)* $W^{\alpha_n} = \{p\epsilon \in W; \epsilon_n = 1\}$

(b) For $1 \le i \le n-1$, W^{α_i} is a disjoint union of the following three sets:

(i) $\{p\epsilon \in W; \quad \epsilon_i = \epsilon_{i+1} = 1,\ p(i) < p(i+1)\}$;
(ii) $\{p\epsilon \in W; \quad \epsilon_i = 1, \epsilon_{i+1} = -1\}$;
(iii) $\{p\epsilon \in W; \quad \epsilon_i = \epsilon_{i+1} = -1,\ p(i) > p(i+1)\}$. $\square$

Recall that ${}^{\alpha_i}W = \left\{w \in W; w^{-1}\alpha_i > 0\right\}$. We get also in a completely analogous way

Lemma 4.2. *(a)* ${}^{\alpha_n}W = \{p\epsilon \in W; \epsilon_{p^{-1}(n)} = 1\}$

(b) For $1 \le i \le n-1$, ${}^{\alpha_i}W$ is a disjoint union of the following three sets:

(i) $\{p\epsilon \in W; \quad \epsilon_{p^{-1}(i)} = \epsilon_{p^{-1}(i+1)} = 1,\ p^{-1}(i) < p^{-1}(i+1)\}$;
(ii) $\{p\epsilon \in W; \quad \epsilon_{p^{-1}(i)} = 1, \epsilon_{p^{-1}(i+1)} = -1\}$;
(iii) $\{p\epsilon \in W : \quad \epsilon_{p^{-1}}(i) = \epsilon_{p^{-1}(i+1)} = -1,\ p^{-1}(i) > p^{-1}(i+1)\}$. $\square$

The same results hold for $\mathrm{GSp}(n, F)$.

Recall that $[W_\Omega \backslash W] = \underset{\alpha\in\Omega}{\cap} {}^{\alpha}W$, for $\Omega \subseteq \Delta$.

Lemma 4.3. *Let $1 \le i \le n$ and let $0 \le j \le i$. Denote by X_j^i the set of all $p\epsilon \in W$ such that the following six conditions are satisfied*

(i) $\epsilon_{p^{-1}(k)} = 1$, *for* $1 \le k \le j$;
(ii) $p^{-1}(k_1) < p^{-1}(k_2)$, *for* $1 \le k_1 < k_2 \le j$;
(iii) $\epsilon_{p^{-1}(k)} = -1$, *for* $j+1 \le k \le i$;
(iv) $p^{-1}(k_1) > p^{-1}(k_2)$, *for* $j+1 \le k_1 < k_2 \le i$;
(v) $\epsilon_{p^{-1}(k)} = 1$, *for* $i+1 \le k \le n$;
(vi) $p^{-1}(k_1) < p^{-1}(k_2)$, *for* $i+1 \le k_1 < k_2 \le n$.

Then $\left[W_{\Delta\backslash\{\alpha_i\}} \backslash W\right] = \underset{0\le j\le i}{\cup} X_j^i$.

Proof. Let $p\epsilon \in \left[W_{\Delta\backslash\{\alpha_i\}} \backslash W\right] = \underset{j\neq i}{\cap} {}^{\alpha_j}W$. If $i < n$, then (a) of Lemma 4.2 implies that $\epsilon_{p^{-1}(n)} = 1$. Further, (b) of Lemma 4.2 implies $\epsilon_{p^{-1}(i+1)} = \epsilon_{p^{-1}(i+2)} = \dots = \epsilon_{p^{-1}(n)} = 1$. By (b) of Lemma 4.2, we have $p^{-1}(i+1) < p^{-1}(i+2) < \dots < p^{-1}(n)$. Thus $\epsilon_{p^{-1}(k)} = 1$, for $i+1 \le k \le n$ and $p^{-1}(k_1) < p^{-1}(k_2)$, for $i+1 \le k_1 < k_2 \le n$. Note that if $i = n$, the above condition is empty. Therefore, $p\epsilon \in \left[W_{\Delta\backslash\{\alpha_n\}} \backslash W\right]$ also satisfies the above condition.

Choose the greatest $j \in \{0, 1, \dots, i\}$ such that $\epsilon_{p^{-1}(j)} = 1$ (if there does not exist such a j, then one takes $j = 0$). In the same way as before we conclude from (b) of Lemma 4.2 that $\epsilon_{p^{-1}(k)} = 1$, for $1 \le k \le j$ and $p^{-1}(k_1) < p^{-1}(k_2)$, for $1 \le k_1 < k_2 \le j$. Certainly $\epsilon_{p^{-1}(k)} = -1$, for $j+1 \le k \le i$. Then by (b) of Lemma 4.2, we have $p^{-1}(k_1) > p^{-1}(k_2)$, for $j+1 \le k_1 < k_2 \le i$. Thus $p\epsilon \in X_j^i$ where $0 \le j \le n$. This proves the inclusion $\left[W_{\Delta\backslash\{\alpha_i\}} \backslash W\right] \subseteq \underset{0\le j\le i}{\cup} X_j^i$.

We shall now prove the other inclusion. Let $j \in \{0, 1, \dots, i\}$ and let $p\epsilon \in X_j^i$. Take $\ell \neq i$. Then it is enough to prove that $p\epsilon \in {}^{\alpha_\ell}W$. We consider several possibilities.

Suppose that $1 \le \ell \le j-1$. Then conditions (i) and (ii) of Lemma 4.3 imply that $p\epsilon \in^{\alpha_\ell} W$. More precisely, $p\epsilon$ is in the set (i) of (b) of Lemma 4.2.

Now take $\ell = j$. Then $\epsilon_{p^{-1}(\ell)} = 1$ by condition (i) of Lemma 4.3 Since $\ell \neq i$ we have that $j < i$. Now $\epsilon_{p^{-1}(\ell+1)} = -1$ by condition (iii) of the Lemma. Therefore $p\epsilon \in^{\alpha_\ell} W$. This $p\epsilon$ is in the set (ii) of (b) of Lemma 4.2.

Now suppose that $j+1 \leq \ell \leq i-1$. Then conditions (iii) and (iv) of the Lemma imply $p\epsilon \in {}^{\alpha_\ell}W$. In particular, $p\epsilon$ is in the set (iii) of (b) of Lemma 4.2.

It remains to consider the case $\ell > i$. If $\ell < n$, then conditions (v) and (vi) of the Lemma imply that $p\epsilon \in {}^{\alpha_\ell}W$. The element $p\epsilon$ is in the set (i) of (b) of Lemma 4.2. If $\ell = n$, then $i < n$. By condition (v) of the Lemma, we have that $p\epsilon \in {}^{\alpha_\ell}W$. $\square$

Recall that $[W/W_\Omega] = \bigcap_{\alpha\in\Omega} W^\alpha$, for $\Omega \subseteq \Delta$. Using the relation $\left[W_{\Delta\backslash\{\alpha_i\}}\backslash W\right]^{-1} = \left[W/W_{\Delta\backslash\{\alpha_i\}}\right]$, one obtains from Lemma 4.3 the following

Lemma 4.4. *Let $1 \leq i \leq n$ and let $0 \leq j \leq i$. Denote by Y_j^i the set of all $p\epsilon \in W$ such that the following six conditions are satisfied*

(i) $\epsilon_k = 1$, *for* $1 \leq k \leq j$;
(ii) $p(k_1) < p(k_2)$, *for* $1 \leq k_1 < k_2 \leq j$;
(iii) $\epsilon_k = -1$, *for* $j+1 \leq k \leq i$;
(iv) $p(k_1) > p(k_2)$, *for* $j+1 \leq k_1 < k_2 \leq i$;
(v) $\epsilon_k = 1$, *for* $i+1 \leq k \leq n$;
(vi) $p(k_1) < p(k_2)$, *for* $i+1 \leq k_1 < k_2 \leq n$.

Then $\left[W/W_{\Delta\backslash\{\alpha_i\}}\right] = \bigcup_{0\leq j\leq i} Y_j^i$. $\square$

Let i_1 and i_2 be an integers which satisfy $1 \leq i_1, i_2 \leq n$. Take an integer d such that $0 \leq d \leq \min\{i_1, i_2\}$. Suppose that an integer k satisfies

$$\max\{0, (i_1 + i_2 - n) - d\} \leq k \leq \min\{i_1, i_2\} - d.$$

We now define a permutation $p_n(d,k)_{i_1,i_2}$ by the following formula:

$$p_n(d,k)_{i_1,i_2}(j) = \begin{cases} j, & \text{for} \quad 1 \leq j \leq k; \\ j + i_1 - k, & \text{for} \quad k+1 \leq j \leq i_2 - d; \\ (i_1 + i_2 - d + 1) - j, & \text{for} \quad i_2 - d + 1 \leq j \leq i_2; \\ j - i_2 + k, & \text{for} \quad i_2 + 1 \leq j \leq i_1 + i_2 - d - k; \\ j, & \text{for} \quad i_1 + i_2 - d - k + 1 \leq j \leq n. \end{cases}$$

For a graphical illustration of $p_n(d,k)_{i_1,i_2}$ see the last page of the paper. The conditions on d and k imply that $p = p_n(d,k)_{i_1,i_2}$ is well defined. Either the above drawing, or a simple direct computation implies $p_n(d,k)_{i_1,i_2}^{-1} = p_n(d,k)_{i_2,i_1}$.

For $k \geq 0$, set $\mathbf{1_k} = \underbrace{1, 1, \dots, 1}_{k \text{ times}}$ and $-\mathbf{1_k} = \underbrace{-1, -1, \dots, -1}_{k \text{ times}}$. We define the following element of W: $\quad q_n(d,k)_{i_1,i_2} = p_n(d,k)_{i_1,i_2}(\mathbf{1_{i_2-d}}, -\mathbf{1_d}, \mathbf{1_{n-i_2}})$.

Lemma 4.5. *Let $i_1, i_2 \in \{1, 2, \dots, n\}$. Suppose that integers j_1 and j_2 satisfy $1 \leq j_1 \leq i_1$ and $1 \leq j_2 \leq i_2$. We have*

(i) *If $i_1 - j_1 \neq i_2 - j_2$, then $X_{j_1}^{i_1} \cap Y_{j_2}^{i_2} = \phi$.*

(ii) *Suppose that* $i_1-j_1=i_2-j_2$. *Denote this number by* d. *Then* $0\le d\le min\{i_1,i_2\}$. *The set* $X^{i_1}_{j_1}\cap Y^{i_2}_{j_2} = X^{i_1}_{i_1-d}\cap Y^{i_2}_{i_2-d}$ *consists of all* $q_n(d,k)_{i_1,i_2}$ *with* k *an integer which satisfies* $max\{0,(i_1+i_2-n)-d\}\le k\le min\{i_1,i_2\}-d$.

Proof. Take $p\epsilon\in W$. Then $p\epsilon\in X^{i_1}_{j_1}\cap Y^{i_2}_{j_2}$ if and only if the following twelve conditions are satisfied:

(1) $\epsilon_{p^{-1}(\ell)}=1$, for $1\le\ell\le j_1$;
(2) $p^{-1}(\ell_1)<p^{-1}(\ell_2)$, for $1\le\ell_1<\ell_2\le j_1$;
(3) $\epsilon_{p^{-1}(\ell)}=-1$, for $j_1+1\le\ell\le i_1$;
(4) $p^{-1}(\ell_1)>p^{-1}(\ell_2)$, for $j_1+1\le\ell_1<\ell_2\le i_1$;
(5) $\epsilon_{p^{-1}(\ell)}=1$, for $i_1+1\le\ell\le n$;
(6) $p^{-1}(\ell_1)<p^{-1}(\ell_2)$, for $i_1+1\le\ell_1<\ell_2\le n$;
(7) $\epsilon_\ell=1$, for $1\le\ell\le j_2$;
(8) $p(\ell_1)<p(\ell_2)$, for $1\le\ell_1<\ell_2\le j_2$;
(9) $\epsilon_\ell=-1$, for $j_2+1\le\ell\le i_2$;
(10) $p(\ell_1)>p(\ell_2)$, for $j_2+1\le\ell_1<\ell_2\le i_2$;
(11) $\epsilon_\ell=1$, for $i_2+1\le\ell\le n$;
(12) $p(\ell_1)<p(\ell_2)$, for $i_2+1\le\ell_1<\ell_2\le n$.

Suppose that there exists $p\epsilon\in X^{i_1}_{j_1}\cap Y^{i_2}_{j_2}$. The number of -1's which appear in ϵ must be i_1-j_1 by (1), (3) and (5). This number is i_2-j_2 by (7), (9) and (11). Therefore $X^{i_1}_{j_1}\cap Y^{i_2}_{j_2}\ne\phi$ implies $i_1-j_1=i_2-j_2$. This proves (i).

Now, suppose that $i_1-j_1=i_2-j_2=d$. Clearly, $0\le d\le\min\{i_1,i_2\}$. Let $p\epsilon\in X^{i_1}_{j_1}\cap Y^{i_2}_{j_2}$..Take an integer k such that $0\le k\le\min\{j_1,j_2\}=\min\{i_1,i_2\}-d$, and such that k is maximal with the property $p(\ell)=\ell$ for all $1\le\ell\le k$. If $p(1)\ne1$, then one takes $k=0$.

From (7), (9) and (11) we see that $\epsilon=(\mathbf{1_{i_2-d}},-\mathbf{1_d},\mathbf{1_{n-i_2}})$. We shall prove that $k\ge i_1+i_2-n-d$ and $p=p_n(d,k)_{i_1,i_2}$. This will imply that $p\epsilon=q_n(d,k)_{i_1,i_2}$.

From (3) and (9), and also (1), (5), (7), (11), we see that

$$p^{-1}(\{\ell;j_1+1\le\ell\le i_1\})=\{\ell;j_2+1\le\ell\le i_2\}.$$

This implies $\{\ell;j_1+1\le\ell\le i_1\}=p(\{\ell;j_2+1\le\ell\le i_2\})$. By (10), p is order-reversing as a mapping $p:\{\ell;j_2+1\le\ell\le i_2\}\to\{\ell;j_1+1\le\ell\le i_1\}$. Since $i_2-j_2=i_1-j_1$, we obtain directly that $p(\ell)=i_1+j_2+1-\ell,\ \ j_2+1\le\ell\le i_2$, i.e.,

$$p(\ell)=i_1+i_2-d+1-\ell,\ \ i_2-d+1\le\ell\le i_2.$$

We already know that $p(\ell)=\ell$, for $1\le\ell\le k$.

We are now going to prove that $p(\ell)=\ell+i_1-k$ for $k+1\le\ell\le i_2-d$. It is enough to consider only the case $k+1\le i_2-d$. We shall assume it.

First, we claim that $p(\{\ell;k+1\le\ell\le i_2-d\})\subseteq\{\ell;i_1+1\le\ell\le n\}$. We already know that $p(\{\ell;1\le\ell\le k\})=\{\ell;1\le\ell\le k\}$ and

$$p(\{\ell;i_2-d+1\le\ell\le i_2\})=\{\ell;i_1-d+1\le\ell\le i_1\}.$$

Thus $p(\{\ell; k+1 \leq \ell \leq i_2 - d\}) \subseteq \{\ell; k+1 \leq \ell \leq i_1 - d\} \cup \{\ell; i_1+1 \leq \ell \leq n\}$. If $k = i_1 - d$, then the above relation implies our claim. It remains to consider the case $k < i_1 - d$. Suppose that our claim does not hold in that situation. Then $k+1 \leq p(\ell_0) \leq i_1 - d$ for some ℓ_0 which satisfies $k+1 \leq \ell_0 \leq i_2 - d$. Since $p(\ell_1) < p(\ell_2)$ for $1 \leq \ell_1 < \ell_2 \leq i_2 - d$ by (8), we must have $k+1 \leq p(k+1) \leq i_1 - d$. By the choice of k, $p(k+1) \neq k+1$. Then clearly $p^{-1}(k+1) \neq k+1$. This implies $k+1 < p(k+1)$ and $k+1 < p^{-1}(k+1)$. Since $k+1, p(k+1) \in \{\ell; 1 \leq \ell \leq i_1 - d\}$, (2) implies $p^{-1}(k+1) < k+1$. This contradiction proves our claim.

We shall now use that $p(\{\ell; k+1 \leq \ell \leq i_2 - d\}) \subseteq \{\ell; i_1+1 \leq \ell \leq n\}$. If $k < i_2 - d$, then the above relation implies $i_2 - d - k - 1 \leq n - i_1 - 1$, so that $i_1 + i_2 - d - n \leq k$. If $k \geq i_2 - d$, then $k = i_2 - d$ and the above inequality is obvious. Therefore, the above inequality holds in general.

Since p is monotone on $\{\ell; k+1 \leq \ell \leq i_2 - d\}$ by (8), we have

$$(i_2 - d) - (k+1) \leq p(i_2 - d) - p(k+1).$$

Since p^{-1} is monotone on $\{\ell; i_1+1 \leq \ell \leq n\}$ by (6), we have

$$p(i_2 - d) - p(k+1) \leq p^{-1}(p(i_2 - d)) - p^{-1}(p(k+1)) = (i_2 - d) - (k+1).$$

Thus $(i_2 - d) - (k+1) = p(i_2 - d) - p(k+1)$. Again using that p is monotone on $\{\ell; k+1 \leq \ell \leq i_2 - d\}$, one obtains $p(\{\ell; k+1 \leq \ell \leq i_2 - d\}) = \{\ell; p(k+1) \leq \ell \leq p(i_2 - d)\}$.

We now claim that $p(k+1) = i_1 + 1$. Suppose that $p(k+1) \neq i_1 + 1$. This implies $i_1 + 1 < p(k+1)$. Condition (6) implies $p^{-1}(i_1+1) < k+1$. But we know that for $\ell \leq k$, $p(\ell) = \ell$. Thus $p^{-1}(i_1+1) = i_1+1$, and further $i_1 + 1 < k+1$, i.e., $i_1 < k$. This contradicts the assumption $k \leq \min\{j_1, j_2\} = \min\{i_1, i_2\} - d$ which we had on k. Thus $p(k+1) = i_1 + 1$. Since we have

$$p(\{\ell; k+1 \leq \ell \leq i_2 - d\}) = \{\ell; p(k+1) \leq \ell \leq p(i_2 - d)\}$$

and p is monotone on this set, we have that $p(\ell) = \ell + i_1 - k$, for $k+1 \leq \ell \leq i_2 - d$.

We are now going to prove that $p(\ell) = \ell - i_2 + k$ for $i_2 + 1 \leq \ell \leq i_1 + i_2 - d - k$. It is enough to consider the case when $i_2 + 1 \leq i_1 + i_2 - d - k$, i.e., $k+1 \leq i_1 - d$.

Consider now $p^{-1}(\{\ell; k+1 \leq \ell \leq i_1 - d\})$. Since p and p^{-1} are bijective, we have

$$p^{-1}(\{\ell; k+1 \leq \ell \leq i_1 - d\}) \subseteq \{\ell; i_2+1 \leq \ell \leq n\}.$$

Since p^{-1} is monotone on $\{\ell; k+1 \leq \ell \leq i_1 - d\}$ by (2) and p is monotone on $\{\ell; k+1 \leq \ell \leq i_1 - d\}$ by (12), we obtain that $p^{-1}(i_1 - d) - p^{-1}(k+1) = (i_1 - d) - (k+1)$, as before. Suppose that $p^{-1}(k+1) \neq i_2 + 1$. Since p^{-1} is bijective, we have $i_2 + 1 < p^{-1}(k+1)$. Condition (12) implies $p(i_2+1) < k+1$. The choice of k implies $p(i_2+1) = i_2+1$. Thus $i_2 < k$. This contradicts the choice of k. Thus, we have proved that $p^{-1}(k+1) = i_2 + 1$.

We can now conclude that $p^{-1}(\ell) = \ell + i_2 - k$, for $k+1 \leq \ell \leq i_1 - d$. Therefore $p(\ell) = \ell - i_2 + k$, for $i_2 + 1 \leq \ell \leq i_1 + i_2 - d - k$.

Since p is bijective, we have

$$p(\{\ell; i_1 + i_2 - d - k + 1 \leq \ell \leq n\}) = \{\ell; i_1 + i_2 - d - k + 1 \leq \ell \leq n\}.$$

By (6), p is monotone on this set. Thus, $p(\ell) = \ell$ for $i_1 + i_2 - d - k + 1 \le \ell \le n$. This finishes the proof that $p = p_n(d,k)_{i_1,i_2}$.

It remains to prove now that $q_n(d,k)_{i_1,i_2} \in X^{i_1}_{i_1-d} \cap Y^{i_2}_{i_2-d}$ when $0 \le d \le \min\{i_1, i_2\}$ and

$$\max\{0, (i_1 + i_2 - n) - d\} \le k \le \min\{i_1, i_2\} - d.$$

Taking into account that $i_1 - j_1 = i_2 - j_2 = d$, one sees from the definition of $q_n(d,k)_{i_1,i_2}$ directly that conditions (7)-(12) are satisfied. In the same way one sees that conditions (1)-(6) are satisfied. $\square$

Let us recall that for $\Omega_1, \Omega_2 \subseteq \Delta$, $\quad [W_{\Omega_1} \backslash W / W_{\Omega_2}] = [W_{\Omega_1} \backslash W] \cap [W / W_{\Omega_2}]$.

Proposition 4.6. *Let $i_1, i_2 \in \{1, 2, \ldots, n\}$. The set $\left[W_{\Delta \backslash \{\alpha_{i_1}\}} \backslash W / W_{\Delta \backslash \{\alpha_{i_2}\}}\right]$ consists of all $q_n(d,k)_{i_1,i_2}$ where d, k are integers which satisfy $0 \le d \le min\{i_1, i_2\}$ and*

$$max\{0, (i_1 + i_2 - n) - d\} \le k \le min\{i_1, i_2\} - d.$$

Proof. From Lemmas 4.3, 4.4 and 4.5 we have $\left[W_{\Delta \backslash \{\alpha_{i_1}\}} \backslash W / W_{\Delta \backslash \{\alpha_{i_2}\}}\right] =$

$$\left[W_{\Delta \backslash \{\alpha_{i_1}\}} \backslash W\right] \cap \left[W / W_{\Delta \backslash \{\alpha_{i_2}\}}\right] = \Big(\bigcup_{0 \le j_1 \le i_1} X^{i_1}_{j_1} \Big) \cap \Big(\bigcup_{0 \le j_2 \le i_2} Y^{i_2}_{j_2} \Big) =$$

$$\bigcup_{0 \le j_1 \le i_1} \ \bigcup_{0 \le j_2 \le i_2} (X^{i_1}_{j_1} \cap Y^{i_2}_{j_2}) = \bigcup_{0 \le d \le \min\,\{i_1, i_2\}} (X^{i_1}_{i_1-d} \cap Y^{i_2}_{i_2-d}).$$

Now, (b) of Lemma 4.5 implies the proposition. $\square$

Lemma 4.7. *Fix $i_1, i_2 \in \{1, 2, \ldots, n\}$. Suppose that integers d, d', d'' and k, k', k'' satisfy the following conditions $\quad 0 \le d, d', d'' \le min\{i_1, i_2\}$,*

$$max\{0, (i_1 + i_2 - n) - d\} \le k \le min\{i_1, i_2\} - d,$$

$$max\{0, (i_1 + i_2 - n) - d'\} \le k' \le min\{i_1, i_2\} - d',$$

$$max\{0, (i_1 + i_2 - n) - d''\} \le k'' \le min\{i_1, i_2\} - d''.$$

Then:

(i) *If $q_n(d',k')_{i_1,i_2} = q_n(d'',k'')_{i_1,i_2}$, then $d' = d''$ and $k' = k''$.*
(ii) $(p_n(d,k)_{i_1,i_2})^{-1} = p_n(d,k)_{i_2,i_1}$
(iii) $(q_n(d,k)_{i_1,i_2})^{-1} = q_n(d,k)_{i_2,i_1}$

Proof. (i) Suppose $q_n(d',k')_{i_1,i_2} = q_n(d'',k'')_{i_1,i_2}$. Recall that

$$q_n(d,k)_{i_1,i_2} = p_n(d,k)_{i_1,i_2}(\mathbf{1_{i_2-d}}, -\mathbf{1_d}, \mathbf{1_{n-i_2}}).$$

Thus d is the number of -1's in $(\mathbf{1_{i_2-d}}, -\mathbf{1_d}, \mathbf{1_{n-i_2}})$. This implies $d' = d''$. Therefore $p_n(d',k')_{i_1,i_2} = p_n(d',k'')_{i_1,i_2}$. The definition of $p_n(d,k)_{i_1,i_2}$ (also see the proof of Lemma 4.5) implies that k is the maximal integer which satisfies $0 \le k \le \min\{i_1, i_2\} - d$ and

$p_n(d,k)_{i_1,i_2}(\ell) = \ell$ for all $1 \le \ell \le k$. This implies $k' = k''$. Thus, if we fix n, i_1 and i_2, then $q_n(d,k)_{i_1,i_2} \in W$ completely determines d and k.

(ii) The relation follows directly either from the definition of $p_n(d,k)_{i_1,i_2}$, or from the graphical interpretation of $p_n(d,k)_{i_1,i_2}$.

(iii) Note that $(q_n(d,k)_{i_1,i_2})^{-1} \in \left[W_{\Delta\backslash\{\alpha_{i_2}\}}\backslash W/W_{\Delta\backslash\{\alpha_{i_1}\}}\right]$. Thus $(q_n(d,k)_{i_1,i_2})^{-1} = q_n(d_1,k_1)_{i_2,i_1}$ by Lemma 4.6, where $0 \le d_1 \le \min\{i_1,i_2\}$ and

$$\max\{0,(i_1+i_2-n)-d_1\} \le k_1 \le \min\{i_1,i_2\} - d_1.$$

Since d_1 is the number of -1's , and since $q_n(d,k)_{i_1,i_2}$ and its inverse have the same number -1's, we have $d = d_1$. Thus, by the proof of Lemma 4.5 we have

$$(q_n(d,k)_{i_1,i_2})^{-1} = p(\mathbf{1_{i_1-d}}, -\mathbf{1_d}, \mathbf{1_{n-i_1}})$$

Note that $p = (p_n(d,k)_{i_1,i_2})^{-1}$. At this point we may apply (ii). This finishes the proof of (iii).

Another possibility is to note that k is the maximal integer which satisfies $0 \le k \le \min\{i_1,i_2\} - d$ and $(p_n(d,k)_{i_1,i_2})^{-1}(\ell) = \ell, \quad 1 \le \ell \le k$. This implies $(q_n(d,k)_{i_1,i_2})^{-1} = (q_n(d,k)_{i_2,i_1})$. From this we also get another proof of (ii). $\square$

Lemma 4.8. *Let $i_1, i_2 \in \{1,\dots,n\}$. Suppose that d and k are integers which satisfy $0 \le d \le min\ \{i_1,i_2\}$ and $max\{0,(i_1+i_2-n)-d\} \le k \le min\{i_1,i_2\} - d$. Then*

$$(\Delta\backslash\{\alpha_{i_1}\}) \cap q_n(d,k)_{i_1,i_2}(\Delta\backslash\{\alpha_{i_2}\}) = \Delta\backslash\{\alpha_\ell; \ell \in \{k, i_1-d, i_1, i_1+i_2-d-k\}\backslash\{0\}\}.$$

Proof. It is easy to get from the conditions on d and k that

$$0 \le k \le i_1 - d \le i_1 \le i_1 + i_2 - d - k \le n.$$

We need to calculate when $q_n(d,k)_{i_1,i_2}(\alpha_r) \in \Delta\backslash\{\alpha_{i_1}\}$ if $1 \le r \le n$ and $r \ne i_2$. If this is the case, we need to find what $q_n(d,k)_{i_1,i_2}(\alpha_r)$ is. We consider several cases.

(i) Suppose that $1 \le r \le k-1$. Note that in this case $r \ne i_2$. Also $r \ne i_1$. Since $r < n$, we have $q_n(d,k)_{i_1,i_2}(\alpha_r) = p_n(d,k)_{i_1,i_2}(\mathbf{1_{i_2-d}}, -\mathbf{1_d}, \mathbf{1_{n-i_2}})(a_r^*/a_{r+1}^*)$. Since $k \le i_2 - d$ and $r+1 \le k$, we have $q_n(d,k)_{i_1,i_2}(\alpha_r) = p_n(d,k)_{i_1,i_2}(a_r^*/a_{r+1}^*) = a_r^*/a_{r+1}^* = \alpha_r$.

The conclusion of (i) is that for any k as in the lemma $\{\alpha_r; 1 \le r \le k-1\} \subseteq \ \Delta\backslash\{\alpha_{i_2}\}$ and $q_n(d,k)_{i_1,i_2}(\{\alpha_r; 1 \le r \le k-1\}) = \{\alpha_r; 1 \le r \le k-1\} \subseteq \ \Delta\backslash\{\alpha_{i_1}\}$.

(ii) Suppose that $r = k$. We need to assume that $r \ne i_2$, i.e., $k < i_2$.

Consider first the case of $k < i_2 - d$. Then $k < n$ and we have $q_n(d,k)_{i_1,i_2}(\alpha_k) =$

$$p_n(d,k)_{i_1,i_2}(\mathbf{1_{i_2-d}}, -\mathbf{1_d}, \mathbf{1_{n-i_2}})(a_k^*/a_{k+1}^*) = p_n(d,k)_{i_1,i_2}(a_k^*/a_{k+1}^*) = a_k^*/a_{i_1+1}^*.$$

If $a_k^*/a_{i_1+1}^* \in \Delta$, then $i_1 = k$. But then $q_n(d,k)_{i_1,i_2}(\alpha_k) = \alpha_k = \alpha_{i_1}$ and this is not in $\Delta\backslash\{\alpha_{i_1}\}$.

Now consider the case when $k = i_2 - d$. Since $r = k \ne i_2$, we have $d \ge 1$. Thus $k < n$. Therefore $q_n(d,k)_{i_1,i_2}(\alpha_k) = p_n(d,k)_{i_1,i_2}(\mathbf{1_{i_2-d}}, -\mathbf{1_d}, \mathbf{1_{n-i_2}})(a_k^*/a_{k+1}^*) =$

$$p_n(d,k)_{i_1,i_2}(a_k^* a_{k+1}^*) = a_k^* a_{p_n(d,k)_{i_1,i_2}(k+1)}^*.$$

This is never in Δ.

The conclusion of (ii) is that for any k as in the lemma, $k \geq 1$ and $\alpha_k \in \Delta\backslash\{\alpha_{i_2}\}$ we have $q_n(d,k)_{i_1,i_2}(\alpha_k) \notin \Delta\backslash\{\alpha_{i_1}\}$.

(iii) Assume $k+1 \leq r \leq i_2 - d - 1$.

Now $r \neq i_2$ and $r < n$. Further, we have $q_n(d,k)_{i_1,i_2}(\alpha_r) =$

$$p_n(d,k)_{i_1,i_2}(\mathbf{1_{i_2-d}}, -\mathbf{1_d}, \mathbf{1_{n-i_2}})(a_r^*/a_{r+1}^*) = p_n(d,k)_{i_1,i_2}(a_r^*/a_{r+1}^*) = \alpha_{r+i_1-k}.$$

Since $1 \leq r-k$, we have $\alpha_{r+i_1-k} \neq \alpha_{i_1}$.

The conclusion of (iii) is that $\{\alpha_r; k+1 \leq r \leq i_2-d-1\} \subseteq \Delta\backslash\{\alpha_{i_2}\}$ and

$$q_n(d,k)_{i_1,i_2}(\{\alpha_r; k+1 \leq r \leq i_2-d-1\}) = \{\alpha_\ell; i_1+1 \leq \ell \leq i_1+i_2-d-k-1\} \subseteq \Delta\backslash\{\alpha_{i_1}\}.$$

(iv) Suppose that $r = i_2 - d$. Since $r \neq i_2$, we have $d \geq 1$. Thus $r < n$. Now,

$$q_n(d,k)_{i_1,i_2}(\alpha_{i_2-d}) = p_n(d,k)_{i_1,i_2}(a_{i_2-d}^* a_{i_2-d+1}^*) = (a_{i_2-d+i_1-k}^* a_{i_1}^*) \notin \Delta.$$

The conclusion is that $i_2 - d \geq 1$ and $\alpha_{i_2-d} \in \Delta\backslash\{\alpha_{i_2}\}$ imply $q_n(d,k)_{i_1,i_2}(\alpha_{i_2-d}) \notin \Delta\backslash\{\alpha_{i_1}\}$.

(v) Consider the case when $i_2 - d + 1 \leq r \leq i_2 - 1$. Clearly, $r \neq i_2$ and $r < n$. Now

$$q_n(d,k)_{i_1,i_2}(\alpha_r) = p_n(d,k)_{i_1,i_2}((a_r^*)^{-1} a_{r+1}^*) = a_{-r+i_1+i_2-d+1}^* a_{-r+i_1+i_2-d}^* = \alpha_{i_1+i_2-r-d}.$$

Note that $i_2 - r - d \leq -1$. Thus $q_n(d,k)_{i_1,i_2}(\alpha_r) \in \Delta\backslash\{\alpha_{i_1}\}$.

The conclusion of (v) is that $\{\alpha_r; i_2-d+1 \leq r \leq i_2-1\} \subseteq \Delta\backslash\{\alpha_{i_2}\}$ and

$$q_n(d,k)_{i_1,i_2}(\{\alpha_r; i_2-d+1 \leq r \leq i_2-1\}) = \{\alpha_\ell; i_1-d+1 \leq \ell \leq i_1-1\} \subseteq \Delta\backslash\{\alpha_{i_1}\}.$$

(vi) The next case is $i_2 + 1 \leq r \leq i_1 + i_2 - d - k - 1$, since $r \neq i_2$. Note that $i_1 + i_2 - d - k \leq n$. Thus $r < n$. Now,

$$q_n(d,k)_{i_1,i_2}(\alpha_r) = p_n(d,k)_{i_1,i_2}(a_r^*/a_{r+1}^*) = \alpha_{r-i_2+k}.$$

Note that $r - i_2 + k \leq i_1 - d - 1$. Thus $\alpha_{r-i_2+k} \neq \alpha_{i_1}$.

The conclusion of (vi) is that $\{\alpha_r; i_2+1 \leq r \leq i_1+i_2-d-k-1\} \subseteq \Delta\backslash\{\alpha_{i_2}\}$ and

$$q_n(d,k)_{i_1,i_2}(\{\alpha_r; i_2+1 \leq r \leq i_1+i_2-d-k-1\}) =$$

$$\{\alpha_\ell; k+1 \leq \ell \leq i_1-d-1\} \subseteq \Delta\backslash\{\alpha_{i_1}\}.$$

(vii) Let $r = i_1 + i_2 - d - k$. Note that in general $i_2 \leq i_1 + i_2 - d - k$. Since $r \neq i_2$, we assume $i_2 + 1 \leq i_1 + i_2 - d - k$.

We consider two cases.

Suppose $r < n$. Now $q_n(d,k)_{i_1,i_2}(\alpha_{i_1+i_2-d-k}) =$

$$p_n(d,k)_{i_1,i_2}(a_{i_1+i_2-d-k}^*/a_{i_1+i_2-d-k+1}^*) = a_{i_1-d}^*/a_{i_1+i_2-d-k+1}^*.$$

From $k \leq i_2-d$ we get that $i_1-d \leq i_1+i_2-d-k+1$. Thus, $a^*_{i_1-d}/a^*_{i_1+i_2-d-k+1} \in \Delta$ implies $i_1+i_2-d-k+1 = i_1-d+1$, i.e. $i_2-k=0$. But since $k \leq \min\{i_1,i_2\}-d$, we have $k \leq i_2-d$, i.e. $d \leq i_2-k$. Since $d \geq 0$, we have that $d=0$. Now $q_n(d,k)_{i_1,i_2}(\alpha_{i_1+i_2-d-k}) = \alpha_{i_1}$. Note that this is not in $\Delta\backslash\{\alpha_{i_1}\}$.

We now consider the case $r=n$. Then $i_1+i_2-d-k=n$. Since $r \neq i_2$, we assume $i_2<n$. Now since $i_2<n$, we have

$$q_n(d,k)_{i_1,i_2}(\alpha_n) = q_n(d,k)_{i_1,i_2}((a^*_n)^2) = (p_n(d,k)_{i_1,i_2}(a^*_n))^2.$$

Since $i_2+1 \leq n = i_1+i_2-d-k$, we have $q_n(d,k)_{i_1,i_2}(\alpha_n) = (a^*_{i_1-d})^2$. Thus, to have $q_n(d,k)_{i_1,i_2}(\alpha_n) \in \Delta$, we must have $i_1-d=n$. Now $i_1+i_2-d-k=n$ implies $i_2=k$. Since $k \leq i_2-d$ and $d \geq 0$, we know $d=0$. Since $d \geq 0$, we know $d=0$. But then $i_1=n$ and we have $q_n(d,k)_{i_1,i_2}(\alpha_n) = (a^*_n)^2 = \alpha_n \notin \Delta\backslash\{\alpha_{i_1}\}$.

The conclusion in (vii) is that $i_1+i_2-d-k>0$ and $\alpha_{i_1+i_2-d-k} \in \Delta\backslash\{\alpha_{i_2}\}$ imply $q_n(d,k)_{i_1,i_2}(\alpha_{i_1+i_2-d-k}) \notin \Delta\backslash\{\alpha_{i_1}\}$.

(viii) It remains to consider the case of $i_1+i_2-d-k+1 \leq r \leq n$. Since $k \leq i_1-d, k \leq i_2-d$, we have $i_1+1 \leq r$ and $i_2+1 \leq r$. Thus $\alpha_r \neq \alpha_{i_1}$ and $\alpha_r \neq \alpha_{i_2}$.

One considers now two cases. The first one is $i_1+i_2-d-k+1 \leq r \leq n-1$. The second one is $i_1+i_2-d-k+1 \leq r=n$. In both cases one gets directly $q_n(d,k)_{i_1,i_2}(\alpha_r) = \alpha_r$.

The conclusion of (viii) is that $\{\alpha_r; i_1+i_2-d-k+1 \leq r \leq n\} \subseteq \Delta\backslash\{\alpha_{i_1}\}$ and

$$q_n(d,k)_{i_1,i_2}(\{\alpha_r; i_1+i_2-d-k+1 \leq r \leq n\}) = \{\alpha_r; i_1+i_2-d-k+1 \leq r \leq n\} \subseteq \Delta\backslash\{\alpha_{i_1}\}.$$

We can finish the proof now. Recall that $0 \leq k \leq i_1-d \leq i_1 \leq i_1+i_2-d-k \leq n$. From (i), (vi), (v), (iii) and (viii), we get that

$$\Delta\backslash\{\alpha_\ell; \ell \in \{k, i_1-d, i_1, i_1+i_2-d-k\}\backslash\{0\}\} \subseteq (\Delta\backslash\{\alpha_{i_1}\}) \cap q_n(d,k)_{i_1,i_2}(\Delta\backslash\{\alpha_{i_2}\}).$$

Note that in (i)-(viii) we have examined all $\alpha_r \in \Delta\backslash\{\alpha_{i_2}\}$. Since in cases (ii), (iv) and (vii) we do not get elements in $\Delta\backslash\{\alpha_{i_1}\}$, we have actually an identity

$$\Delta\backslash\{\alpha_\ell; \ell \in \{k, i_1-d, i_1, i_1+i_2-d-k\}\backslash\{0\}\} = (\Delta\backslash\{\alpha_{i_1}\}) \cap q_n(d,k)_{i_1,i_2}(\Delta\backslash\{\alpha_{i_2}\}). \quad \square$$

5. Jacquet modules of induced representations, the case of C_n

A positive integer n will be fixed in this section. Let $i_1 \in \{1,2,\ldots,n\}$. Suppose that π is an admissible representation of $\mathrm{GL}(i_1,F)$ and suppose that σ is an admissible representation of $\mathrm{GSp}(n-i_1,F)$.

Take $i_2 \in \{1,2,\ldots,n\}$. Let d and k be an integers which satisfy

$$0 \leq d \leq \min\{i_1,i_2\} \text{ and } \max\{0,(i_1+i_2-n)-d\} \leq k \leq \min\{i_1,i_2\}-d.$$

By Lemma 4.8

$$(\Delta\backslash\{\alpha_{i_1}\}) \cap q_n(d,k)_{i_1,i_2}(\Delta\backslash\{\alpha_{i_2}\}) = \Delta\backslash\{\alpha_\ell; \ell \in \{k, i_1-d, i_1, i_1+i_2-d-k\}\backslash\{0\}\}.$$

By Lemmas 4.7 and 4.8 we have $(q_n(d,k)_{i_1,i_2})^{-1}(\Delta\backslash\{\alpha_{i_1}\})\cap(\Delta\backslash\{\alpha_{i_2}\}) =$

$$q_n(d,k)_{i_2,i_1}(\Delta\backslash\{\alpha_{i_1}\})\cap(\Delta\backslash\{\alpha_{i_2}\}) = \Delta\backslash\{\alpha_\ell;\ell\in\{k,i_2-d,i_2,i_1+i_2-d-k\}\backslash\{0\}\}.$$

Let $w = q_n(d,k)_{i_1,i_2}$. Then, note that

$$w\ \left(\text{q-diag}\,(g_1,g_2,g_3,g_4,h,\ \psi(h)\,{}^\tau g_4^{-1},\ \psi(h)\,{}^\tau g_3^{-1},\ \psi(h)\,{}^\tau g_2^{-1},\ \psi(h)\,{}^\tau g_1^{-1})\ \right)w^{-1} =$$

$$\text{q-diag}\,(g_1,g_4,\ \psi(h)\,{}^\tau g_3^{-1},g_2,h,\ \psi(h)\,{}^\tau g_2^{-1},g_3,\ \psi(h)\,{}^\tau g_4^{-1},\ \psi(h)\,{}^\tau g_1^{-1})$$

for $g_1\in\mathrm{GL}(k,F), g_2\in\mathrm{GL}(i_2-d,F), g_3\in\mathrm{GL}(d,F), g_4\in\mathrm{GL}(i_1-d-k,F)$ and $h\in\mathrm{GSp}(n-i_1-i_2+d+k,F)$. It is now easy to prove

Lemma 5.1. *Let* $s.s.(r_{(k,i_1-d-k,d),(i_1)}(\pi)) = \sum_i \pi_i^{(1)}\otimes\pi_i^{(2)}\otimes\pi_i^{(3)}$, *and let*

$$s.s.(s_{(i_2-d-k),(0)}(\sigma)) = \sum_j \pi_j^{(4)}\otimes\sigma_j.$$

Set $\Theta=\Delta\backslash\{\alpha_{i_1}\}$, $\Omega=\Delta\backslash\{\alpha_{i_2}\}$ *and* $w=q_n(d,k)_{i_1,i_2}$. *Then*

$$s.s.(Ind_{w^{-1}P_\Theta w\cap M_\Omega}^{M_\Omega}(w^{-1}(r_{M_{\Theta\cap w\Omega},M_\Theta}(\pi\otimes\sigma)))) =$$

$$\sum_i\sum_j \pi_i^{(1)}\times\pi_j^{(4)}\times\tilde\pi_i^{(3)}\otimes\pi_i^{(2)}\rtimes(\omega_{\pi_i^{(3)}}\sigma_j) =$$

$$\sum_i\sum_j \pi_i^{(1)}\times\tilde\pi_i^{(3)}\times\pi_j^{(4)}\otimes\pi_i^{(2)}\rtimes(\omega_{\pi_i^{(3)}}\sigma_j) = \sum_i\sum_j \tilde\pi_i^{(3)}\times\pi_i^{(1)}\times\pi_j^{(4)}\otimes\pi_i^{(2)}\rtimes(\omega_{\pi_i^{(3)}}\sigma_j).$$

Proof. We have $\text{s.s.}(r_{M_{\Theta\cap w\Omega},M_\Theta}(\pi\otimes\sigma)) = (\sum_i \pi_i^{(1)}\otimes\pi_i^{(2)}\otimes\pi_i^{(3)})\otimes(\sum_j \pi_j^{(4)}\otimes\sigma_j)$. Further, by previous calculations $w^{-1}(\pi_1\otimes\pi_2\otimes\pi_3\otimes\pi_4\otimes\sigma) = \pi_1\otimes\pi_4\otimes\tilde\pi_3\otimes\pi_2\otimes\omega_{\pi_3}\sigma$. We need now to induce from $w^{-1}P_\Theta w\cap M_\Omega$ to M_Ω. Recall that a Levi factor of $w^{-1}P_\Theta w\cap M_\Omega$ is M_Π, where $\Pi = w^{-1}\Theta\cap\Omega = \Delta\backslash\{\alpha_\ell;\ell\in\{k,i_2-d,i_2,i_1+i_2-d-k\}\backslash\{0\}\}$. Thus $\text{s.s.}(\text{Ind}_{w^{-1}P_\Theta w\cap M_\Omega}^{M_\Omega}(w^{-1}(r_{M_{\Theta\cap w\Omega},M_\Theta}(\pi\otimes\sigma)))) = \sum_i\sum_j \pi_i^{(1)}\times\pi_j^{(4)}\times\tilde\pi_i^{(3)}\otimes\pi_i^{(2)}\rtimes(\omega_{\pi_i^{(3)}}\sigma_j)$. The other equalities in the lemma follow from the commutativity of R. □

Let π_i be an irreducible smooth representation of $\mathrm{GL}(n_i,F)$ for $i=1,2,3,4$. Let σ be an irreducible smooth representation of $\mathrm{GSp}(m,F)$. Set

$$(\pi_1\otimes\pi_2\otimes\pi_3)\ \tilde\rtimes\ (\pi_4\otimes\sigma) = \tilde\pi_1\times\pi_2\times\pi_4\otimes\pi_3\rtimes\omega_{\pi_1}\sigma.$$

One extends $\tilde\rtimes$ to a $\mathbb{Z}$-bilinear mapping $\tilde\rtimes : (R\otimes R\otimes R)\times(R\otimes R(G))\to R\otimes R(G)$. Let $s: R\otimes R\to R\otimes R$ be the homomorphism determined by $s(r_1\otimes r_2) = r_2\otimes r_1,\ \ r_1,r_2\in R$. Now, we have the following

Theorem 5.2. *Let π be an admissible representation of $GL(i_1, F)$ of finite length and let σ be an admissible representation of $GSp(n-i_1, F)$ of finite length. Set*

$$\mathfrak{M}^* = (1 \otimes m^*) \circ s \circ m^*.$$

Then

$$\mu^*(\pi \rtimes \sigma) = \mathfrak{M}^*(\pi) \tilde{\rtimes} \mu^*(\sigma).$$

Proof. We write in $R \otimes R$, $m^*(\pi) = \sum_{q=0}^{i_1} (\sum_{j=1}^{j_q} \alpha_j^{(q)} \otimes \beta_j^{(i_1-q)}) \in \sum_{q=0}^{i_1} R_q \otimes R_{i_1-q}$. Further $s \circ m^*(\pi) = \sum_{q=0}^{i_1} (\sum_{j=1}^{j_q} \beta_j^{(i_1-q)} \otimes \alpha_j^{(q)})$. Write

$$m^*(\alpha_j^{(q)}) = \sum_{r=0}^{q} (\sum_{u=1}^{u_r(j,q)} (\gamma_j^{(q)})_u^{(r)} \otimes (\delta_j^{(q)})_u^{(q-r)}) \in \sum_{r=0}^{q} R_r \otimes R_{q-r}.$$

Now $\mathfrak{M}^*(\pi) = \sum_{q=0}^{i_1} \sum_{j=1}^{j_q} \sum_{r=0}^{q} \sum_{u=1}^{u_r(j,q)} \beta_j^{(i_1-q)} \otimes (\gamma_j^{(q)})_u^{(r)} \otimes (\delta_j^{(q)})_u^{(q-r)}$. Introduce a new index $\ell = i_1 - q + r$. Then we have

$$\mathfrak{M}^*(\pi) = \sum_{\ell=0}^{i_1} (\sum_{\substack{q,r,\\ 0\le q\le i_1, 0\le r\le q,\\ i_1-q+r=\ell}} (\sum_{j=1}^{j_q} \sum_{u=1}^{u_r(j,q)} \beta_j^{(i_1-q)} \otimes (\gamma_j^{(q)})_u^{(r)} \otimes (\delta_j^{(q)})_u^{(q-r)})).$$

Write $\mu^*(\sigma) = \sum_{p=0}^{n-i_1} (\sum_{v=1}^{v_p} \tau_v^{(p)} \otimes \sigma_v^{(n-i_1-p)}) \in \sum_{p=0}^{n-i_1} R_p \otimes R_{n-i_1-p}(G)$. We have

$$\mathfrak{M}^*(\pi) \tilde{\rtimes} \mu^*(\sigma) = \sum_{m=0}^{n} (\sum_{\substack{\ell,p,\\ 0\le \ell\le i_1, 0\le p\le n-i_1,\\ l+p=m}} \sum_{\substack{q,r,\\ 0\le q\le i_1, 0\le r\le q,\\ i_1-q+r=\ell}} \sum_{j=1}^{j_q} \sum_{u=1}^{u_r(j,q)} \sum_{v=1}^{v_p}$$

$$(\beta_j^{(i_1-q)})^{\sim} \times (\gamma_j^{(q)})_u^{(r)} \times \tau_v^{(p)} \otimes (\delta_j^{(q)})_u^{(q-r)} \rtimes \omega_{\beta_j^{(i_1-q)}} \sigma_v^{(n-i_1-p)}) \in \sum_{m=0}^{n} R_m \otimes R_{n-m}(G).$$

A term of the above sum corresponding to m is denoted by A_m. Consider $1 \le i_2 \le n$. Now

$$A_{i_2} = \sum_{\substack{\ell,p,\\ 0\le \ell\le i_1, 0\le p\le n-i_1,\\ \ell+p=i_2}} \sum_{\substack{q,r,\\ 0\le q\le i_1, 0\le r\le q,\\ i_1-q+r=\ell}} \sum_{j=1}^{j_q} \sum_{u=1}^{u_r(j,q)} \sum_{v=1}^{v_p}$$

$$(\beta_j^{(i_1-q)})^{\sim} \times (\gamma_j^{(q)})_u^{(r)} \times \tau_v^{(p)} \otimes (\delta_j^{(q)})_u^{(q-r)} \rtimes \omega_{\beta_j^{(i_1-q)}} \sigma_v^{(n-i_1-p)}.$$

Since $\ell = i_2 - p$ and $r = \ell + q - i_1 = i_2 - p + q - i_1$, we have

$$A_{i_2} = \sum_{\substack{p,\\ 0\le p\le n-i_1,\\ 0\le i_2-p\le i_1}} \sum_{\substack{q,\\ 0\le q\le i_1,\\ 0\le i_2-p+q-i_1\le q}} \sum_{j=1}^{j_q} \sum_{u=1}^{u_{i_2-p+q-i_1}(j,q)} \sum_{v=1}^{v_p}$$

$$(\beta_j^{(i_1-q)})^\sim \times (\gamma_j^{(q)})_u^{(i_2-p+q-i_1)} \times \tau_v^{(p)} \otimes (\delta_j^{(q)})_u^{(p+i_1-i_2)} \rtimes \omega_{\beta_j^{(i_1-q)}} \sigma_v^{(n-i_1-p)}.$$

Introduce a new index $d = i_1 - q$. Then, the conditions on q are equivalent to the conditions $0 \leq d \leq i_1$ and $d \leq i_2 - p$. Therefore,

$$A_{i_2} = \sum_{\substack{p,d,\\ 0\leq p\leq n-i_1, 0\leq i_2-p\leq i_1,\\ 0\leq d\leq i_1, d\leq i_2-p}} \sum_{j=1}^{j_{i_1-d}} \sum_{u=1}^{u_{i_2-p-d}(j,i_1-d)} \sum_{v=1}^{v_p}$$

$$(\beta_j^{(d)})^\sim \times (\gamma_j^{(i_1-d)})_u^{(i_2-p-d)} \times \tau_v^{(p)} \otimes (\delta_j^{(i_1-d)})_u^{(p+i_1-i_2)} \rtimes \omega_{\beta_j^{(d)}} \sigma_v^{(n-i_1-p)}$$

$$= \sum_{\substack{d,\\ 0\leq d\leq i_1}} \sum_{\substack{p,\\ 0\leq p\leq n-i_1, 0\leq i_2-p\leq i_1,\\ p\leq i_2-d}} \sum_{j=1}^{j_{i_1-d}} \sum_{u=1}^{u_{i_2-p-d}(j,i_1-d)} \sum_{v=1}^{v_p}$$

$$(\beta_j^{(d)})^\sim \times (\gamma_j^{(i_1-d)})_u^{(i_2-p-d)} \times \tau_v^{(p)} \otimes (\delta_j^{(i_1-d)})_u^{(p+i_1-i_2)} \rtimes \omega_{\beta_j^{(d)}} \sigma_v^{(n-i_1-p)}.$$

We introduce a new index $k = i_2 - p - d$, i.e., $p = i_2 - d - k$. Now, the relations

$$0 \leq d \leq i_1, \quad 0 \leq p \leq n - i_1, \quad 0 \leq i_2 - p \leq i_1, \quad p \leq i_2 - d$$

are equivalent to the relations

$$0 \leq d \leq i_1, \quad 0 \leq i_2 - k - d \leq n - i_1, \quad 0 \leq d + k \leq i_1, \quad i_2 - d - k \leq i_2 - d.$$

We can rewrite the last relations as

$$0 \leq d \leq i_1, \quad 0 \leq k \quad -d \leq k \leq i_1 - d, \quad i_1 - n \leq k + d - i_2 \leq 0,$$

and further $0 \leq d \leq i_1, \quad 0 \leq k \leq i_1 - d, \quad i_1 + i_2 - n - d \leq k \leq i_2 - d$, since $d \geq 0$. The last three relations are obviously equivalent to

$$0 \leq d \leq i_1, \quad \max\{0, (i_1 + i_2 - n) - d\} \leq k \leq \min\{i_1, i_2\} - d.$$

The last relation implies $0 \leq k \leq \min\ \{i_1, i_2\} - d$. Thus $d \leq \min\ \{i_1, i_2\}$. Therefore, our relations imply

$$0 \leq d \leq \min\ \{i_1, i_2\}, \quad \max\{0, (i_1 + i_2 - n) - d\} \leq k \leq \min\{i_1, i_2\} - d.$$

Obviously, $0 \leq d \leq \min\ \{i_1, i_2\}$ implies $0 \leq d \leq i_1$.

We can now write

$$A_{i_2} = \sum_{d=0}^{\min\ \{i_1,i_2\}} \sum_{k=\max\{0,(i_1+i_2-n)-d\}}^{\min\{i_1,i_2\}-d} \sum_{j=1}^{j_q} \sum_{u=1}^{u_k(j,i_1-d)} \sum_{v=1}^{v_{i_2-d-k}}$$

$$(\beta_j^{(d)})^{\sim} \times (\gamma_j^{(i_1-d)})_u^{(k)} \times \tau_v^{(i_2-d-k)} \otimes (\delta_j^{(i_1-d)})_u^{(i_1-d-k)} \rtimes \omega_{\beta_j^{(d)}} \sigma_v^{(n-i_1-i_2+d+k)}.$$

Write $\mu^*(\pi \rtimes \sigma) = \sum_{m=0}^n A'_m \in \sum_{m=0}^n R_m \otimes R_{n-m}(G)$ where A'_m is the component of $\mu^*(\pi \rtimes \sigma)$ which comes from $R_m \otimes R_{n-m}(G)$. By the definition of $\mu^*(\pi \rtimes \sigma)$, we have $A'_m = \text{s.s.}(s_{(m),(0)}(\pi \rtimes \sigma))$. Let $i_2 \in \{1, \dots, n\}$. We shall now compute A'_{i_2}. Fix $0 \leq d \leq \min\{i_1, i_2\}$ and

$$\max\{0, (i_1 + i_2 - n) - d\} \leq k \leq \min\{i_1, i_2\} - d.$$

We have s.s. $(r_{(i_1-d,d),(i_1)}(\pi)) = \sum_{j=1}^{j_{i_1-d}} (\alpha_j^{(i_1-d)} \otimes \beta_j^{(d)})$. Further,

$$\text{s.s. } (r_{(k,i_1-d-k,d),(i_1)}(\pi)) = \sum_{j=1}^{j_{i_1-d}} \sum_{u=1}^{u_k(j,i_1-d)} (\gamma_j^{(i_1-d)})_u^{(k)} \otimes (\delta_j^{(i_1-d)})_u^{(i_1-d-k)} \otimes \beta_j^{(d)}.$$

Also, s.s. $(s_{(i_2-d-k),(0)}(\sigma)) = \sum_{v=1}^{v_{i_2-d-k}} \tau_v^{(i_2-d-k)} \otimes \sigma_v^{(n-i_1-i_2+d+k)}$. By the first section, Proposition 4.6 and Lemma 5.1, we have

$$A'_{i_2} = \sum_{d=0}^{\min\{i_1,i_2\}} \sum_{k=\max\{0,(i_1+i_2-n)-d\}}^{\min\{i_1,i_2\}-d} \sum_{j=1}^{j_{i_1-d}} \sum_{u=1}^{u_k(j,i_1-d)} \sum_{v=1}^{v_{i_2-d-k}}$$

$$(\beta_j^{(d)})^{\sim} \times (\gamma_j^{(i_1-d)})_u^{(k)} \times \tau_v^{(i_2-d-k)} \otimes (\delta_j^{(i_1-d)})_u^{(i_1-d-k)} \rtimes \omega_{\beta_j^{(d)}} \sigma_v^{(n-i_1-i_2+d+k)}.$$

Thus $A_{i_2} = A'_{i_2}$ for $1 \leq i_2 \leq n$. It remains to prove $A_0 = A'_0$. Note that $A'_0 = 1 \otimes \pi \rtimes \sigma$. We have $\mathfrak{M}^*(\pi) =$

$$(1 \otimes m^*)(1 \otimes \pi + \pi \otimes 1 + \sum_{q=1}^{i_1-1} (\sum_{j=1}^{j_q} \beta_j^{(i_1-q)} \otimes \alpha_j^{(q)})) = 1 \otimes 1 \otimes \pi + \sum_i \rho_i \otimes \rho'_i \otimes \rho''_i$$

where for any i in the above sum there exists some $t \geq 1$ such that $\rho_i \in R_t$ or $\rho'_i \in R_t$. Also $\mu^*(\sigma) = 1 \otimes \sigma + \sum_{p=1}^{n-i_1} (\sum_{v=1}^{v_p} \tau_v^{(p)} \otimes \sigma_v^{(n-i_1-p)})$. The definition of $\mathfrak{M}^*(\pi) \tilde{\rtimes} \mu^*(\sigma)$ implies $A_0 = 1 \otimes \pi \rtimes \sigma$. Thus $A_0 = A'_0$. This finishes the proof. $\square$

For irreducible smooth representations π_i of $\mathrm{GL}(n_i, F), i = 1, 2, 3, 4$, and for an irreducible smooth representation σ of $\mathrm{Sp}(m, F)$, put

$$(\pi_1 \otimes \pi_2 \otimes \pi_3) \tilde{\rtimes} (\pi_4 \otimes \sigma) = \tilde{\pi}_1 \times \pi_2 \times \pi_4 \otimes \pi_3 \times \sigma.$$

Extend $\tilde{\rtimes}$ to a $\mathbb{Z}$-bilinear mapping $\tilde{\rtimes} : (R \otimes R \otimes R) \times (R \otimes R(S)) \to R \otimes R(S)$. Now we get in the same way

Theorem 5.3. *Let π be an admissible representation of $GL(i_1,F)$ of finite length and let σ be an admissible representation of $Sp(n-i_1,F)$ of finite length. Then*

$$\mu^*(\pi \rtimes \sigma) = \mathfrak{M}^*(\pi) \tilde{\rtimes} \mu^*(\sigma). \quad \square$$

For $r_1 \otimes r_2 \in R \otimes R$ and $r \otimes s \in R \otimes R(S)$ set

$$(r_1 \otimes r_2) \rtimes (r \otimes s) = (r_1 \times r) \otimes (r_2 \rtimes s).$$

Extend $\rtimes$ $\mathbb{Z}$-bilinearly to $\rtimes : (R \otimes R) \times (R \otimes R(S)) \to R \otimes R(S)$. Set

$$M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*.$$

Theorem 5.3 is equivalent to

Theorem 5.4. *For an admissible representation π of $GL(i_1,F)$ of finite length and for an admissible representation σ of $GL(n-i_1,F)$ of finite length we have*

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \ \rtimes \ \mu^*(\sigma). \quad \square$$

6. SO$(2n+1,F)$

Let $n \in \mathbb{Z}_+$. Denote by $\mathrm{SO}(2n+1,F)$ the group of all $(2n+1)\times(2n+1)$ matrices X of determinant one with entries in F, which satisfy ${}^\tau XX = I_{2n+1}$. We fix minimal the parabolic subgroup $P_{\min}$ in $\mathrm{SO}(2n+1,F)$ consisting of all upper triangular matrices in the group (the notation P_ϕ will be also used). In $\mathrm{SO}(2n+1,F)$, we consider the maximal split torus A consisting of all diagonal matrices the groups. The maximal split torus A can be parameterized by $a : (F^\times)^n \to A$, $a(x_1,\dots,x_n) = \mathrm{diag}(x_1,\dots,x_n,1,x_n^{-1},\dots,x_1^{-1})$. The simple roots Δ determined by $P_{\min}$ are

$$\alpha_i : a(x_1,..,x_n) \mapsto x_i/x_{i+1},\ 1 \le i \le n-1, \quad \alpha_n : a(x_1,..,x_n) \mapsto x_n.$$

The positive roots are $\quad a(x_1,..,x_n) \mapsto x_i/x_j,\ 1 \le i < j \le n,$

$$a(x_1,..,x_n) \mapsto x_ix_j,\ 1 \le i < j \le n, \quad a(x_1,..,x_n) \mapsto x_i,\ 1 \le i \le n.$$

The Weyl group determined by the maximal split torus is denoted by W.

Let $\alpha = (n_1,...,n_k)$ be a partition of $m \le n$. One denotes by M_α a subgroup in $\mathrm{SO}(2n+1,F)$ consisting of all q-diag$\,(g_1,\cdots,g_k,h,\ {}^\tau g_k^{-1},\cdots,\ {}^\tau g_1^{-1})$ where $g_i \in \mathrm{GL}(n_i,F)$ and $h \in \mathrm{SO}(2(n-m)+1,F)$. Then $P_\alpha = M_\alpha P_{\min}$ is a parabolic subgroup in $\mathrm{SO}(2n+1,F)$. Note that $M_{(m)}$ is naturally isomorphic to $\mathrm{GL}(m,F) \times \mathrm{SO}(2(n-m)+1,F)$.

Let π be an admissible representation of $\mathrm{GL}(m,F)$, and let σ be an admissible representation of $\mathrm{SO}(2(n-m)+1,F)$. Set

$$\pi \rtimes \sigma \ = \ Ind_{P_{(m)}}^{\mathrm{SO}(2n+1,F)}(\pi \otimes \sigma), \quad \pi \underline{\rtimes} \sigma \ = \ Ind_{{}^tP_{(m)}}^{\mathrm{SO}(2n+1,F)}(\pi \otimes \sigma).$$

Now one can prove directly

Proposition 6.1. *Let π, π_1, π_2 be admissible representations of $GL(n, F), GL(n_1, F)$ and $GL(n_2, F)$ respectively. Let σ be an admissible representation of $SO(2m+1, F)$. Then*

(i) $\pi_1 \rtimes (\pi_2 \rtimes \sigma) \cong (\pi_1 \times \pi_2) \rtimes \sigma$;
(ii) $(\pi \rtimes \sigma)^{\sim} \cong \tilde{\pi} \rtimes \tilde{\sigma}$;
(iii) $\pi \underline{\rtimes} \sigma \cong {}^{\tau}\pi^{-1} \rtimes \sigma$. $\square$

Define $R(S)$ to be the sum of all Grothendieck groups of categories of admissible representations of $\mathrm{SO}(2n+1, F)$ of finite length. Lift $\rtimes$ to a $\mathbb{Z}$-bilinear mapping $\rtimes : R \times R(S) \to R(S)$ in the same way as we did it in the symplectic case.

Proposition 6.2. *For $\pi \in R$ and $\sigma \in R(S)$ we have $\pi \rtimes \sigma = \tilde{\pi} \rtimes \sigma$.* $\square$

Let σ be an admissible representation of finite length of $\mathrm{SO}(2n+1, F)$. The Jacquet module of σ for the parabolic subgroup $P_\alpha = M_\alpha N_\alpha$ is denoted by $s_{\alpha,(0)}(\sigma)$. Set

$$\mu^*(\sigma) = \sum_{k=0}^{n} \mathrm{s.s.}(s_{(k),(0)}(\sigma)).$$

We consider $\mu^*(\sigma)$ as an element of $R \otimes R(S)$. We lift μ^* $\mathbb{Z}$-linearly to

$$\mu^* : R(S) \to R \otimes R(S).$$

Then μ^* is coassociative, i.e., $(1 \otimes \mu^*) \circ \mu^* = (m^* \otimes 1) \circ \mu^*$.

Let $a_i^* : A \to F^\times$, $a(x_1, ..., x_n) \mapsto x_i, \quad 1 \le i \le n$. Then $\alpha_i = a_i^*/a_{i+1}^*, 1 \le i \le n-1, \alpha_n = a_n^*$. Positive roots are

$$a_i^*, \quad 1 \le i \le n, \quad a_i^*/a_j^*, \quad 1 \le i < j \le n, \quad a_i^* a_j^*, \quad 1 \le i < j \le n.$$

We may identify $W = \{\pm 1\}^n \rtimes \mathrm{Sym}(n)$. Then $pa(x_1, \ldots, x_n) = a(x_{p^{-1}(1)}, \ldots, x_{p^{-1}(n)})$, for $p \in$ Sym(n). Also, for $\epsilon = (\epsilon_i) \in \{\pm 1\}^n$ $\epsilon a(x_1, \ldots, x_n) = a(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$. Further, $(p\epsilon)\ (a_i^*) = (a_{p(i)}^*)^{\epsilon_i}$.

A simple observation tells us that Lemmas 4.1 and 4.2 hold for $\mathrm{SO}(2n+1, F)$. Then obviously, Lemmas 4.3 and 4.4, and further, Lemma 4.5 and Proposition 4.6 hold as well. Clearly, Lemma 4.7 holds-it does not depend on B_n or C_n setting. Lemma 4.8 holds. The proof is practically the same, except that for B_n, one takes $\alpha_n = a_n^*$ instead of $\alpha_n = (a_n^*)^2$, as was the case for C_n.

Fix $i_1 \in \{1, \ldots, n\}$. Let π be an admissible representation of $\mathrm{GL}(i_1, F)$. Let σ be an admissible representation of $\mathrm{SO}(2(n-i_1)+1, F)$. Take integers i_2, d, k which satisfy

$$1 \le i_2 \le n, \quad 0 \le d \le \min\{i_1, i_2\}, \quad \max\{0, (i_1 + i_2 - n) - d\} \le k \le \min\{i_1, i_2\} - d.$$

Analogously, as we got Lemma 5.1, we get

Lemma 6.3. *Let*

$$s.s.(r_{(k,i_1-d-k,d),(i_1)}(\pi)) = \sum_i \pi_i^{(1)} \otimes \pi_i^{(2)} \otimes \pi_i^{(3)}, \quad s.s.(s_{(i_2-d-k),(0)}(\sigma)) = \sum_j \pi_j^{(4)} \otimes \sigma_j.$$

Set $\Theta = \Delta\backslash\{\alpha_{i_1}\}$, $\Omega = \Delta\backslash\{\alpha_{i_2}\}$ *and* $w = q_n(d,k)_{i_1,i_2}$. *Then*

$$s.s.\ (Ind_{w^{-1}P_\Theta w\cap M_\Omega}^{M_\Omega}(w^{-1}(r_{M_{\Theta\cap w\Omega},M_\Theta}(\pi\otimes\sigma)))) = \sum_i\sum_j \pi_i^{(1)} \times \pi_j^{(4)} \times \tilde{\pi}_i^{(3)} \otimes \pi_i^{(2)} \rtimes \sigma_j =$$

$$\sum_i\sum_j \pi_i^{(1)} \times \tilde{\pi}_i^{(3)} \times \pi_j^{(4)} \otimes \pi_i^{(2)} \rtimes \sigma_j = \sum_i\sum_j \tilde{\pi}_i^{(3)} \times \pi_i^{(1)} \times \pi_j^{(4)} \otimes \pi_i^{(2)} \rtimes \sigma_j. \quad \square$$

Let π_i be an irreducible smooth representation of $\mathrm{GL}(n_i,F), i = 1,2,3,4$, and let σ be an irreducible smooth representation of $\mathrm{SO}(2m+1,F)$. We define

$$(\pi_1\otimes\pi_2\otimes\pi_3)\tilde{\rtimes}(\pi_4\otimes\sigma) = \tilde{\pi}_1\times\pi_2\times\pi_4\otimes\pi_3\rtimes\sigma.$$

Now, Theorem 5.3 holds in this situation.

Theorem 6.4. *Let* π *be an admissible representation of* $GL(i_1,F)$ *of finite length and let* σ *be an admissible representation of* $SO(2(n-i_1)+1,F)$ *of finite length. Set*

$$\mathfrak{M}^* = (1\otimes m^*)\circ s\circ m^*.$$

Then

$$\mu^*(\pi\rtimes\sigma) = \mathfrak{M}^*(\pi)\ \tilde{\rtimes}\ \mu^*(\sigma). \quad \square$$

Make $R\otimes R(S)$ an $R\otimes R$-module, as we did in the symplectic case. Set

$$M^* = (m\otimes 1)\circ(\sim\otimes m^*)\circ s\circ m^*.$$

We can now say the last theorem in the following way

Theorem 6.5. *Let* π *be an admissible representation of* $GL(i_1,F)$ *of finite length and let* σ *be an admissible representation of* $SO(2(n-i_1)+1,F)$ *of finite length. Then*

$$\mu^*(\pi\rtimes\sigma) = M^*(\pi)\rtimes\mu^*(\sigma).$$

7. Relation with Hopf modules

Let $\mathcal{H}$ be a Hopf algebra (over $\mathbb{Z}$) with multiplication $m : \mathcal{H}\otimes\mathcal{H}\to\mathcal{H}$ and comultiplication $m^* : \mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$. Consider $\mathcal{H}\otimes\mathcal{H}$ as an algebra in a natural way. The mapping $s : \mathcal{H}\otimes\mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$, $\sum x_i\otimes y_i \to \sum y_i\otimes x_i$ is a ring endomorphism. One proves the following lemma directly.

7.1. Lemma. *Suppose that the multiplication is commutative. Let $\psi : \mathcal{H} \to \mathcal{H}$ be a ring endomorphism (preserving unit). Set*

$$M(\psi) = (m \otimes 1) \circ (\psi \otimes m^*) : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}.$$

Then $M(\psi)$ is a ring endomorphism. $\square$

Suppose that $\mathcal{M}$ is a module over $\mathcal{H}$, and that it is also a comodule over $\mathcal{H}$ with the comodule structure map $\mu^* : \mathcal{M} \to \mathcal{H} \otimes \mathcal{M}$ (see [Sw] for these definitions). Note that $\mathcal{H} \otimes \mathcal{M}$ is a $\mathcal{H} \otimes \mathcal{H}$-module in an obvious way. One can supply $\mathcal{H}$-module structure on $\mathcal{H} \otimes \mathcal{M}$ in different ways. Let

$$\Psi : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$$

be a ring homomorphism. Define a $\mathcal{H}$-module structure on $\mathcal{H} \otimes \mathcal{M}$ by $h'.(h \otimes m) = \Psi(h')(h \otimes m)$. Then we shall say that $\mathcal{H} \otimes \mathcal{M}$ is a $\mathcal{H}$-module with respect to Ψ.

We shall say that a module and a comodule $\mathcal{M}$ over a Hopf algebra $\mathcal{H}$ is a Ψ-Hopf module over $\mathcal{H}$ if $\mu^* : \mathcal{M} \to \mathcal{H} \otimes \mathcal{M}$ is a homomorphism of $\mathcal{H}$-modules, where we consider $\mathcal{H}$-module structure on $\mathcal{H} \otimes \mathcal{M}$ with respect to Ψ. Note that taking for $\Psi = m^*$ we get the definition of a Hopf module, i.e. m^*-Hopf module is just Hopf module. If $\mathcal{M}$ is a Ψ-Hopf module over a graded Hopf algebra $\mathcal{H}$, if $\mathcal{M}$ is a $\mathbb{Z}_+$-graded abelian group and if the module and the comodule structure maps are $\mathbb{Z}_+$-graded, then we shall say that $\mathcal{M}$ is a graded Ψ-Hopf module over $\mathcal{H}$.

Since $\sim: R \to R$ is a ring automorphism, $M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^* : R \to R \otimes R$ is a ring homomorphism by previous lemma. Now we have the following description of the structure of $R(S)$:

7.2. Theorem. *$R(S)$ is a graded M^*-Hopf module over R*

7.3. Remark. The above theorem implies that $R(S)$ is not a Hopf module (over R). We could see easily that without using the above theorem. Namely, if $R(S)$ would be a Hopf module, then the representation theory of groups $\mathrm{Sp}(n, F)$ and $\mathrm{SO}(2n+1, F)$ would be pretty different. We shall show now how one can see from the representation theory, that $R(S)$ is not a Hopf module. Suppose that $R(S)$ is a Hopf module. We shall show that this implies that unitary principal series representations of $\mathrm{Sp}(n, F)$ (and $\mathrm{SO}(2n+1, F)$) are irreducible. Since it is known that unitary principal series representations of $\mathrm{Sp}(n, F)$ are not irreducible in general (not even for $\mathrm{Sp}(1, F)$=$\mathrm{SL}(2,F)$), this would imply that $R(S)$ is not a Hopf module. Let $\chi_1, \chi_2, \dots, \chi_n$ be unitary characters of $F^\times$. Since we suppose that $R(S)$ is a Hopf module, we can easily compute

$$\text{s.s.}(s_{(n),(0)}(\chi_1 \times \chi_2 \times \cdots \times \chi_n \rtimes 1)) = \chi_1 \times \chi_2 \times \cdots \times \chi_n \otimes 1.$$

The representation on the right hand side is irreducible ([Z1]). Let π be an irreducible subquotient of $\chi_1 \times \chi_2 \times \cdots \times \chi_n \rtimes 1$ such that $\chi_1 \times \chi_2 \times \cdots \times \chi_n \otimes 1$ is a subquotient of $s_{(n),(0)}(\pi)$. Then the multiplicity of π in $\chi_1 \times \chi_2 \times \cdots \times \chi_n \rtimes 1$ is one. Suppose that there is some other irreducible subquotient τ of $\chi_1 \times \chi_2 \times \cdots \times \chi_n \rtimes 1$. The exactness of the Jacquet functor implies $s_{(n),(0)}(\tau)$=0. This implies that the Jacquet module for the standard minimal parabolic subgroup is trivial. This cannot happen since a non-trivial subquotient of a non-unitary principal series representation has always non-trivial Jacquet module for the standard minimal parabolic subgroup (this fact follows easily from Corollary 6.3.9, (b) of [C], and Frobenius reciprocity). Therefore, $\pi = \chi_1 \times \chi_2 \times \cdots \times \chi_n \rtimes 1$. So, we have proved that assumption that $R(S)$ is a Hopf module implies that the unitary principal series of these groups are always irreducible. Since this is not the case, we see that $R(S)$ is not a Hopf module.

The above remark provides a very simple example for understanding how the structure of $R(S)$ determines some properties of the representation theory of the corresponding groups.

In comparison with the structure of R, the structure of $R(S)$ over R has one substantially new ingredient. This is M^*. It is not in the range of the Hopf algebra R, since $M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*$ is defined using the contragredient map $\sim: R \to R$, which does not enter the definition of R as a Hopf algebra (note that the contragredient map is an involutive anti-automorphism of the Hopf algebra R). The development of the representation theory of groups $\mathrm{Sp}(n,F)$ and $\mathrm{SO}(2n+1,F)$ should give a new understanding of the structure of $R(S)$ (we expect that also this structure will help to this development). In a similar way, the development of the representation theory of general linear groups done by J. Bernstein and A.V. Zelevinsky helped a lot to our understanding of the structure of the Hopf algebra R (see [Z1]).

The fact that $R(S)$ is a M^*-Hopf module over R must contain some information about R itself (besides the structure studied in this paper, we expect that there will be a number of other such structures coming from other series of classical groups). It remains to be seen what kind of information about R one will get from this fact. Two things can help for getting such information. One is a better understanding of such structures which we called Ψ-Hopf modules (with perhaps a more specific Ψ, as we have in the case of $R(S)$). The other one is a better understanding of the the representation theory of $\mathrm{Sp}(n,F)$ and $\mathrm{SO}(2n+1,F)$ (which should give an information about internal structure of $R(S)$). The understanding of both topics is in the moment relatively poor.

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Graphical interpretation of $p = p_n(d, k)_{i_1,i_2}$

We use the following notation in the drawing: $u = i_2 - d$, $v = i_2$, $u' = i_1 - d$, $v' = i_1$ and $w = i_1 + i_2 - d - k$.

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