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REPRESENTATIONS OF CLASSICAL p-ADIC GROUPS

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Contents

Preface

This paper is based on the notes of the course "Representations of classical p-adic groups" given on the European School of Group Theory in C.I.R.M. Luminy-Marseille, in 1991. To the original notes of the course we have added, at some places in this revised version, more detailed explanations. Also, we have corrected a number of misprints which we noted in the original notes. Most of them were in the ninth section. At this point we want to thank to M. Duflo, J. Faraut and J.-L. Waldspurger for an excellent organization of the school, and to the participants of the school for the interest that they have shown.

There exist excellent papers which can be used for an introduction to the representation theory of reductive p-adic groups. Particularly interesting for this paper are the paper [Cs1]

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of W. Casselman and the series of papers [BeZe1], [BeZe2], [Ze1] of J. Bernstein and A. V. Zelevinsky. In this paper our interest is to introduce only to some of the ideas of the representation theory of reductive p-adic groups. To keep the size of this paper moderate, we have tried to avoid technical details as much as possible. The size of this paper is only a fraction of the size of the papers that we wanted to cover. In general, detailed proofs are omitted in this paper. At the most places we have tried either to explain the ideas of the proofs, or to give a proof in same simple case where one should be able to catch the idea of the general proof. Unfortunately, because of the size of the paper, and of the attempt to keep the paper on an introductory level, we were forced to omit a lot of topics which would be treated naturally with the subject of the representation theory of reductive p-adic groups. Let us mention just a few of them: the structure theory of reductive groups, Tits' systems, Bruhat-Tits' structure theory of reductive p-adic groups, the representation theory of general locally compact groups, the theory of automorphic forms and the Langlands' program, e.t.c. Never the less, these topics are implicitly present in the paper. This omission is a reason for including of a wider list of the references where these other topics are considered. Some of the references are included also because of the historical reasons.

We shall describe now the content of this paper in more details. In the first section we introduce the classical groups whose representation theory we shall consider. We have also described the fields over which the classical groups will be considered. The second section introduces the notion of the parabolic induction and discusses the place of this notion in the representation theory of reductive groups over local fields.

The theory of admissible representations of reductive p-adic groups is essentially the language of the theory of general locally compact groups adapted to the reductive p-adic groups. We introduce this theory in the third section. The Jacquet modules, which are crucial in the study of the parabolically induced representations, are studied in the fourth section.

The fifth section is devoted to the computation of the composition series of the Jacquet modules of the parabolically induced representations of $SL(2)$. This illustrates Casselman's, and also Bernstein's and Zelevinsky's calculation in the general case ([Cs1], [BeZe2]). We gave an example of a representation which can not be reached by the parabolic induction in the sixth section, namely an example of a cuspidal representation. We have also shown how one can use the calculation of the fifth section to get a preliminary classification of irreducible representations of $GL(2)$ over a finite field. This is a simple introduction to the analysis of the parabolically induced representations of $SL(2)$ and $GL(2)$ which has been done in the seventh section. This analysis implies a preliminary classification of the irreducible representations of these groups. In this section we have also explained the classification of the irreducible unitary representations of these groups. In the eighth section we wrote down the general consequences which may be obtained from the computation of composition series of the Jacquet modules of the parabolically induced representations of general reductive p-adic groups. We have proved that results for the case of $SL(2)$ in the previous section.

The ninth section considers $GL(n)$. We follow here Bernstein and Zelevinsky. They have used the Hopf algebra structure to get a control of the Jacquet modules of the parabolically induced representations of $GL(n)$. The algebra structure is defined using the parabolic induction, while the definition of the coalgebra structure uses the Jacquet modules. We tried to explain here how one can use this Hopf algebra structure in the representation theory of p-adic $GL(n)$. At the end of this section we describe the Langlands' classification for $GL(n)$. The unitary dual is also described there ([Td6]). This section is a good introduction to the tenth section where the symplectic groups are studied. Representations of these groups are studied as modules over the representations of the general linear groups, via the parabolic induction. The Jacquet modules define a comodule structure. The connection of these two structures is explained here. These two structures are used here to construct some new square integrable representations. The last section explains how one can use these structures in the study of the irreducibility of the parabolically induced representations of the symplectic groups.

The first nine sections can be used as an introduction to the already mentioned introductory papers of Casselman and of Bernstein and Zelevinsky. A complete proofs of the facts discussed in these sections could be found mainly in the papers [Cs], [BeZe1], [BeZe2] and [Ze1]. Most of these facts this author has learned either from that papers, or from several people, primarily from D. Miličić. For the unitary dual of $GL(n)$ one should consult [Td6]. The last two sections, together with the ninth section, introduces to the ideas of the papers [Td13]-[Td15], [SaTd] and forthcoming papers on the representations of classical groups. We hope that these ideas will play a role in a number of unsolved problems of the representation theory of classical p-adic groups. Most of the problems of the classifications for the classical p-adic groups other than $GL(n)$ are still unsolved.

1. Classical groups

Let G be a locally compact group. Suppose that V is a complex vector space. Let π be a homomorphism of G into the group of all invertible operators on V. Then (π, V), or simply π, is called a **representation** of G. Suppose that (π, H) is a representation of G where H is a Hilbert space. Suppose that the mapping

$$(g, v) \mapsto \pi(g)v$$

from $G \times H$ to H is continuous. Then (π, H) is called a **continuous representation** of G. A continuous representation (π, H) is called **irreducible** if there does not exist non-trivial closed subspaces of H which are invariant for all $\pi(g)$ when $g \in G$. Sometimes such representations are called also **topologically irreducible**. A continuous representation on a one-dimensional space is called a **character** of G. Clearly, a character is always an irreducible representation. A **unitary representation** of G is a continuous representation of G such that all operators $\pi(g), g \in G$, are unitary operators. Let (π_1, H_1) and (π_2, H_2) be unitary representations of G. They are called **unitarily equivalent** if there exists a Hilbert space isomorphism $\varphi : H_1 \to H_2$ such that $\pi_2(g)\varphi = \varphi\pi_1(g)$ for all $g \in G$. Denote by $\hat{G}$ the set of all unitarily equivalence classes of irreducible unitary representations of G on non-trivial Hilbert spaces. Then $\hat{G}$ is called the **dual space** of G, or the **unitary dual** of G.

The first step in building of harmonic analysis on G is classification of irreducible unitary representations of G. The next step is description of interesting and important unitary representations of G in terms of irreducible unitary representations, i.e. in terms of $\hat{G}$.

We shall be interested in the problem of classification of irreducible unitary representations of G mostly when G is a classical simple group over a locally compact non-discrete field F. Very soon we shall restrict ourselves to the case when F is not isomorphic to $\mathbb{R}$ or $\mathbb{C}$.

A locally compact non-discrete field will be called a **local field**. First, we are going to describe local fields. Each connected local field is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$. These fields are called **archimedean**. A local field which is not isomorphic to $\mathbb{R}$ or $\mathbb{C}$ will be called **non-archimedean**.

Let p be a prime rational integer. Write $q \in \mathbb{Q}^\times$ as $q = p^\alpha \frac{u}{v}$, where $u, v, \alpha \in \mathbb{Z}$ and where p does not divide uv. Set

$$|q|_p = p^{-\alpha},$$

and $|0|_p = 0$. Denote by $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to $|\ \ |_p$. Then $\mathbb{Q}_p$ is called the **field of p-adic numbers**. Let E be a finite field extension of $\mathbb{Q}_p$. Then E has a natural topology of vector space over $\mathbb{Q}_p$. With this topology, E becomes a local field. The topology on E can be introduced with the following absolute value

$$|x|_E = |N_{E/\mathbb{Q}_p}(x)|_p,$$

where $N_{E/\mathbb{Q}_p}$ denotes the norm map of the extension $E \supset \mathbb{Q}_p$. Note that this absolute value does not extend in general the absolute value $|\ \ |_p$ from $\mathbb{Q}_p$. The absolute value which extends $|\ \ |_p$ is

$$|x|'_E = |N_E/\mathbb{Q}_p(x)|^{\frac{1}{[E:\mathbb{Q}_p]}}.$$

For our purposes it is more convenient to deal with $|\ \ |_E$ instead of $|\ \ |'_E$. We shall see soon what is the reason. These absolute values are **ultrametric**, i.e. the following inequality holds

$$|x + y|_E \leqslant \max\{|x|_E, |y|_E\}, \quad x, y \in E.$$

This inequality is stronger then the usual triangle inequality. Obviously, for $t \geqslant 0$, the ball

$$\{x \in E; |x|_p \leqslant t\}$$

is an additive subgroup. If $t \leqslant 1$, then it is moreover closed for multiplication. Therefore, E is a totally disconnected topological space. Each non-archimedean local field of characteristic 0 is isomorphic to some finite extension E of some $\mathbb{Q}_p$.

We could come to these fields also in the following more arithmetic way. Let K be a number field (i.e., a finite extension of $\mathbb{Q}$). Let $\mathcal{O}_K$ be the ring of integers of K (i.e. the set of elements of K integral over $\mathbb{Z} \subseteq K$). Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_K$. Consider the topology on K having

$$\mathfrak{p}^i, \quad i = 1, 2, 3, ...$$

for a basis of neighborhoods of 0. Then the completion of K with respect to corresponding uniform structure is a local non-archimedean field.

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field with q elements. Let $\mathbb{F}((X))$ be the field of formal power series over $\mathbb{F}$. For $f \in \mathbb{F}((X))^\times$ set

$$|f|_{\mathbb{F}((X))} = q^{-n},$$

where

$$f = \sum_{k=n}^{\infty} a_k X^k$$

and $a_n \neq 0$. One sees directly that $|\ \ |_{\mathbb{F}((X))}$ is an ultrametric absolute value. Now $\mathbb{F}((X))$ with respect to the above absolute value is a local non-archimedean field. Each non-archimedean field of positive characteristic is isomorphic to some $\mathbb{F}((X))$.

For the above classification of local fields one can consult the first chapter of the fundamental Weil's book [We2].

On a local field F there exists a positive measure

$$f \mapsto \int_F f(x)dx \tag{1.1}$$

which is invariant under translations by elements of F. Such measure is unique up to a multiplication with a positive constant. If $a \in F$ and f is a compactly supported continuous function on F, then

$$\int f(ax)dx = |a|_F^{-1} \int_F f(x)dx.$$

In the case of $F=\mathbb{R}$ one takes for $|\ \ |_{\mathbb{R}}$ the usual absolute value. If $F=\mathbb{C}$, then one takes $|x+iy|_{\mathbb{C}}=x^2+y^2, x,y\in\mathbb{R}$.

We shall now describe the groups.

Let V be a finite dimensional vector space over a local field F. The group of all regular linear operators on V is denoted by $GL(V)$. This group is called the **general linear group** of V. The **special linear group** of V consists of operators in $GL(V)$ which have determinant equal to one. It is denoted by $SL(V)$. If V possesses a non-degenerate symplectic form, then the subgroup of $GL(V)$ of operators which preserves this form is called the **symplectic group** of V. It is denoted by $Sp(V)$. Suppose that V is supplied with a non-degenerate orthogonal form which possesses isotropic subspaces of maximal possible dimension. Then the subgroup of $SL(V)$ of all operators which preserve the orthogonal form is denoted by $SO(V)$. It is called the **orthogonal group** of V.

We have the following matrix realizations of these groups. First, $GL(n,F)$ denotes the group of all $n\times n$ regular matrices with entries in F. Further, $SL(n,F)$ denotes the subgroup of those matrices which have determinant equal to one. Consider the following $n\times n$ matrix

$$J_n=\begin{bmatrix}00 & \dots & 01\\ 00 & \dots & 10\\ \vdots & & \\ 10 & \dots & 0\end{bmatrix}.$$

Let

$$Sp(n,F)=\left\{S\in GL(2n,F);\ \ {}^tS\begin{bmatrix}0 & J_n\\ -J_n & 0\end{bmatrix}S=\begin{bmatrix}0 & J_n\\ -J_n & 0\end{bmatrix}\right\}.$$

Here ${}^tS$ denotes the transposed matrix of S. We shall denote by ${}^{\tau}S$ the transposed matrix of S with respect to the second diagonal. Denote by I_n the identity matrix in $GL(n,F)$. Let

$$SO(n,F)=\{S\in SL(n,F);\ \ {}^{\tau}S\,S=I_n\}.$$

We shall always work with matrix forms of classical groups.

These groups are topological groups in the natural way. If F is a non-archimedean field, then these groups are totally disconnected. One can write a basis of neighborhoods of identity consisting of open (and then also closed) compact subgroups.

We shall say now a few words about the structure of these groups. Let G be either $GL(n,F)$ or $Sp(n,F)$ or $SO(n,F)$. We shall now introduce subgroups of G which play a very important role in the representation theory of G. Roughly, these subgroups enable a reduction of some of the problems of the representation theory of G to a groups of similar type of lower dimension (and complexity).

The subgroup of all upper triangular matrices in G will be denoted by $P_{\min}$. It will be called the **standard minimal parabolic subgroup**. Note that $P_{\min}$ is a solvable group. The standard minimal parabolic subgroup is maximal with respect to this property. A subgroup of G containing $P_{\min}$ is called a **standard parabolic subgroup**.

One can describe standard parabolic subgroups of $GL(n,F)$ in the following way. Let $\alpha=(n_1,...,n_k)$ be an ordered partition of n into positive integers. Look at elements of $GL(n,F)$ as block matrix with blocks of sizes $n_i\times n_j$. Let P_α (resp. M_α), be the upper block-triangular matrices (resp. block-diagonal matrices) in $GL(n, F)$. Let N_α be the matrices in $\mathcal{P}_\alpha$ which have identity matrices on the block-diagonal. Then

$$\alpha \mapsto P_\alpha$$

is an one-to-one mapping of the set of all partitions of n onto the set of all standard parabolic subgroups of $GL(n, F)$. One has also $P_\alpha = M_\alpha N_\alpha$. More precisely,

$$P_\alpha = M_\alpha \ltimes N_\alpha.$$

This will be called the **standard Levi decomposition** of P_α. The group M_α is called the **standard Levi factor** of P_α. The group N_α is called the **unipotent radical** of P_α. Note that M_α is a direct product of some general linear groups.

Let

$$\alpha = (n_1, ..., n_k)$$

be a partition of some $0 \leqslant m \leqslant n$. In the case of the group $Sp(n, F)$ we denote

$$\alpha' = (n_1, ..., n_k, 2n - 2m, n_k, ..., n_1).$$

In the case of the group $SO(2n + 1, F)$ we denote

$$\alpha' = (n_1, ..., n_k, 2n + 1 - 2m, n_k, ..., n_1).$$

Let G denotes either the group $Sp(n, F)$ or $SO(2n + 1, F)$. Set

$$P_\alpha^S = P_{\alpha'} \cap G,$$

$$M_\alpha^S = M_{\alpha'} \cap G,$$

$$N_\alpha^S = N_{\alpha'} \cap G.$$

Then $\alpha \mapsto P_\alpha^S$ is a parametrization of all standard parabolic subgroups of G. One has also

$$P_\alpha^S = M_\alpha^S \ltimes N_\alpha^S.$$

This is called the **standard Levi decomposition** of P_α^S, M_α^S is called the **standard Levi factor** of P_α^S, N_α^S is called the **unipotent radical** of P_α^S. If $G = Sp(n, F)$ (resp. $G = SO(2n+1, F)$), then M_α^S is a direct product of a group of Sp-type (resp. $SO(2k+1, F)$-type) and of general linear groups.

For $SO(2n, F)$ there is a slightly different situation in parametrization of the standard parabolic subgroups.We shall omit this case here.

Let G be either $GL(n, F)$, or $Sp(n, F)$, or $SO(2n + 1, F)$. **Parabolic subgroups** of G are conjugates of the standard parabolic subgroups of G. **Levi decompositions** are conjugates of the corresponding standard Levi decompositions of the standard parabolic subgroups.

A Levi factor of a parabolic subgroup of the group G is either a direct product of general linear groups if G is a general linear group, or in the other case, a direct product of a classical group of the same type with general linear groups. We define **parabolic subgroups** (resp. **standard parabolic subgroups**) of a Levi factor (resp. standard Levi factor) to be direct products of parabolic subgroups (resp. standard parabolic subgroups). Analogously, we define **Levi decompositions** (**standard Levi decompositions**) of such parabolic subgroups (resp. standard parabolic subgroups), etc.

For a general locally compact group G there exists a positive measure on G which is invariant under right translations. Such a measure will be called a **right Haar measure** on G. Two right Haar measures on G are proportional. The integral of a continuous compactly supported function f with respect to a fixed right Haar measure will be denoted by

$$\int_G f(g)dg.$$

There exists a continuous positive-valued character Δ_G of G such that

$$\int_G f(xg)dg = \Delta_G(x)^{-1} \int_G f(g)dg$$

for each continuous compactly supported function f on G and each $x \in G$. The character Δ_G is called the **modular character** of G. If $\Delta_G \equiv 1_G$, then one says that G is a **unimodular group** (by 1_X we shall denote in this paper the constant function on X which is equal to 1 everywhere).

For a proof of these facts one may consult [Bb3]. Let me mention that the definition of the modular character in [Bb3] is diferent from the one that we use here. The modular character in [Bb3] corresponds to Δ_G^{-1} in our notation. The proof of existence of Haar measures is very simple for totally disconnected groups (see [BeZe1]).

1.1. Examples.

(i) The (right) Haar measure on $(F, +)$ is the invariant measure (1.1).

(ii) We have the following Haar measure on F^m

$$\int_{F^m} f(x)dx = \int_F \cdots \int_F f(x_1, \dots, x_m)dx_1, \dots dx_m.$$

(iii) The Haar measure on $GL(n, F)$ is

$$\int_{GL(n,F)} f(g)dg = \int_{F^{(n^2)}} f(x)dx/|\det(g)|_F^n.$$

As one can see easily from (iii), the group $GL(n, F)$ is unimodular. Moreover, all classical groups are unimodular. In general, proper parabolic subgroups are not unimodular. Never the less, Levi factors and unipotent radicals are unimodular groups.

1.2. Remark.

A good framework when one works with representations of classical groups are **reductive groups**. The classical groups are the most important examples of reductive groups. We have already used some of the general notions from the theory of reductive groups (a linear algebraic group is reductive if it does not contain a normal unipotent subgroup of positive dimension). Usually in this paper we shall not give general definitions, but we shall rather present objects explicitly.

The theory that we present here is directed to the representations of classical groups over local fields (after the following section we shall assume that the local field is non-archimedean). For the technical reasons it is useful to develop the theory for products of classical groups of type $Sp(n)$ or $SO(n)$ with general linear groups. This is useful in order to include in the theory the Levi factors of parabolic subgroups. In the sequel, we shall denote by G one of such groups. Most of the general results that we present in these notes apply to general reductive groups. For a theory of reductive groups one may consult [BlTi1] and [BlTi2].

2. Parabolic induction

If one wants to classify $\hat{G}$, one should have some method of construction of irreducible unitary representations of G, or at least for the first step, of irreducible continuous representations of G.

Let us consider for a moment a finite group G and a subgroup $P \leqslant G$. One of the simplest functors which one can consider on representations of G is the restriction functor to the subgroup P. An interesting question is usually does some functor have an adjoint functor. The right adjoint functor to the restriction functor is the induction functor from P to G. Clearly, the induction functor assigns to representations of the smaller group P, representations of the whole group G.

The notion of induction can be generalized to the case of locally compact groups. For reductive groups over local fields a particular type of induction is pretty simple, but a very powerful tool of producing of irreducible continuous representations. This tool is parabolic induction. We shall now define this notion.

Let G be one of the classical groups over local fields introduced in the first section, or one of the Levi factors of parabolic subgroups of such classical groups. Let P be a (standard) parabolic subgroup with (the standard) Levi decomposition $P = MN$. Then there exists a maximal compact subgroup K_o of G such that

$$G = P_{\min} K_o.$$

This is called the **Iwasawa decomposition** of G.

2.1. Examples.

(i) If $G = GL(n, F)$ and if F is a non-archimedean field, then one may take $K_o = GL(n, \mathcal{O}_F)$ where $\mathcal{O}_F = \{x \in F; |x|_F \leqslant 1\}$ is the **ring of integers of** F. If $F = \mathbb{R}$ (resp. $\mathbb{C}$) one may take for K_o the group of orthogonal (resp. unitary) matrices in $GL(n, F)$.

(ii) Let F be a local non-archimedean field. Since $GL(n, \mathcal{O}_F)$ is an open (and then closed) subgroup of $GL(n, F)$, the restriction of the Haar measure of $GL(n, F)$ (see Examples 1.1., (i)) to $GL(n, \mathcal{O}_F)$, is a Haar measure on $GL(n, \mathcal{O}_F)$. Note that this Haar measure on $GL(n, \mathcal{O}_F)$ is just a restriction of the standard invariant measure on $F^{(n^2)}$ to $GL(n, \mathcal{O}_F)$.

Let (σ, H) be a continuous representation of M. Denote by $\mathbf{Ind}_P^G(\sigma)$ the Hilbert space of all (classes of) measurable functions

$$f : G \longrightarrow H$$

which satisfy

(i) $f(mng) = \Delta_P(m)^{1/2} \sigma(m) f(g)$, for any $m \in M, n \in N$ and $g \in G$;

(ii)

$$\int_{K_o} \|f(k)\|^2 dk < \infty.$$

For $g \in G$ and $f \in \mathbf{Ind}_P^G(\sigma)$ denote by $R_g f$ the function

$$(R_g f)(x) = f(xg). \tag{2.1}$$

We shall say that G acts on $\mathbf{Ind}_P^G(\sigma)$ by the **right translations**. Then $R_g f \in \mathbf{Ind}_P^G(\sigma)$ and $(R\,,\mathbf{Ind}_P^G(\sigma))$ is a continuous representation of G. If σ was a unitary representation, then $\mathbf{Ind}_P^G(\sigma)$ is also a unitary representation. The representation $\mathbf{Ind}_P^G(\sigma)$ is called a **parabolically induced representation** of G by σ from P.

The integration over K_o which was a condition in the definition of the parabolically induced representations may appear a little bit mysterious for somebody. For an explanation of this condition one may consult Remarks 2.2, (i). There is also an explanation of the factor $\Delta_P^{1/2}$ which appears in the definition of the parabolically induced representations.

Let $(\pi, H) \in \hat{G}$. Then π, as a representation of K_o, decomposes into a direct sum of irreducible representation

$$\pi|K_o \cong \underset{\delta\in\hat{K}_o}{\oplus} n_\delta \delta,$$

where $n_\delta \in \mathbb{Z}_+ \cup \{\infty\}$. A fundamental property of the representation theory of G is that $n_\delta \in \mathbb{Z}_+$, i.e. K_o-multiplicities are finite. This fact has a very important technical consequences, as well as qualitative consequences for the harmonic analysis on G (G is a type I group, see [Dx]). Roughly, this finiteness condition is a consequence of the fact that certain convolution algebras of functions on G are not too far from being commutative. This result was proved by Harish-Chandra for the archimedean fields (see 4.5. in [Wr]), and by J. Bernstein in [Be1] for the non-archimedean fields. In the following two sections there is an outline of the proof of that Bernstein's result. Because of this Harish-Chandra's and Bernstein's result, we shall assume at the rest of this section that the irreducible continuous representations of G that we consider, have all K_o-multiplicities finite.

Suppose for a moment that F is a non-archimedean field. Since each $GL(n,\mathbb{C})$ has a neighborhood of identity which does not contain a non-trivial subgroup, each irreducible unitary representation of K_o factors through a representation of K_o/K where K is an open normal subgroup of K_o. It is easy to see from this that the above fact about K_o-multiplicities is equivalent to the following fact: for any $(\pi, H) \in \hat{G}$ and for any open compact subgroup K of G the space of K-**invariants**

$$H^K = \{v \in H; \pi(k)v = v \text{ for any } k \in K\}$$

is finite dimensional.

The problem of the classification of the irreducible unitary representations of G appeared to be much harder than it was expected. An easier problem appeared to be the problem of the classification of the irreducible continuous representations of G, which have finite K_o-multiplicities. To get the unitary dual $\hat{G}$ one needs then to extract from irreducible representations those representations which are actually unitary.

From the point of view of the parabolic induction, there are two classes of irreducible continuous representations of G. The first class is formed of the representations which are equivalent to the irreducible subrepresentations of the parabolically induced representations from the proper parabolic subgroups by the irreducible continuous representations. Certainly, it is very important to make precise what means equivalent. Since we shall pass very soon to an algebraic treatment of the theory, we shall omit the precise definition here (see Remarks 2.2.,(ii)). The second class consists of remaining irreducible representations. Let us call for a moment the representations from the second class primitive.

One approach to the irreducible continuous representations of G may be first to classify primitive representations of Levi factors of parabolic subgroups, and then to classify representations in the first class. In this way the representations in the first class reduce to the primitive representations of smaller reductive groups, modulo an understanding of parabolic induction. Our principal aim in this lectures will be to describe some methods of the study of the parabolic induction in the non-archimedean case. One further step in the strategy that we have described here is introduction of the Langlands classification (see the fourth section).

Suppose that F is an archimedean field. Then the Casselman's subrepresentation theorem (Theorem 8.21. in [CsMi]), which generalizes the famous Haris-Chandra's subquotient theorem, tells that each irreducible continuous representation is equivalent to a subrepresentation of $\mathbf{Ind}_{P_{\min}}^{G}(\sigma)$, where σ is an irreducible continuous representation of (the standard) Levi factor of $P_{\min}$. In the case of the classical groups, σ is a character.

Let us take a look at one of the simplest non-trivial cases, the case of $SL(2,\mathbb{R})$. Here one may take

$$K_o = \left\{ \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}; \varphi \in \mathbb{R}/2\pi\mathbb{Z} \right\}.$$

Note that K_o is commutative. Because of that, each irreducible unitary representation of K_o is one dimensional. Now $\hat{K}_o = \{\delta_n; n \in \mathbb{Z}\}$, where

$$\delta_n : \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \mapsto e^{in\varphi}.$$

Take

$$P_{\min} = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}; a \in \mathbb{R}^\times, b \in \mathbb{R} \right\},$$

$$M_{\min} = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}; a \in \mathbb{R}^\times \right\}.$$

Let χ be a character of $M_{\min}$. Restriction to K_o gives an isomorphism of $\mathbf{Ind}_{P_{\min}}^{SL(2,\mathbb{R})}(\sigma)$ to

$$\bigoplus_{n \in 2\mathbb{Z}} \delta_n$$

$$(\text{resp. } \bigoplus_{n \in (2\mathbb{Z}+1)} \delta_n)$$

if

$$\sigma\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) = 1$$

$$\text{(resp.}\quad \sigma\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) = -1)$$

as representations of K_o. Suppose that H is a (closed) irreducible subrepresentation of $\mathbf{Ind}_{P_{\min}}^{SL(2,\mathbb{R})}(\sigma)$. Then it is completely determined by $X \subseteq 2\mathbb{Z}$ (resp. $X \subseteq (2\mathbb{Z}+1)$). Denote by $\mathrm{Ind}_{P_{\min}}^{SL(2,\mathbb{R})}(\sigma)$ the algebraic span of irreducible K_o-subrepresentations of $\mathbf{Ind}_{P_{\min}}^{SL(2,\mathbb{R})}(\sigma)$. Then

$$\pi(X)f = \frac{d}{dt}(\pi(\exp tX)f)|_{t=0}$$

defines an action of the Lie algebra

$$\mathfrak{sl}(2,\mathbb{R}) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}; a,b,c,d \in \mathbb{R}, a+d=0\right\}$$

of the Lie group $SL(2,\mathbb{R})$ on $\mathrm{Ind}_{P_{\min}}^{SL(2,\mathbb{R})}(\sigma)$. One can find basis of $\mathfrak{sl}(2,\mathbb{R})$ in such a way that the formulas for the action of the elements of the basis are particularly simple (see §5. of ch. VI of [Lang]). Certainly, $H \bigcap \mathrm{Ind}_{P_{\min}}^{SL(2,\mathbb{R})}(\sigma)$ is invariant for the action of the Lie algebra and it is dense in H. In this way, the problem of the classification of the irreducible continuous representations reduces to a combinatorial problem which is now not very hard to solve. For general reductive groups over archimedean fields, such type of approach leads to the theory of $(\mathfrak{g}, K)$-modules (see [Vo2]).

In the case of the non-archimedean fields, such type of approach to parabolically induced representations fails. First of all, we do not know what is $\hat{K}_o$ here. Then, we do not have the action of the Lie algebra. In the non-archimedean case the most powerful tool in the study of the parabolically induced representations are Jacquet modules. Before we introduce them, we shall introduce an algebraic version of the representation theory of reductive groups over non-archimedean fields which is very convenient in the study of the questions related to the problems of the classifications of the unitary duals.

2.2. Remarks. :

(i) In the definition of the parabolically induced representations in the theory of admissible representations, which will be given in the following section, we shall not need integration over K_o. Nevertheless, this type of integration will appear at some other places related to the parabolic induction. Therefore, we shall explain the background of this integration. The parabolic induction extends the notion of the unitary induction. In general, the unitary induction assigns to a unitary representation of a subgroup a unitary representation of the whole group. The first problem that one faces with the unitary induction is that there does not need to exist a nontrivial measure on $P\backslash G$ which is invariant for the right translations by elements of G. But there is another useful form which we shall describe now. Let $C_c(G)$ be the vector space of all compactly supported continuous functions on G. Denote by

X the space of all continuous functions on G which satisfy $f(pg) = \Delta_P(p)f(g)$ for any $p \in P$ and $g \in G$. The mapping

$$q : C_c(G) \to X$$

defined by

$$(qf)(g) = \int_P f(pg)\Delta_P^{-1}(p)dp$$

is surjective (here the measure that we consider on P is a right Haar measure). If $qf_1 = qf_2$, then

$$\int_G f_1(g)dg = \int_G f_2(g)dg.$$

Therefore, we can define integration of elements of X by

(2.2) $$\int_{P\backslash G} (qf)(x)dx = \int_G f(g)dg.$$

In this way we get a positive form on X which is obviously invariant for right translations since q commutes with right translations and the measure on G is invariant for right translations. A positive linear form on X invariant for right translations is unique up to a positive multiple. Sometimes, it is called the Haar measure on $P\backslash G$.

It is possible to build the Haar measure on G from the Haar measures on K_o and P by the formula

$$\int_G f(g)dg = \int_{K_o} \left(\int_P f(pk)\Delta_P(p)^{-1}dp \right) dk$$

i.e.

(2.3) $$\int_G f(g)dg = \int_{K_o} (qf)(k)dk.$$

Because of the definition of the Haar measure on $P\backslash G$, from (2.2) and (2.3) we have

$$\int_{P\backslash G} \varphi(x)dx = \int_{K_o} \varphi(k)dk$$

for $\varphi \in X$. Thus, integration over K_o is simply the Haar measure on $P\backslash G$. For more details and proofs concerning these facts one can consult [Bb3].

Let σ be a unitary representation of M. Take a continuous functions $f_1, f_2 \in \mathbf{Ind}_P^G(\sigma)$. Then the function

$$g \mapsto (f_1(g), f_2(g))$$

belongs to X. Therefore,

$$(f_1, f_2) \mapsto \int_{K_o} (f_1(k), f_2(g))dk$$

defines a G-invariant inner product on $\mathbf{Ind}_P^G(\sigma)$ and because of that, $\mathbf{Ind}_P^G(\sigma)$ is unitary.

(ii) The equivalence that we have mentioned when we were talking about the classification of continuous irreducible representations, is the Naimark equivalence. For a precise definition of this notion one should consult [Wr]. Roughly, two irreducible continuous representations are Naimark equivalent if there is densely defined closed intertwining between them. The important fact is that two irreducible unitary representations which are Naimark equivalent are unitarily equivalent.

3. Admissible representations

Suppose that G is a totally disconnected locally compact group. Then G has a basis of neighborhoods of identity consisting of open compact subgroups.

Let (π, V) be a representation of G. If U is a subspace of V invariant for all $\pi(g), g \in G$, then U is called a **subrepresentation** of V. If V does not contain subrepresentations different from $\{0\}$ and V, then one says that V is an **irreducible representations**.

For two representations (π_1, V_1) and (π_2, V_2) a linear map $\varphi : V_1 \to V_2$ is called **G-intertwining** or **morphism of G-modules** if

$$\varphi\pi_1(g) = \pi_2(g)\varphi$$

for all $g \in G$. Representations π_1 and π_2 are called **isomorphic** or **equivalent** if there exists a G-intertwining φ which is a one-to-one mapping onto.

We may talk of **representations of finite length** in a usual way (at least, representations of G are just modules over suitably defined group algebras).

Let (π, V) be a representation of G. A vector $v \in V$ is called **smooth** if there exists an open subgroup K of G such that $\pi(k)v = v$ for all $k \in K$. The vector subspace of all smooth vectors in V will be denoted by V^∞. Then V^∞ is invariant for the action of G. The representation of G on V^∞ is denoted by π^∞. Then (π^∞, V^∞) is called the **smooth part** of (π, V). A representation (π, V) is called a **smooth representation** if $V = V^\infty$. If K is an open compact subgroup of G, then we denote **K-invariants** by

$$V^K = \{v \in V; \pi(k)v = v \ \text{ for any } \ k \in K\}.$$

It is easy to see that for a compact subgroup K of G

$$(\pi, V) \mapsto V^K$$

is an exact functor from the category of all smooth representations and G-intertwinings into the category of complex vector spaces.

A smooth representation (π, V) of G is called **admissible** if

$$\dim_{\mathbb{C}} V^K < \infty$$

for any open compact subgroup K of G.

An admissible representation (π, V) of G is called **unitarizable** if there exists an inner product $(\ , \ )$ on V such that

$$(\pi(g)v, \pi(g)w) = (v, w)$$

for all $v, w \in V$ and $g \in G$. Each unitarizable representation is completely reducible. We shall see soon that for an irreducible admissible representation (π, V) of G the space of a G-invariant Hermitian forms on V is at most one dimensional.

We shall assume in further that G is one of the groups introduced in the first section and that it is defined over a non-archimedean local field.

Denote by $\tilde{G}$ the set of all equivalence classes of non-zero irreducible admissible representations of G. The set $\tilde{G}$ is called the **non-unitary dual** or the **admissible dual** of G. From the Bernstein's result [Be1] it follows that the mapping

$$(\pi, H) \mapsto (\pi^{\infty}, H^{\infty})$$

is a one-to-one mapping of $\hat{G}$ into $\tilde{G}$ (see Remarks 4.2., (iv)). Moreover, it maps $\hat{G}$ onto the set of all unitarizable classes in $\tilde{G}$. Because of that, we shall in further identify the unitary dual $\hat{G}$ with the subset of all unitarizable classes in $\tilde{G}$. It means that we assume in further

$$\hat{G} \subseteq \tilde{G}.$$

In this way the problem of the classification of the unitary dual of G splits into two parts. The first part is the problem of the classification of the non-unitary dual $\tilde{G}$. The second part is the **unitarizability problem**: determination of the subset $\hat{G}$ of $\tilde{G}$.

We shall say now a few words about representations of certain algebras which are very useful in the study of the smooth representations. Denote by $C_c^{\infty}(G)$ the space of all compactly supported locally constant functions on G. It is an associative algebra for the convolution which is defined by

$$(f_1 * f_2)(x) = \int_G f_1(xg^{-1}) f_2(g) dg.$$

This algebra does not have identity. For an open compact subgroup K of G denote by $C_c(G//K)$ the vector space of all compactly supported functions f which satisfy

$$f(k_1 g k_2) = f(g)$$

for all $k_1, k_2 \in K$ and $g \in G$. Then $C_c(G//K)$ is a subalgebra of $C_c^{\infty}(G)$. Denote by

$$\Xi_K$$

the characteristic function of the set K, divided by the (Haar) measure of K. Then $\Xi_K \in C_c(G//K)$ and it is identity of the algebra $C_c(G//K)$. Moreover

$$C_c(G//K) = \Xi_K * C_c^{\infty}(G) * \Xi_K. \tag{3.1}$$

Note that each $f \in C_c^{\infty}(G)$ is in some $C_c(G//K)$.

Algebra $C_c(G//K)$ is called the **Hecke algebra** of G with respect to K. There are also a more general Hecke algebras (see [HoMo]).

Let (π, V) be a smooth representation of G. Take $f \in C_c^{\infty}(G)$ and $v \in V$. Since the function

$$g \mapsto f(g)\pi(g)v$$

is compactly supported locally constant, we can find open compact subsets $K_1, \cdots, K_n$ such that the above function is constant on each K_i, and that it vanishes outside the union of all K_i's. Set

$$\pi(f)v = \sum_{i=1}^{n} f(g_i)\left(\int_{K_i} dg\right)\pi(g_i)v$$

where g_i is some element of K_i. Then $\pi(f)$ is a linear operator on V and we write

$$\pi(f) = \int_G f(g)\pi(g)dg.$$

The operator $\pi(f)$ is characterized by the condition that for any $v \in V$ and any linear form v^* on V we have

$$v^*(\pi(f)v) = \int_K f(g)v^*(\pi(g)v)dg.$$

In this way one gets a representation of the algebra $C_c^\infty(G)$ on V.

It is easy to get that a subspace $W \subseteq V$ is a G-subrepresentation if and only if it is a $C_c^\infty(G)$-submodule. Also, a linear map φ between two representations of G is a G-intertwining if and only if it is a homomorphism of a $C_c^\infty(G)$-modules. The $C_c^\infty(G)$-modules which are coming from the smooth representations of G are characterized among all $C_c^\infty(G)$-modules by the condition

$$\mathrm{span}_{\mathbb{C}}\ \{fv; f \in C_c^\infty(G), v \in V\} = V. \tag{3.2}$$

The $C_c^\infty(G)$-modules satisfying the above condition are called **non-degenerate $C_c^\infty(G)$-modules**. Suppose that K is an open compact subgroup of G. For a smooth representation (π, V) of G we have

$$V^K = \pi(\Xi_K)V.$$

Moreover,

$$V^K = \pi(C_c(G//K))V.$$

Thus, V^K is a $C_c(G//K)$-module. Since $\pi(\Xi_K)$ is an idempotent, we have

$$V = \mathrm{Im}\,\pi(\Xi_K) \oplus \mathrm{Ker}\,\pi(\Xi_K).$$

Also

$$\mathrm{Ker}\,\pi(\Xi_K) = \mathrm{span}_{\mathbb{C}}\ \{\pi(k)v - v; k \in K, v \in V\}.$$

We shall prove now that if π is irreducible and $V^K \neq \{0\}$, then V^K is an irreducible $C_c(G//K)$-module. Let $\{0\} \neq W \subseteq V^K$ be a $C_c(G//K)$-submodule. Take $v \in V^K$. Let $w \in W$, $w \neq 0$. Since V is irreducible $\mathbb{C}_c^\infty(G)$-module, there exists $f \in C_c(G)$ such that $v = \pi(f)w$. Now

$$v = \pi(\Xi_K)v = \pi(\Xi_K)\pi(f)w = \pi(\Xi_K)\pi(f)\pi(\Xi_K)w = \pi(\Xi_K * f * \Xi_K)w.$$

Since $\Xi_K * f * \Xi_K \in C_c(G//K)$ and W is a $C_c(G//K)$-submodule, we have $v \in W$. Thus $W = V^K$, what proves the irreducibility.

Suppose that (π, V) is an irreducible admissible representation of G. Let φ be in the commutator of the representation π Then $\varphi(V^K) \subseteq V^K$. Since V^K is irreducible $C_c(G//K)$-module, φ is a scalar operator on V^K. From this one gets that φ is a scalar operator on the whole V.

Algebras $C_c(G//K)$ play a very important role in the representation theory of G. It can be easily shown that each irreducible $C_c(G//K)$-module is isomorphic to a $C_c(G//K)$-module V^K for some irreducible smooth representation (π, V) of G (see the Proposition 2.10. of [BeZe1]).

Suppose that $f \in C_c^\infty(G//K)$ acts trivially in every irreducible $C_c(G//K)$-module. Then also $f^* * f$ acts trivially, where $f^*(g) = \overline{f(g^{-1})}$. The theory of non-commutative rings implies that $f * f^*$ is nilpotent. Since $\varphi \in C_c^\infty(G)$ and $\varphi \neq 0$ implies $(\varphi * \varphi^*)(1) \neq 0$, we have that $f = 0$. Thus, every $f \in C_c(G//K)$, $f \neq 0$, acts non-trivially in same irreducible $C_c(G//K)$-module.

If (π, V_1) and (π_2, V_2) are two irreducible smooth representations such that $C_c(G//K)$-modules are non-trivial and isomorphic, then $\pi_1 \cong \pi_2$.

For an admissible representation (π, V) of G, $\pi(f)$ is an operator of finite rank for $f \in C_c^\infty(G)$. Thus

$$f \mapsto \operatorname{Trace} \pi(f)$$

defines a linear form on $C_c^\infty(G)$. This linear form is called the **character of the representation** π. Characters of representations in $\tilde{G}$ are linearly independent. Therefore, if two admissible representations of finite lengths have equal characters, then they have the same Jordan-Hölder series.

The problem of computation of the characters of $\tilde{G}$, in particular of $\hat{G}$, is a very important problem. In general, not too much is known about this problem. We shall not discuss more about this problem in these notes.

Let (π, V) be a smooth representation of G. By $(\overline{\pi}, \overline{V})$ we denote the **complex conjugate representation** of the representation (π, V). It is the representation by the same operators on the same space, except that the vector space structure is conjugate to the previous one. The new multiplication with scalars is given by

$$z \cdot v = \overline{z} v.$$

Clearly, π is irreducible (resp. admissible) if and only if $\overline{\pi}$ is irreducible (resp. admissible).

Denote by V^* the dual space of V. Define a representation π^* on V^* by

$$[\pi^*(g)(v^*)](v) = v^*(\pi(g^{-1})v).$$

Let $(\tilde{\pi}, \tilde{V})$ be the smooth part of (π^*, V^*). Then $(\tilde{\pi}, \tilde{V})$ is called the **contragredient representation** of (π, V). Now

$$(\pi, V) \mapsto (\tilde{\pi}, \tilde{V})$$

becomes a contravariant functor in a natural way on the category of all smooth representations of G and G-intertwinings.

Let K be an open compact subgroup of G. One can see directly that the linear map

$$f \mapsto f \circ \pi(\Xi_K), \tag{3.3}$$

goes from $(V^K)^*$ to $(V^*)^K = (\tilde{V})^K$ and that it is an isomorphism of vector spaces. Since the natural linear map

$$\varphi : V \mapsto \tilde{\tilde{V}}$$

defined by $[\varphi(v)](\tilde{v}) = \tilde{v}(v), \tilde{v} \in \tilde{V}$, is G-intertwining, we have

$$(\pi, V) \cong (\tilde{\tilde{\pi}}, \tilde{\tilde{V}})$$

if (π, V) is an admissible representation of G.

The functor $\pi \mapsto \tilde{\pi}$ is an exact functor on the category of all smooth representations of G.

In general, for a smooth representation (π, V) of G the form

$$(\tilde{v}, v) \mapsto \tilde{v}(v)$$

on $\tilde{V} \times V$ is a non-degenerate bilinear G-invariant form.

Suppose that $W \neq \{0\}$ is a proper subrepresentation of a smooth representation (π, V). Then

$$W^\perp = \left\{\tilde{v} \in \tilde{V}; \tilde{v}(w) = 0 \text{ for any } w \in W\right\}$$

is a proper non-zero subrepresentation of $\tilde{\pi}$. Therefore, for an admissible representation π we have that π is irreducible if and only if $\tilde{\pi}$ is irreducible. Moreover, π is an admissible representation of length n if and only if $\tilde{\pi}$ is an admissible representation of length n.

Suppose that (π, V) is an irreducible unitarizable admissible representation of G. Then

$$v \mapsto \overline{(\cdot, v)}$$

is a non-trivial G-intertwining from V to $\overline{\tilde{V}}$. Thus

$$\pi \cong \overline{\tilde{\pi}}.$$

An irreducible admissible representation (π, V) is called **Hermitian** if $\overline{\tilde{\pi}} \cong \pi$. Thus, every $\pi \in \hat{G}$ is Hermitian.

For each irreducible admissible representation (π, V) of G with a non-trivial G-invariant Hermitian form Ψ, the mapping

$$v \mapsto \overline{\Psi(\cdot, v)}$$

is an isomorphism of V onto $\overline{\tilde{V}}$. Since the commutator of an irreducible representation consists of scalars only, we see that each two G-invariant Hermitian forms on V are proportional. This explains also why a G-invariant inner product on $(\pi, V) \in \tilde{G}$ is unique up to a positive multiple, if it exists.

Let $P = MN$ be a parabolic subgroup of G. Take a smooth representation (σ, U) of M. Let $\underline{\mathrm{Ind}}_P^G(\sigma)$ be the space of all functions

$$f \colon G \to U$$

such that

$$f(nmg) = \Delta_P(m)^{1/2}\sigma(m)f(g)$$

for all $m \in M$, $n \in N$, $g \in G$. Define the action R of G on $\underline{\mathrm{Ind}}_P^G(\sigma)$ by the right translations

$$(R_g f)(x) = f(xg),$$

where $x, g \in G$. The smooth part of this representation is denoted by

$$(R, \mathrm{Ind}_P^G(\sigma)).$$

The representation $\mathrm{Ind}_P^G(\sigma)$ is called a **parabolically induced representation** of G from P by σ. It is easy to see that for a continuous representation (σ, H) of M we have

$$\Big(\mathbf{Ind}_P^G(\sigma)\Big)^{\infty} \cong \mathrm{Ind}_P^G(\sigma^{\infty}).$$

From this we see that there is a natural relationship between the two parabolic inductions that we have introduced. Namely, the parabolic induction that we have just introduced is just an algebraic version of the parabolic induction that we have introduced in the second section.

Suppose that K is an open compact subgroup of G. Then $f \in (\mathrm{Ind}_P^G(\sigma))^K$ is completely determined by values on any set of representatives for $P\backslash G/K$. A consequence of the Iwasawa decomposition is that $P\backslash G$ is compact. Since K is open, $P\backslash G/K$ is a finite set. If σ is admissible, then the values of $f \in (\mathrm{Ind}_P^G(\sigma))^K$ are contained in a certain finite dimensional subspaces of invariants in U. This implies that $\mathrm{Ind}_P^G(\sigma)$ is admissible if σ is admissible. Moreover, if σ is an admissible representation of finite length, then $\mathrm{Ind}_P^G(\sigma)$ has finite length (for additional comments about this fact see the section eight).

Suppose that

$$\varphi : U_1 \to U_2$$

is an M-intertwining between smooth M-representations σ_1 and σ_2. Define

$$\mathrm{Ind}_P^G(\varphi) : \mathrm{Ind}_P^G(\sigma_1) \to \mathrm{Ind}_P^G(\sigma_2),$$

by the formula

$$f \mapsto \varphi \circ f.$$

It is easy to see that $\mathrm{Ind}_P^G(\varphi)$ is G-intertwining. In this way Ind_P^G becomes a functor from the category of all smooth representations of M to the category of all smooth representations of G. Considering a description of K-invariants in induced representations, it is easy to see that Ind_P^G is an exact functor. Further, if $\varphi \neq 0$ then $\mathrm{Ind}_P^G(\varphi) \neq 0$. Moreover, if

$\mathrm{Ind}_P^G(\varphi)$ is a one-to-one mapping (resp. mapping onto), then φ is also one-to-one mapping (resp. mapping onto).

Let (σ, U) be a smooth representation of M. Let $f \in \mathrm{Ind}_P^G(\sigma)$ and $\tilde{f} \in \mathrm{Ind}_P^G(\tilde{\sigma})$. One directly checks that the function

$$g \mapsto [\tilde{f}(g)](f(g))$$

belongs to the space X introduced in Remarks 2.2, (i). Therefore the formula

$$\Psi(\tilde{f}, f) = \int_{K_o} [\tilde{f}(k)](f(k))dk$$

defines a G-invariant bilinear form on $\mathrm{Ind}_P^G(\tilde{\sigma}) \times \mathrm{Ind}_P^G(\sigma)$. A direct consequence is that the mapping

$$\tilde{f} \mapsto \Psi(\tilde{f}, \cdot)$$

defines a G-intertwining

$$\mathrm{Ind}_P^G(\tilde{\sigma}) \to (\mathrm{Ind}_P^G(\sigma))\tilde{}.$$

It is not hard to show that the above intertwining is an isomorphism. Thus

$$(\mathrm{Ind}_P^G(\sigma))\tilde{} \cong \mathrm{Ind}_P^G(\tilde{\sigma}).$$

In the same way one gets that $\mathrm{Ind}_P^G(\sigma)$ is unitarizable if σ is unitarizable (see Remarks 2.2., (i)). These two facts are the reason why $\Delta_P^{1/2}$ appears in the definition of the induced representations. Clearly

$$(\mathrm{Ind}_P^G(\sigma))^- \cong \mathrm{Ind}_P^G(\overline{\sigma}).$$

Suppose that one has a parabolic subgroup $P = MN$ in G and a smooth representation (σ, U) of M. Let $g \in G$. Let σ' be an admissible representation of gMg^{-1} given by

$$\sigma'(gmg^{-1}) = \sigma(m) \ , \qquad m \in M.$$

Then it is easy to see that representations $\mathrm{Ind}_P^G(\sigma)$ and $\mathrm{Ind}_{gPg^{-1}}^G(\sigma')$ are equivalent. We shall say that pairs (P, σ) and (gPg^{-1}, σ') are **conjugate**.

Let P be a standard parabolic subgroup of G with the standard Levi decomposition $P = MN$. Suppose that P' is a standard Levi subgroup of M with the standard Levi decomposition $P' = M'N'$. Let σ be an admissible representation of M'. There exists a standard parabolic subgroup P'' in G whose standard Levi factor is M'. Then it is not hard to prove that

$$\mathrm{Ind}_{P''}^G(\sigma) \cong \mathrm{Ind}_P^G(\mathrm{Ind}_{P'}^M(\sigma)).$$

We may say that the parabolic induction does not depend on the stages of induction. This property will be illustrated on examples in the ninth and tenth sections.

Suppose that $P_1 = M_1N_1$ and $P_2 = M_2N_2$ are parabolic subgroups in G. Suppose that $M_1 = M_2$ and that σ is an admissible representation of M_1 of finite length. Then representations $\mathrm{Ind}_{P_1}^G(\sigma)$ and $\mathrm{Ind}_{P_2}^G(\sigma)$ have the same Jordan-Hölder sequences ([BeDeKz]).

This fact is much harder to prove than the previous one. It follows from the equality of the characters of two induced representations. We shall prove this fact for $SL(2,F)$ in the seventh section. If additionally (P',σ') (resp. P') is conjugated to (P_2,σ) (resp. P_2), then we say that (P_1,σ) and (P',σ') (resp. P_1 and P') are **associate**. Therefore, the parabolic induction from associate pairs gives the same Jordan-Hölder series.

3.1. Remarks.

(i) An interesting question may be what are the irreducible smooth representations of G (without the assumption of the admissibility). The answer is very simple: each irreducible smooth representation is admissible. This fact, which was first proved by H. Jacquet as far as this author knows, is also important in the proof of the Bernstein's result that each irreducible unitary representation has finite K_o-multiplicities. For more explanations regarding these topics one should consult Remarks 4.2., (ii), in the following section.

(ii) One could prove also directly (without use of the Hecke algebras $C_c(G//K)$) that the commutator of an irreducible smooth representation of G consists only of the scalars operators. It follows directly from the fact from the linear algebra that the commutator of an irreducible family of linear operators on a countable dimensional vector space V over $\mathbb{C}$ consists of scalar operators. Let us outline the proof. Suppose that C is the commutator of such a family. Since kernels and images are invariant subspaces, C is a division ring. Suppose, that $L \in C$ is not scalar. Then $L-\lambda\cdot\mathrm{id}_V \in C$ and it is different from 0. Therefore

$$P(L)v \neq 0$$

for any polynomial $P \neq 0$ and any $v \neq 0$. Fix $v \neq 0$. Since

$$(L-\lambda\,\mathrm{id}_V)^{-1}v, \quad \lambda \in \mathbb{C},$$

is a linearly dependent set, there exist $\lambda_1,\cdots,\lambda_k \in \mathbb{C}$, mutually different, and $\mu_1,\ldots,\mu_k \in \mathbb{C}$ which are not all equal to 0, such that

$$\sum_{i=1}^{k} \mu_i (L-\lambda_i\,\mathrm{id}_V)^{-1}v = 0.$$

Acting on the last relation by

$$\prod_{i=1}^{k} (L-\lambda_i\,\mathrm{id}_V),$$

one gets that $P(L)v = 0$ for a polynomial $P \neq 0$, what is a contradiction. This proves that only the scalar operators can be in the commutator of an irreducible smooth representation of G.

4. Jacquet modules and cuspidal representations

We shall now define a left adjoint functor to the functor Ind_P^G. Let (π, V) be a smooth representation of G and let $P = MN$ be a parabolic subgroup of G. Set

$$V(N) = \mathrm{span}_{\mathbb{C}} \{\pi(n)v - v; n \in N, v \in V\}.$$

The group N has the property that it is the union of its open compact subgroups. In the case of $SL(2, F)$ or $GL(2, F)$, and a proper parabolic subgroup $P = MN$, we have $N \cong F$. Therefore the above property is evident in these cases. The above property of N has for a consequence that

$$V(N) = \bigcup \mathrm{Ker}\, \pi\left(\Xi_{N_o}\right) \tag{4.1}$$

when N_o runs over all open compact subgroups of N. In the above formula we consider π also as a representation of N only. Therefore, $\pi\left(\Xi_{N_o}\right)$ is well defined. Note that we have also

$$V(gNg^{-1}) = \pi(g)V(N)$$

for $g \in G$. Since M normalizes N, $V(N)$ is invariant for the action of M. Set

$$V_N = V/V(N).$$

We consider the natural quotient action of M on V_N

$$\pi_N(m)(v + V(N)) = \pi(m)v + V(N).$$

The M-representation (π_N, V_N) is called the **Jacquet module** of (π, V) with respect to $P = MN$. It is easy to see that

$$(\pi, V) \mapsto (\pi_N, V_N)$$

is a functor from the category of smooth G-representations to the category of smooth M-representations. It is not hard to show that this functor is exact.

One could consider the Jacquet functor as a functor from the category of smooth P-representations to the category of smooth M-representations. In this setting, it is also an exact functor.

If (π, V) is a finitely generated smooth G-representation, then one can prove directly that $(\pi|P, V)$ is a finitely generated P-representation . This is a consequence of the compactness of $P\backslash G$ and the smoothness of the action of G. By the above observation, the Jacquet functor carries finite generated G-representations to a finite generated M-representations.

A little bit more work is required to prove that the Jacquet functor carries the admissible representations to the admissible ones. Moreover, one has that the Jacquet functor carries the admissible representations of finite length again to the representation of finite length (see (8.4)).

Let π be a smooth representation of G and let σ be a smooth representation of M. Then we have a canonical isomorphism

$$\operatorname{Hom}_G(\pi, \operatorname{Ind}_P^G(\sigma)) \cong \operatorname{Hom}_M(\pi_N, \Delta_P^{1/2}\sigma).$$

This isomorphism is called the **Frobenius reciprocity**. Let us explain how one gets it. Denote by

$$\Lambda : \operatorname{Ind}_P^G(\sigma) \to \Delta_P^{1/2}\sigma$$

the mapping

$$f \to f(1).$$

Then composition with Λ gives

$$\operatorname{Hom}_G(\pi, \operatorname{Ind}_P^G(\sigma)) \cong \operatorname{Hom}_P(\pi, \Delta_P^{1/2}\sigma).$$

Since N acts trivially on σ, one gets $\operatorname{Hom}_P(\pi, \Delta_P^{1/2}\sigma) \cong \operatorname{Hom}_M(\pi_N, \Delta_P^{1/2}\sigma)$.

Suppose that we have a standard Levi subgroup P in G, the standard Levi decomposition $P = MN$ of P, a standard Levi subgroup P' of M with the standard Levi decomposition $P' = M'N'$. Let P'' be the standard parabolic subgroup of G which has M' for the standard Levi factor. Let $P'' = M'N''$ be the standard Levi decomposition of P''. Suppose that π is a smooth representation of G. Then we have the following transitivity of Jacquet modules

$$\pi_{N''} \cong (\pi_N)_{N'}.$$

In a number of applications it is more convenient to work with normalized Jacquet modules. Denote by

$$\left(r_M^G(\pi), r_M^G(V)\right)$$

the representation $\Delta_P^{-1/2}\pi_N$ on V_N. The representation $(r_M^G(\pi), r_M^G(V))$ is called the **normalized Jacquet module** of (π, V) with respect to $P = MN$. Now the Frobenius reciprocity becomes

$$\operatorname{Hom}_G(\pi, \operatorname{Ind}_P^G(\sigma)) \cong \operatorname{Hom}_M(r_M^G(\pi), \sigma).$$

Normalized Jacquet modules have again the above transitivity property.

The Frobenius reciprocity indicates how interesting is to understand the Jacquet modules. But this is only one of the very important information that are contained in the Jacquet module. Very soon we shall see some of the others.

An admissible representation (π, V) of G is called **cuspidal** (or **supercuspidal**, or **absolutely cuspidal** by some authors) if for any proper parabolic subgroup $P = MN$ of G and for any smooth representation σ of M we have

$$\operatorname{Hom}_G(\pi, \operatorname{Ind}_P^G(\sigma)) = 0.$$

By the Frobenius reciprocity (π, V) is cuspidal if and only if

$$V_N = 0$$

for any proper parabolic subgroup $P = MN$ of G. Because of the transitivity of the Jacquet modules, it is enough to prove for cuspidality that V_N=0 for all maximal proper parabolic subgroups. If π is irreducible,then the cuspidality of π is equivalent to the following fact: π is not equivalent to a subrepresentation of $\mathrm{Ind}_P^G(\sigma)$ for any proper parabolic subgroup $P = MN$ and any smooth representation σ of M.

Suppose that $(\pi, V) \in \tilde{G}$. Then there exists a parabolic subgroup $P = MN$ of G such that $r_M^G(\pi) \neq 0$. The case of $P = G$ is not excluded. Choose a minimal P with the property that $r_M^G(\pi) \neq 0$. Then the transitivity of the Jacquet modules implies that $r_{M'}^M(r_M^G(\pi)) = 0$ for any proper parabolic subgroup $P' = M'N'$ of M. Since $r_M^G(\pi)$ is finitely generated, it has an irreducible quotient, say σ. Since $r_M^G(\pi)$ is admissible, σ is admissible. The exactness of the Jacquet functor implies that σ is a cuspidal representation of M. Now from the Frobenius reciprocity one obtains that for irreducible admissible representation π of G there exist a parabolic subgroup $P = MN$ of G and an irreducible cuspidal representation σ of M such that π is equivalent to a subrepresentation of $\mathrm{Ind}_P^G(\sigma)$.

Let (π, V) be a smooth representation of G. A character ω of the center $Z(G)$ of G is called a central character of V if

$$\pi(z) = \omega(z)\,\mathrm{id}_V$$

for all $z \in Z(G)$. The central character of π, if it exists, is denoted ω_π. We have seen that each irreducible admissible representation of G has a central character.

Suppose that π is a cuspidal representation of G which has a central character, say ω. Then π is a projective object in the category of all smooth representations which have the central character equal to ω (see Remarks 4.2., (i) in the end of this section). Using the contragredient functor one gets that π is also an injective object in the same category. These facts imply that $\mathrm{Ind}_P^G(\sigma)$ does not have cuspidal subquotients if P is a proper parabolic subgroup.

These facts about projectivety and injectivety of cuspidal representations imply directly the following fact. Let (π, V) be a smooth representation of G. Assume that the center of G is compact (then it must be finite in our case). Then there exists a decomposition of $V = V_c \oplus V_n$ as a representation of G such that each irreducible subquotient of V_c is cuspidal while no one irreducible subquotient of V_n is cuspidal. Such decomposition is unique. A little additional analysis gives that the above result holds without the assumption of the compactness of the center of G. This decomposition is just one of many decompositions which may be obtained using [BeDe] (see also [Td10]).

Let (π, V) be a smooth representation of G. Take $v \in V$ and $\tilde{v} \in \tilde{V}$. The function

$$c_{v,\tilde{v}} : g \mapsto \tilde{v}(\pi(g)v)$$

is called a matrix coefficient of G. There is a nice description of the cuspidal representations of G in terms of the matrix coefficients of G ([Jc1],[Cs]):

4.1.Theorem. : *An admissible representation of G is cuspidal if and only if all matrix coefficients are compactly supported functions on G modulo the center (i.e. for each matrix coefficient c there exists a compact subset X of G such that the support of c is contained in $XZ(G)$.*

We may say that a vanishing of the Jacquet modules forces a vanishing of the matrix coefficients. This holds without assumption of admissibility. Thus, each smooth representation whose Jacquet modules for all proper parabolic subgroups are trivial, has matrix coefficients compactly supported modulo the center.

To give an idea of the relationship between the Jacquet modules and the matrix coefficients we shall prove in the case of $SL(2,F)$ that a vanishing of the Jacquet modules implies a compactness of the supports of the matrix coefficients (the proof for a general reductive group G is the same, modulo the structure of the group). We shall first fix some notation for $SL(2,F)$.

Let

$$P = P_{\min} \tag{4.2}$$

be the parabolic subgroup of all upper triangular matrices in $SL(2,F)$. Let

$$M = M_{\min} = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} ; a \in F^{\times} \right\}. \tag{4.3}$$

We shall often identify M with $F^{\times}$ using the isomorphism

$$a \mapsto \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}.$$

Denote

$$N_{\min} = N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} ; x \in F \right\}.$$

Set $K_o = SL(2, \mathcal{O}_F)$. Then we have the **Cartan decomposition** for $SL(2,F)$

$$SL(2,F) = K_o A^- K_o,$$

where

$$A^- = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} ; a \in F \quad \text{and} \quad |a|_F \leqslant 1 \right\}.$$

We shall also use the following notation in the further calculations:

$$\mathbf{d}(a) = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix},$$

for $a \in F^{\times}$.

We can now present the proof of the above implication. Let (π, V) be a smooth representation of $SL(2,F)$ such that $V_N = 0$. Take $v \in V$ and $\tilde{v} \in \tilde{V}$. Since $\pi(K_o)v$ is a finite subset of $V(N)=V$, (4.1) implies that there exists an open compact subgroup N_1 of N such that $\pi(K_o)v \subseteq \mathrm{Ker}\pi(\Xi_{N_1})$. Take an open compact subgroup N_2 of N such that $\tilde{\pi}(K_o)\tilde{v} \subseteq \tilde{V}^{N_2}$. One can prove directly that there exists $t>0$ such that if $a \in F$ and $|a|<t$, then $\mathbf{d}(a)N_1\mathbf{d}(a)^{-1} \subseteq N_2$. Take now $k_1, k_2 \in K_o$ and $a \in F$ such that $|a|_F < t$. In the following calculations we shall write $x \backsim y$ for $x, y \in \mathbb{C}$, if there exists $z \in \mathbb{C}^\times$ such that $x = zy$. A simple calculation gives

$$
\begin{gathered}
c_{v,\tilde{v}}\left(k_1\mathbf{d}(a)k_2\right) = \tilde{v}\left(\pi\left(k_1\mathbf{d}(a)k_2\right)v\right) = \\
(\tilde{\pi}(k_1^{-1})\tilde{v})\left(\pi(\mathbf{d}(a)k_2)v\right) = \\
\left(\tilde{\pi}\left(\Xi_{N_2}\right)\tilde{\pi}(k_1^{-1})\tilde{v}\right)\left(\pi(\mathbf{d}(a)k_2)v\right) \backsim \\
\left(\left(\int_{N_2}\tilde{\pi}(n)dn\right)\tilde{\pi}(k_1^{-1})\tilde{v}\right)\left(\pi(\mathbf{d}(a)k_2)v\right) = \\
\int_{N_2}\left(\tilde{\pi}(n)\tilde{\pi}(k_1^{-1})\tilde{v}\right)\left(\pi(\mathbf{d}(a)k_2)v\right)dn = \\
\int_{N_2}(\tilde{\pi}(k_1^{-1})\tilde{v})\left(\pi(n^{-1})\pi(\mathbf{d}(a)k_2)v\right)dn = \\
\int_{N_2}(\tilde{\pi}(k_1^{-1})\tilde{v})\left(\pi(n)\pi(\mathbf{d}(a)k_2)v\right)dn = \\
(\tilde{\pi}(k_1^{-1})\tilde{v})\left(\left(\int_{N_2}\pi(n)dn\right)\pi(\mathbf{d}(a)k_2)v\right) = \\
(\tilde{\pi}(k_1^{-1})\tilde{v})\left(\pi(\mathbf{d}(a))\left(\int_{N_2}\pi\left(\mathbf{d}(a)^{-1}n\mathbf{d}(a)\right)dn\right)\pi(k_2)v\right) \backsim \\
(\tilde{\pi}(k_1^{-1})\tilde{v})\left(\pi(\mathbf{d}(a))\left(\int_{\mathbf{d}(a^{-1})N_2\mathbf{d}(a)}\pi(n)dn\right)\pi(k_2)v\right) \backsim \\
(\tilde{\pi}(k_1^{-1})\tilde{v})\left(\pi(\mathbf{d}(a))\pi\left(\Xi_{\mathbf{d}(a^{-1})N_2\mathbf{d}(a)}\right)\pi(k_2)v\right).
\end{gathered}
$$

A direct calculation shows that $N_1 \subseteq \mathbf{d}(a)^{-1}N_2\mathbf{d}(a)$ implies

$$
\Xi_{\mathbf{d}(a^{-1})N_2\mathbf{d}(a)} * \Xi_{N_1} = \Xi_{\mathbf{d}(a^{-1})N_2\mathbf{d}(a)}.
$$

Since $\pi(k_2)v \in \mathrm{Ker}\,\pi(\Xi_{N_1})$ we get $c_{v,\tilde{v}}(k_1\mathbf{d}(a)k_2)=0$. By the Cartan decomposition the support of $c_{v,\tilde{v}}$ is contained in

$$
K_o\{\mathbf{d}(a); a \in F \ \text{ and } \ t \leqslant |a|_F \leqslant 1\}K_o,
$$

which is a compact set. This finishes the proof of the implication.

The proof of the other implication in the theorem is more technical.

There exists a strong connection between the asymptotic properties of the matrix coefficients and the Jacquet modules. A nice elaboration of that connection can be found in the fourth section of [Cs1]. An application of this connection to the square integrable representations will be given now. Let us first define the square integrable representations.

Suppose that (π, V) is an admissible representation of G which has a unitary central character. Then the absolute value of each matrix coefficient

$$|c_{v,\tilde{v}}| \ : g \mapsto |c_{v,\tilde{v}}(g)|$$

is a function on $G/Z(G)$. The representation π is called **square integrable** if all functions $|c_{v,\tilde{v}}|$ are square integrable functions on $G/Z(G)$. If for an admissible representation τ of G there exists a character χ of G such that the representation

$$\chi\tau : g \mapsto \chi(g)\tau(g)$$

is square integrable, then τ will be called an **essentially square integrable representation**. It is easy to see that each irreducible cuspidal representation is an essentially square integrable representation.

Suppose that (π, V) is an irreducible square integrable representation of G. Take $\tilde{v}_o \in \tilde{V}, \quad \tilde{v}_o \neq 0$. For $u, v \in V$ define

$$(u, v) = \int_{G/Z(G)} \tilde{v}_o(\pi(g)u)\overline{\tilde{v}_o(\pi(g)v)}dg.$$

This defines a G-invariant inner product on V. Thus, each irreducible square integrable representation is unitarizable.

There is a very useful criterion for the square integrability ([Cs1]). Let us explain it in the case of $SL(2, F)$. Take $\pi \in SL(2, F)\check{}$. Then $r_M^{SL(2,F)}(\pi)$ is a finite dimensional representation. Irreducible subquotients of $r_M^{SL(2,F)}(\pi)$ are characters of $M = M_{\min}$. Let p be a generator of the unique maximal ideal $\mathfrak{p}_F = \{x \in F : |x|_F < 1\}$ in the ring of integers $\mathcal{O}_F = \{x \in F; |x|_F \leqslant 1\}$ in F. Then π is square integrable if and only if for each irreducible subquotient χ of $r_M^{SL(2,F)}(\pi)$ we have

$$\text{(SI)} \qquad \left|\chi\left(\begin{bmatrix} p & 0 \\ 0 & p^{-1} \end{bmatrix}\right)\right|_F < 1.$$

An irreducible admissible representation π of G will be called an irreducible tempered representation of G, if there exist a parabolic subgroup $P = MN$ and a square integrable representation δ of M such that π is equivalent to a subrepresentation of $\mathrm{Ind}_P^G(\delta)$. The Langlands classification ([BlWh], [Si1]) reduces parametrization of $\tilde{G}$ to the classification of the tempered representations of the standard Levi factors of the standard parabolic subgroups. More precisely, for each standard parabolic subgroup P with the standard Levi decomposition $P = MN$, each irreducible tempered representation τ of M, and each positive valued character χ of M satisfying certain "positiveness condition", the representation $\mathrm{Ind}_P^G(\chi\sigma)$ has a unique irreducible quotient, say $L(\chi\sigma)$. If $L(\chi\sigma) = L(\chi'\sigma')$, then $P = P', \sigma \cong \sigma'$ and $\chi = \chi'$. Also, each irreducible admissible representation is equivalent to some $L(\chi\sigma)$, for some P, σ and χ as above. The "positiveness condition" will be described explicitly for the general linear groups in the ninth section.

4.2. Remarks.

(i) Suppose that G has a compact center (then the center is finite in our case). Introduction of this condition is done only in order to avoid dealing with the central characters. As it is well known, an irreducible unitary representation π of G is square integrable if and only if π is unitarily equivalent to a subrepresentation of $L^2(G)$, where G acts by right translations. Suppose that (π, V) is an irreducible cuspidal representation of G. Choose any $\tilde{v} \in \tilde{V}, \tilde{v} \neq 0$. Then

$$v \to c_{v,\tilde{v}}$$

defines a non trivial G-intertwining of V into $C_c^\infty(G)$, where G acts an $C_c^\infty(G)$ by right translations. Also, each irreducible subrepresentation of $C_c^\infty(G)$ is cuspidal (this follows from the following remark and the Theorem 4.1.). Therefore, irreducible cuspidal representations of G are exactly the representations which are equivalent to irreducible subrepresentations of $C_c^\infty(G)$. Therefore, we may say that irreducible cuspidal representations are exactly the representations which appear discretely in $C_c^\infty(G)$. Roughly, this fact explains why irreducible cuspidal representations are projective objects.

(ii) We have noted that each irreducible admissible representation π of G is equivalent to a subrepresentation of $\mathrm{Ind}_P^G(\sigma)$, where σ is an irreducible cuspidal representation of M. In the same way it follows that each irreducible smooth representation π of G is equivalent to a subrepresentation of $\mathrm{Ind}_P^G(\sigma)$ where (σ, U) is an irreducible smooth representations of M such that $\sigma_{N'} = 0$ for any proper parabolic subgroup $P' = M'N'$ of M. To prove that π is admissible, it is enough to prove that σ is admissible. Note that by a previous remark σ has matrix coefficients compactly supported modulo the center $Z(M)$. Suppose that σ is not admissible. Chose an open compact subgroup K of M such that U^K is not finite dimensional (certainly, it is of countable dimension). Take $u \in U^K,\ u \neq 0$. Then $\sigma(m)u, m \in M$ generates U. Thus $\sigma(\Xi_K)\sigma(m)u, m \in M$ generates U^K. Choose a sequence (m_k) in M such that the sequence $\sigma(\Xi_K)\sigma(m_k)u$ form a basis of U^K when k runs over positive integers. Choose $\tilde{u}' \in (U^K)^*$ such that $\tilde{u}'(\sigma(\Xi_K)\sigma(m_k)u) \neq 0$ for all $k \geqslant 1$. Then $\tilde{u} = \tilde{u}' \circ \sigma(\Xi_K) \in \tilde{U}^K$ by (3.3) and further

$$0 \neq \tilde{u}'(\sigma(\Xi_K)\sigma(m_k)u) = \tilde{u}(\sigma(\Xi_K)\sigma(m_k)u) =$$

$$(\tilde{\sigma}(\Xi_K)\tilde{u})(\sigma(m_k)u) = \tilde{u}(\sigma(m_k)u) = c_{u,\tilde{u}}(m_k).$$

Now using Remarks 3.1., (ii), one obtains

$$\bigcup_{k=1}^{\infty} Z(G)Km_kK \subseteq \mathrm{supp}\ c_{u,\tilde{u}}.$$

Note that $Z(G)Km_{k_1}K$ and $Z(G)Km_{k_2}K$ are disjoint by the same remark for $k_1 \neq k_2$ because the elements $\sigma(\Xi_K)\sigma(m_k)u$, $k \geq 1$, form a basis of U^K. Therefore, the support of $c_{u,\tilde{u}}$ is not compact modulo the center. This contradiction proves the admissibility of σ, and further, the admissibility of π. So, each irreducible smooth representation of G is admissible.

(iii) Let us now sketch the proof of the Bernstein's result that for any open compact subgroup K of G, the spaces of the K-invariants are finite dimensional in the topologically irreducible unitary representations of G (in Hilbert spaces). From the second section and the previous remark it follows that irreducible $C_c(G//K)$-modules are finite dimensional. The following crucial step in the proof of the Bernstein's result is to show that dimensions of irreducible $C_c(G//K)$-modules are bounded. Note that for a proof of this, it is enough to prove such statement for some open subgroup K' of K.

For a positive integer k set

$$K'_k = \{g \in GL(n, F); g \equiv I_n (\text{mod}\, \mathfrak{p}_F^k)\}.$$

It is possible to embed G in some $GL(n, F)$ in a such way that the following property holds, for any n. Set $K_k = K'_k \cap G$. Then the algebra $C_c(G//K_k)$ satisfies the following condition. There exist $f_1, \ldots, f_q \in C_c(G//K_k)$ and a commutative finitely generated subalgebra A of $C_c(G//K_k)$ such that

$$C_c(G//K_k) = \sum_{i,j=1}^{q} f_i * A * f_j. \tag{4.4}$$

To simplify the notation, we shall denote K_k by K in further. One gets now that the dimensions of the irreducible $C_c(G//K)$-modules are bounded from the following interesting lemma from the linear algebra: there exists a function $g \mapsto p(g)$ from the set of positive integers into the strictly positive real numbers such that if $\mathcal{A}$ is a commutative subalgebra of the algebra of all endomorphisms of an m-dimensional complex vector space generated by g generators, then

$$\dim_{\mathbb{C}} \mathcal{A} \leqslant m^{2-p(g)}$$

(Lemma 4.10. of [BeZe1]). Note that for $g = 1$ one can take $p(1) = 1$. This follows directly from the Hamilton-Cayley Theorem. Thus, in general $0 < p(g) \leqslant 1$. Let g be the cardinality of some generating set of the algebra A in (4.4) which is finite. Suppose that (τ, W) is an irreducible $(C_c(G//K)$-module. Then

$$\tau(C_c(G//K)) = \operatorname{End}_{\mathbb{C}} W.$$

Together with (4.4), this implies

$$(\dim_{\mathbb{C}} W)^2 \leqslant q^2 (\dim_{\mathbb{C}} W)^{2-p(g)}.$$

Thus

$$\dim_{\mathbb{C}} W \leqslant q^{2/p(g)}.$$

An algebra R is called n-commutative if

$$\sum (-1)^{p(\sigma)} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)} = 0,$$

for any $x_1, \cdots, x_n \in R$, where the sum runs over all the permutations σ of the set $\{1, 2, \cdots, n\}$ ($p(\sigma)$ denotes the parity of a permutation σ). It is easy to see that the algebra of $n \times n$ matrices is (n^2+1)-commutative, but it is not $(n-1)$-commutative. Since each $f \in C_c(G//K)$ acts non-trivially in some irreducible $C_c(G//K)$-module, $C_c(G//K)$ is (c_K^2+1)-commutative.

Suppose that (π, H) is an irreducible unitary representation of G. Then

$$f \mapsto \pi(f) = \int_G f(g)\pi(g)dg$$

defines a topologically irreducible *-representation of $C_c(G//K)$ on H^K, where $C_c(G//K)$ is a *-algebra for the involution $f^*(g) = \overline{f(g^{-1})}$. Now there is a standard strategy to see that the dimension of H^K is less than or equal to any uniform bound of dimensions of irreducible representations of $C_c(G//K)$. Roughly, $\pi(C_c(G//K))$ must be dense in the space of all bounded linear operators on H^K with respect to the strong operator topology. This follows from the von Neumann density theorem (which may be viewed as a topological version of the Jacobsen density theorem). This implies that the algebra of all bounded linear operators on H^K is n-commutative for some n. This implies that H^K is finite dimensional.

(iv) Suppose that (π, H) is an irreducible unitary representation of G. For an open compact subgroup K of G, the space H^K is a finite dimensional topologically irreducible $C_c(G//K)$-module. This implies that H^K is (algebraically) irreducible. Since the smooth part H^∞ is the union of all such spaces of invariants, H^∞ is irreducible $C_c^\infty(G)$-module. Thus $\pi^\infty \in \tilde{G}$. It is easy to see that H^∞ is dense in H.

(v) Let us try to explain why parabolic induction is so interesting in the constructions of elements of $\hat{G}$, or more generally, of $\tilde{G}$. The previous observations about the finiteness of the dimensions of the spaces of the invariants of the irreducible unitary representations tell us that we are interested in inductions that produce admissible representations (smooth representation which are not admissible are never of finite length). If one induces from a "too small" algebraic subgroup of G, one gets very big and highly reducible representations. One way to provide that algebraic subgroup Q is big in G, is to ask that G/Q is a projective variety. Actually, this is the general definition of parabolic subgroups.

If one induces with a smooth representation σ of a parabolic subgroup $P = MN$ which is not admissible, then the induced representation is never admissible (σ does not need to be trivial on N in this considerations). One can easily show that an admissible representation σ of P must be trivial on N, i.e. σ is essentially the representation of M. This is particularly simple to prove for $G = SL(2, F)$. This explain why it is natural to consider parabolic induction.

Parabolic subgroups are cocompact in the group G. There exist also other "big" subgroups. For example, open compact subgroups are in a certain way also big. Namely, their interior is non-empty (this was not the case for the proper parabolic subgroups). They are not algebraic subgroups. The induction from such subgroups may also produce admissible representations. In the case of non-compact center, one induces from compact modulo center subgroups. Let us suppose for the simplicity that the center is compact. Then an induced admissible representation is unitarizable with a natural inner product and matrix coefficients are compactly supported. Thus, in this way one gets cuspidal representations of G. One such example is outlined in the sixth section.

The proof that multiplicities of irreducible representations of the maximal compact subgroups in the irreducible unitary representations of G are finite, is a nice example of application of non-unitary representations in the study of the unitary ones.

5. Composition series of induced representations of $SL(2,F)$ and $GL(2,F)$

The considerations of the previous section imply that in the classification of $\tilde{G}$ (and further, of $\hat{G}$), one should classify the irreducible cuspidal representations of the standard Levi factors and then one should classify irreducible subrepresentations of parabolically induced representations by cuspidal ones. Having in mind Langlands classification, it is crucial to have methods for analysis of induced representations $\mathrm{Ind}_P^G(\sigma)$ not only when σ is cuspidal.

In rest of the paper we shall present some methods of the analysis of the parabolically induced representations. Crucial tool in this analysis are Jacquet modules. In general, it is hard to give explicitly Jacquet modules of parabolically induced representations. But there is a result of W. Casselman ([Cs1]), and also of J. Bernstein and A.V. Zelevinsky ([BeZe1]) which enables one to compute subquotients of some filtrations of Jacquet modules of parabolically induced representations. This result was also obtained by Harish-Chandra ([Si2]). We shall explain this result on two simple examples. In the calculations in this section we shall follow mainly [Cs1]. Note that Jacquet modules were already helpful in the proof of the Bernstein's result about finiteness of K_o-multiplicities in irreducible unitary representations of G.

Let X be a totally disconnected locally compact topological space. The space of all locally constant compactly supported functions on X is denoted by $C_c^\infty(X)$. This space has very often the role played by the space of all compactly supported C^∞-functions on a real manifold. But there are also some essential differences. The following example illustrates it.

Let Y be a closed subset of X. It is easy to see that the sequence

$$0 \to C_c^\infty(X \backslash Y) \hookrightarrow C_c^\infty(X) \xrightarrow{\text{restrict.}} C_c^\infty(Y) \tag{5.1}$$

is exact. Clearly, this does not hold in general for a submanifold of a real manifold.

Such type of exactness arises also in the setting of representations of reductive groups over local non-archimedean fields. We shall explain now the exactness at this setting. This exactness enables computation of subquotients of some filtrations of the Jacquet modules of parabolically induced representations.

We shall consider one of the lowest dimensional non-trivial cases, the case of $SL(2,F)$. We have already fixed subgroups $P = P_{\min}$, $M = M_{\min}$ and $N = N_{\min}$ in $SL(2,F)$ (see the preceding section). We shall denote $G = SL(2,F)$ in further. Let

$$w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then

$$G = P \cup PwP.$$

This is called the **Bruhat decomposition** for $SL(2,F)$.

Let χ be a character of M. Let X be a subset of G such that $PX = X$. Denote by $I(X)$ the space of all locally constant functions f on X which satisfy $f(mnx) = \Delta_P^{1/2}(m)\chi(m)f(x)$, for all $m \in M$, $n \in N$, $x \in X$ and for which there exists a compact subset C of X such that supp(f) $\subseteq PC$. Now the following sequence is well-defined

$$0 \to I(PwP) \hookrightarrow \mathrm{Ind}_P^G(\chi) \overset{\text{restrict.}}{\longrightarrow} I(P) \to 0. \tag{5.2}$$

Similar reasons that imply the exactness of the sequence (5.1), imply also that the above sequence is exact ([Cs1], for example). The group P acts on $I(PwP)$ and $I(P)$ by right translations. In this way the above exact sequence is an exact sequence of P-representations. Because of the exactness of the Jacquet functor, to compute the Jordan-Hölder series of $(\mathrm{Ind}_P^G(\chi))_N$ it is enough to compute the Jordan-Hölder series of $I(PwP)_N$ and $I(P)_N$. Clearly, $I(P)$ is one dimensional. Also, N acts trivially on $I(P)$. Thus $I(P) \cong I(P)_N$ as M-representations. One checks directly that $f \mapsto f(1)$ gives

$$I(P)_N \cong \Delta_P^{1/2}\chi.$$

A more delicate problem is to examine $I(PwP)_N$. Note first that $f \in I(PwP)$ is completely determined by $f|wP$. Define for $f \in I(PwP)$ a function Φ_f on P by the formula

$$\Phi_f(p) = f(wp) \quad , \quad p \in P.$$

Then one gets directly

$$\Phi_f(mp) = f(wmp) = f(wmw^{-1}wp) = f(m^{-1}wp) =$$

$$\Delta_P^{-1/2}(m)\chi(m^{-1})f(wp) = \Delta_P^{-1/2}(m)\chi^{-1}(m)\Phi_f(p)$$

for $m \in M$ and $p \in P$. Denote by J the space of all locally constant functions φ on P which satisfy $\varphi(mp) = \Delta_P^{-1/2}(m)\chi^{-1}(m)\varphi(p)$ for all $m \in M$, $p \in P$, and for which there exists a compact subset $C \subseteq P$ such that $\operatorname{supp}\varphi \subseteq MC$. The group P acts on J by right translations. If $f \in I(PwP)$, then we have seen that $\Phi_f \in J$. Moreover, one can see easily that

$$I(PwP) \cong J$$

as representations of P. Consider the following mapping from J to functions on P

$$\Psi_f(p) = \int_N f(np)dn = \int_F f\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} p\right) dx.$$

Obviously, Ψ_f is a function on $N\backslash P$. Further, for $m = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in M$

$$\Psi_f(m) = \int_N f(nm)\, dn = \int_F f\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right) dx =$$

$$\int_F f\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} 1 & a^{-2}x \\ 0 & 1 \end{bmatrix}\right) dx =$$

$$\Delta_P^{-1/2}(m)\chi^{-1}(m)\int_F f\left(\begin{bmatrix} 1 & a^{-2} & x \\ 0 & 1 & \end{bmatrix}\right) dx =$$

$$\Delta_P^{-1/2}(m)\chi^{-1}(m)|a|_F^2\int_F f\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right) dx =$$

$$\Delta_P^{-1/2}(m)\chi^{-1}(m)|a|_F^2\Psi_f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

We shall see in the following section that $\Delta_P\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right) = |a|_F^2$. Thus

$$\Psi_f(m) = \Delta_P^{1/2}(m)\chi^{-1}(m)\Psi_f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right). \tag{5.3}$$

Therefore, Ψ_f is completely determined by $\Psi_f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$. Define

$$\hat{\Psi} : J \to \mathbb{C}$$

by

$$\hat{\Psi}(f) = \Psi_f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Since

$$\hat{\Psi}(R_m f) = \Psi_{R_m f}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) =$$

$$\int_N (R_m f)(n)dn = \int_N f(nm)dn = \Psi_f(m),$$

we have

$$\hat{\Psi}(R_m f) = \Delta_P^{1/2}(m)\chi^{-1}(m)\hat{\Psi}(f). \tag{5.4}$$

Thus the above formula implies that we have a P-intertwining

$$\hat{\Psi} : J \to \Delta_P^{1/2}\chi^{-1}, \tag{5.5}$$

where N acts trivially on the right hand side. It is easy to see that $\hat{\Psi}$ is surjective (one constructs explicitly a function $f \in J$ such that $\hat{\Psi}(f) \neq 0$). Exactness of the Jacquet functor implies that

$$\hat{\Psi}_N : J_N \to \Delta_P^{1/2}\chi^{-1}$$

is a mapping onto. The last step is proving that $\hat{\Psi}_N$ is an isomorphism. For that one needs to prove

$$\mathrm{Ker}\hat{\Psi} = J(N).$$

Since N acts trivially on the right hand side of (5.5), we have $J(N) \subseteq \mathrm{Ker}\,\hat{\Psi}$. Let us take a look at $\hat{\Psi}$. By the definition of $\hat{\Psi}$ we have

$$\hat{\Psi}(f) = \int_N f(n)\,dn.$$

It means that $\hat{\Psi}$ is the Haar measure on N (recall that $f \in J$ is completely determined by $f|N$). The kernel of the Haar measure, considered on the functions from $C_c^\infty(G)$, consists of the span of all $R_n f - f$, $f \in C_c^\infty(G)$, $n \in N$. Thus, $\mathrm{Ker}\,\hat{\Psi} \subseteq J(N)$ because a form on $C_c^\infty(G)$ which is invariant for the (right) translations by the elements of N, must be proportional to the Haar measure on N.

So, we have shown at the end that there exists the following exact sequence

$$0 \to \Delta_P^{1/2}\chi^{-1} \to \left(\mathrm{Ind}_P^{SL(2,F)}(\chi)\right)_N \to \Delta_P^{1/2}\chi \to 0.$$

In terms of the normalized Jacquet modules we have the following exact sequence

$$0 \to \chi^{-1} \to r_M^{SL(2,F)}\left(\mathrm{Ind}_P^{SL(2,F)}(\chi)\right) \to \chi \to 0.$$

Consider the case of $GL(2,F)$. Set

$$M = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix};\quad a, b \in F^\times \right\}, \tag{5.6}$$

$$N = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}; n \in F \right\} \tag{5.7}$$

and

$$P = MN.$$

For characters χ_1 and χ_2 of $F^\times$ we denote by $\chi_1 \otimes \chi_2$ the character

$$(\chi_1 \otimes \chi_2)\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \chi_1(a)\chi_2(b)$$

of M. Then the same type of calculation as it was described for $SL(2,F)$, gives the following exact sequence

$$0 \to \chi_2 \otimes \chi_1 \to r_M^{GL(2,F)}\left(\mathrm{Ind}_P^{GL(2,F)}(\chi_1 \otimes \chi_2)\right) \to \chi_1 \otimes \chi_2 \to 0.$$

In general, let $P = MN$ and $P' = M'N'$ be parabolic subgroups in a reductive group G. Suppose that σ is an admissible representation of M. Then the same type of considerations as we did for $SL(2, F)$ give a description of $(\mathrm{Ind}_P^G(\sigma))_{N'}$ in the following way. There exist M'-subrepresentations

$$\{0\} = V_o \subseteq V_1 \subseteq \cdots \subseteq V_k = (\mathrm{Ind}_P^G(\sigma))_{N'}$$

such that it is possible to describe subquotients V_{i+1}/V_i as certain induced representations from suitable Jacquet modules of σ. These M'-representations are indexed by the double cosets

$$P\backslash G/P'.$$

One proceeds similarly as it was done in the case of $SL(2, F)$. There exists an open double class Pw_1P' in G. Then one defines $I(Pw_1P')$ and $I(G\backslash Pw_1P')$ in a similar way as it was done for $SL(2, F)$. One has the exact sequence

$$0 \to I(Pw_1P') \hookrightarrow \mathrm{Ind}_P^G(\sigma) \overset{\text{restrict.}}{\longrightarrow} I(G\backslash Pw_1P') \to 0.$$

With a similar analysis as before, one can describe $I(PwP')_{N'}$ as a certain parabolically induced representation from suitable Jacquet module of σ. Then one can pick another double coset Pw_2P' which is open in $G\backslash Pw_1P'$. One proceeds in a similar way. One finishes when one comes to the double coset PP'.

6. Some examples

Modular characters of $SL(2,F)$ and $GL(2,F)$, reducibility points: Because of the definition of $\mathrm{Ind}_P^G(\sigma)$, the trivial representation is always a subrepresentations of $\mathrm{Ind}_P^G(\Delta_P^{-1/2})$. One of the topics that interests us is the reducibility of $\mathrm{Ind}_P^G(\sigma)$. If $P \neq G$, then $\mathrm{Ind}_P^G(\Delta_P^{-1/2})$ is reducible. We shall now calculate explicitly this reducibility point for $SL(2,F)$.

We shall use now the notation for $SL(2,F)$ which was introduced in the last section. Let us write some Haar measures. The Haar measures on N and M are

$$\int_N f(n)dn = \int_F f\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right) dx$$

and

$$\int_M f(m)dm = \int_{F^\times} f\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right) d^\times a$$

respectively. Here $d^\times a$ denotes a Haar measure on the multiplicative group $F^\times$. The definition of $|\ \ |_F$ implies that

$$\int_M f(m)dm = \int_F f\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right) \frac{da}{|a|_F}.$$

Consider the measure

$$f \to \int_N \int_M f(nm)\, dn\, dm$$

on $P = MN$. It is obvious that the above measure is invariant for right translations by elements of M. Also, for $n' \in N$

$$\int_N \int_M f(nmn')\, dn\, dm =$$

$$\int_N \int_M f\left(n(mn'm^{-1})m\right)\, dn\, dm =$$

$$\int_N \int_M f(nm)\, dn\, dm$$

since M normalizes N. We have used the Fubini's theorem in the above manipulations. Thus

$$f \mapsto \int_N \int_M f(nm)\, dn\, dm$$

is a right Haar measure on P. Since for $n' \in N$

$$\int_M \int_N f(n'nm)dndm = \int_M \int_N f(nm)\, dn\, dm$$

we have $\Delta_P(n) = 1$ for $n \in N$. Let $m' = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in M$. Then

$$\int_M \int_N f(m'nm)\, dn\, dm =$$

$$\int_M \int_F f\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} m\right) dx\, dm =$$

$$\int_M \int_F f\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix} m\right) dx\, dm =$$

$$\int_M \int_F f\left(\begin{bmatrix} 1 & a^2 & x \\ 0 & 1 & \end{bmatrix} m\right) dx\, dm =$$

$$|a|_F^{-2} \int_M \int_F f\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} m\right) dx\, dm =$$

$$|a|_F^{-2} \int_M \int_N f(nm)\, dn\, dm.$$

Since

$$\int_M \int_N f(m'nm)\, dn\, dm = \Delta_P^{-1}(m') \int_M \int_N f(nm)\, dn\, dm,$$

we have

$$\Delta_P\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right) = |a|_F^2.$$

A similar calculation for $GL(2, F)$ gives

$$\Delta_{P_{\min}}\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = |a|_F |b|_F^{-1}.$$

$GL(2)$ **over finite field:** We shall see how the calculations done in the previous section can be used in a relatively simple case, in the study of the representation theory of $GL(2)$ over a finite field $\mathbb{F} = \mathbb{F}_q$. We keep the notation which was introduced in the last section also for $GL(2)$ over the finite field.

In the case of the finite fields, one defines parabolically induced representations, Jacquet modules and cuspidal representations in the same way as it was done in the case of local non-archimedean fields. The same form of Frobenius reciprocity holds for finite fields. Then the same calculations as in the last section give the Jacquet modules of the parabolically induced representation. Note that here representations are completely reducible and all modular functions are trivial.

Let χ_1 and χ_2 be characters of $\mathbb{F}^\times$. Then

$$\left(\operatorname{Ind}_P^{GL(2,\mathbb{F})}(\chi_1 \otimes \chi_2)\right)_N = (\chi_1 \otimes \chi_2) \oplus (\chi_2 \otimes \chi_1).$$

First of all,

$$\dim_{\mathbb{C}} \operatorname{Ind}_P^{GL(2,\mathbb{F})}(\chi_1 \otimes \chi_2) = \frac{\text{card } GL(2,\mathbb{F})}{\text{card } P} = \frac{(q^2-1)(q^2-q)}{(q-1)^2 q} = q+1.$$

The Frobenius reciprocity implies that $\operatorname{Ind}_P^{GL(2,\mathbb{F})}(\chi_1 \otimes \chi_2)$ is irreducible if and only if $\chi_1 \neq \chi_2$. If $\chi_1 = \chi_2 = \chi$, then $\chi \circ \det$ is a subrepresentation of $\operatorname{Ind}_P^{GL(2,\mathbb{F})}(\chi \otimes \chi)$. Frobenius reciprocity gives that $\operatorname{Ind}_P^{GL(2,\mathbb{F})}(\chi \otimes \chi)$ is a sum of two irreducible representations which are clearly not isomorphic. One of them is one dimensional and the other one is not one dimensional. If $\operatorname{Ind}_P^{GL(2,\mathbb{F})}(\chi_1 \otimes \chi_2)$ and $\operatorname{Ind}_P^{GL(2,\mathbb{F})}(\chi_1' \otimes \chi_2')$ have irreducible subrepresentations which are equivalent, then the Frobenius reciprocity implies

$$\chi_1' \otimes \chi_2' = \chi_1 \otimes \chi_2 \ \text{ or } \ \chi_1' \otimes \chi_2' = \chi_2 \otimes \chi_1.$$

Therefore, the irreducible subrepresentations of $\operatorname{Ind}_P^{GL(2,\mathbb{F})}(\chi_1 \otimes \chi_2)$ that we have obtained are the following

(i) $[(q-1)^2 - (q-1)]/2 = (q-1)(q-2)/2$ $(q+1)$-dimensional representations,
(ii) $(q-1)$ q-dimensional representations,
(iii) $(q-1)$ one dimensional representations.

The above representations are not equivalent.

We shall say a few words about cuspidal representations following [PS]. For more details one should consult that nice introductory book. We shall use two well known facts from the representations theory of finite groups. The first fact is that the number of equivalence classes of irreducible representations of a finite group is equal to the number of conjugacy classes of the group. The second fact is that the sum of squares of the dimensions of the equivalence classes of the irreducible representations of a finite group is equal to the cardinality of the group.

Set

$$T = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}; a \in \mathbb{F}^\times, b \in \mathbb{F} \right\}.$$

Then one checks directly that T has q conjugacy classes. Note that we have $(q-1)$ characters

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \mapsto \chi(a), \quad \chi \in (\mathbb{F}^\times)\hat{}.$$

Thus, we have only one additional irreducible representation. Denote this representation by τ_o. Since the sum of squares of all (classes of) irreducible representations is equal to the order of the group, we get

$$\dim_{\mathbb{C}} \tau_o = \sqrt{(q-1)q - (q-1)\cdot 1} = q-1.$$

One can realize τ_o as an induced representation of T from N by any non-trivial character of N.

Let ρ be an irreducible cuspidal representation of $GL(2,\mathbb{F})$. Since N acts trivially in the representations $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \mapsto \chi(a), \quad \chi \in F^\times$, we have $\rho|T \cong n\tau_o$ for some positive integer n. A simple calculation gives that $GL(2,\mathbb{F})$ has q^2-1 conjugacy classes. Thus $GL(2,\mathbb{F})$ has

$$(q^2-1)-(q-1)(q-2)/2-2(q-1)=(q^2-q)/2$$

classes of irreducible cuspidal representations. So, irreducible cuspidal representations for finite fields exist. The sum of squares of their dimensions is

$$(q^2-1)(q^2-q)-\frac{(q-1)(q-2)}{2}(q+1)^2-(q-1)q^2-(q-1)1=(q-1)^2\frac{q^2-q}{2}.$$

This immediately implies that all irreducible cuspidal representations are $(q-1)$-dimen--sional. An explicit construction of cuspidal representations of $GL(2,\mathbb{F})$ one can found in [PS]. The cuspidal representations are parametrized with the primitive characters of the multiplicative group of the quadratic extension of $\mathbb{F}$, modulo the action of the Galois group. We can give one example easily. The group $GL(2,\mathbb{F}_2)=SL(2,\mathbb{F}_2)$ is not commutative and has 6 elements. This implies that dimensions of irreducible representations are 2,1,1. Now the non-trivial character (which is of order two) is a cuspidal representation. This is the only cuspidal representation of $GL(2,\mathbb{F}_2)$.

For a nice introduction to representations of general $GL(n,\mathbb{F})$ one can consult [HoMo] (see the appendices in that book).

A cuspidal representation: We shall see now that non-trivial cuspidal representations do exist also in the case of a local non-archimedean field F. We shall give an example of a cuspidal representation of $SL(2,F)$. This example was done by F. Mautner.

We have denoted by $\mathfrak{p}_F=\{x\in F; |x|_F<1\}$ the only non-zero prime ideal in $\mathcal{O}_F$. Denote by $\mathbb{F}=\mathcal{O}_F/\mathfrak{p}_F$ the **residual field** of F. Clearly, it is a finite field. Set $K_o=SL(2,\mathcal{O}_F)$. Then the projection $\mathcal{O}_F\to\mathbb{F}$ induces a group-homomorphism

$$SL(2,\mathcal{O}_F)\to SL(2,\mathbb{F}).$$

Let (σ,U) be an irreducible cuspidal representation of $SL(2,\mathbb{F})$. We shall consider σ as a representation of K_o. Consider the space of all compactly supported functions

$$f: SL(2,F)\to U$$

which satisfy

$$f(kg)=\sigma(k)f(g), \quad \text{for all } k\in K_o \text{ and } g\in SL(2,F).$$

The group $SL(2,F)$ acts on this space by right translations. Let $\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma)$ be the smooth part of that representation. Then $\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma)$ is an irreducible cuspidal representation of $SL(2,F)$. Let us explain briefly the argument that gives that.

Fix a $SL(2,\mathbb{F})$-invariant inner product $(\ ,\ )$ on U. Then

$$< f_1, f_2 >= \int\limits_{SL(2,F)} (f_1(g), f_2(g))\, dg \tag{6.1}$$

is an $SL(2,F)$-invariant inner product on $\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma)$.

For a positive integer n set

$$K_n = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K_o; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{mod}\ \mathfrak{p}_F^n) \right\}.$$

Then K_n is a normal subgroup of K_o. Write

$$K_o = \bigcup_{i=1}^{m} k_i K_n.$$

Suppose that $f \in (\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma))^{K_n}$. Then for $x \in \mathcal{O}_F$ we have

$$\sigma\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right) f\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} k_i\right) =$$

$$f\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} k_i\right) =$$

$$f\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} 1 & xa^{-2} \\ 0 & 1 \end{bmatrix} k_i\right) =$$

$$f\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} k_i k_i^{-1} \begin{bmatrix} 1 & a^{-2}x \\ 0 & 1 \end{bmatrix} k_i\right).$$

One can find $t > 1$ such that $a^{-2}\mathcal{O}_F \subseteq \mathfrak{p}_F^n$ if $|a|_F \geqslant t$. Thus

$$f\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} k_i\right) = 0$$

since σ is cuspidal. The Cartan decomposition of $SL(2,F)$ implies that the support of f is contained in

$$K_o \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}; a \in F \text{ and } 1 \leqslant |a|_F \leqslant t \right\} K_o.$$

Thus, the supports of the functions in $\left(\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma)\right)^{K_n}$ are contained in the fixed compact subset. Since each function f from that space of the K_n-invariants must take values in a certain finite dimensional space, and it is determined on representatives of $K_o \backslash SL(2,F)/K_n$, we get that the spaces of the K_n-invariants are finite dimensional. This implies the admissibility of the representation. In particular, $\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma)$ is unitarizable. Because all the matrix coefficients of a unitarizable representation (π, V) are of the form

$$g \mapsto (\pi(g)v_1, v_2), v_1, v_2 \in V,$$

we get that

$$x \mapsto \int_{SL(2,F)} (f_1(gx), f_2(g))\, dg, \quad f_1, f_2 \in \mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma),$$

are all matrix coefficients of $\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma)$. Clearly, they are compactly supported. Thus, $\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma)$ is cuspidal.

Note that for $f \in \left(\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma)\right)^{K_1}$ we have that $\mathrm{sup} f \subseteq K_o$. Thus

$$\left(\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma)\right)^{K_1} \cong U.$$

This implies that the multiplicity of σ in $\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma)$ as K_o-representation is less than or equal to one. Actually, it is one because $f \mapsto f(1)$ is a K_o-intertwining of $\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma)$ onto U.

To get the irreducibility observe that we have a linear map

$$\Lambda \mapsto \Lambda',$$

$$\mathrm{Hom}_G\left(\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma), \mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma)\right) \to \mathrm{Hom}_{K_o}\left(\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma), \sigma\right).$$

given by $\Lambda' f = (\Lambda f)(1)$. One sees directly that $\Lambda \neq 0$ implies $\Lambda' \neq 0$. Since the multiplicity of σ in $\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma)$ is one, we have that the commutator of the representation $\mathrm{Ind}_{K_o}^{SL(2,F)}(\sigma)$ consists only of the scalar operators. Since the representation is completely reducible, we get the irreducibility.

There is a conjecture that each cuspidal representation of a reductive p-adic group can be induced from a compact modulo center subgroup. For such constructions of cuspidal representations one may consult [Ho], [Cy]. More informations about expectations in that direction one can find in [Ku2].

7. Parabolically induced representations of $SL(2,F)$ and $GL(2,F)$

We shall make an elementary analysis of the case of the parabolically induced representations of the group $SL(2,F)$.

Let χ be a character of $F^\times$. Then we shall consider χ also as a character of M since we have identified $F^\times$ with M. Recall of the exact sequence

$$0 \to \chi^{-1} \to r_M^{SL(2,F)}\left(\mathrm{Ind}_P^{SL(2,F)}(\chi)\right) \to \chi \to 0. \tag{7.1}$$

The above exact sequence implies that $r_M^{SL(2,F)}\left(\mathrm{Ind}_P^{SL(2,F)}(\chi)\right)$ is of length two. Note that there is only one conjugacy class of proper parabolic subgroups in $SL(2,F)$. Since $\mathrm{Ind}_P^{SL(2,F)}(\chi)$ does not contain cuspidal subquotients, $\mathrm{Ind}_P^{SL(2,F)}(\chi)$ is at most of length two. The Jacquet modules imply that if $\mathrm{Ind}_P^{SL(2,F)}(\chi)$ and $\mathrm{Ind}_P^{SL(2,F)}(\chi')$ have irreducible subquotients which are isomorphic, then

$$\chi = \chi' \ \text{ or } \ \chi^{-1} = \chi'. \tag{7.2}$$

Recall of the Frobenius reciprocity in this situation. For characters χ and χ' of $F^\times$ we have

$$\mathrm{Hom}_{SL(2,F)}\left(\mathrm{Ind}_P^{SL(2,F)}(\chi), \mathrm{Ind}_P^{SL(2,F)}(\chi')\right) \cong \mathrm{Hom}_M\left(r_M^{SL(2,F)}\left(\mathrm{Ind}_P^{SL(2,F)}(\chi)\right), \chi'\right). \tag{7.3}$$

The Frobenius reciprocity and the exact sequence (7.1) imply that

$$\dim_{\mathbb{C}} \mathrm{End}_{SL(2,F)}\left(\mathrm{Ind}_P^{SL(2,F)}(\chi)\right) \leqslant 2. \tag{7.4}$$

Suppose that $\mathrm{Ind}_P^{SL(2,F)}(\chi)$ is reducible and that it is not a multiplicity one representation. Then $\chi = \chi^{-1}$. This implies $\chi^2 = 1$. Thus, χ is a unitary character. Therefore, $\mathrm{Ind}_P^{SL(2,F)}(\chi)$ is a unitarizable representation. Since such representations are completely reducible, we get

$$\dim_{\mathbb{C}} \mathrm{End}_{SL(2,F)}\left(\mathrm{Ind}_P^{SL(2,F)}(\chi)\right) = 4.$$

This is impossible by (7.4). The last contradiction implies that $\mathrm{Ind}_P^{SL(2,F)}(\chi)$ is always a multiplicity one representation.

A character χ of M is called **regular** if $\chi \neq \chi^{-1}$. For a regular character χ we have

$$r_M^{SL(2,F)}\left(\mathrm{Ind}_P^{SL(2,F)}(\chi)\right) = \chi \oplus \chi^{-1}.$$

This implies that

$$\dim_{\mathbb{C}} \operatorname{End}_{SL(2,F)}\left(\operatorname{Ind}_P^{SL(2,F)}(\chi)\right) = 1 \tag{7.5}$$

if χ is a regular character.

Suppose that χ is a unitary regular character. Since $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ is a unitarizable representation, it is completely reducible. Thus, $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ is an irreducible unitarizable representation. This is a special case of a general Bruhat result.

Up to now we have seen what happens with $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ when χ is a unitary character which satisfies $\chi^2 \neq 1_{F^\times}$.

Suppose now that χ is not unitary and that $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ splits. Note that χ is a regular character because it is not unitary. Let π_1 and π_2 be different irreducible subquotients. Then, say

$$r_M^{SL(2,F)}(\pi_1) = \chi, \quad r_M^{SL(2,F)}(\pi_2) = \chi^{-1}.$$

Now the square integrability criterion (SI) implies that either π_1 or π_2 is square integrable. Without a lost of generality, we can suppose that π_1 is square integrable. Therefore, π_1 is unitarizable and $\tilde{\overline{\pi}}_1 \cong \pi_1$ is a subquotient of $\operatorname{Ind}_P^{SL(2,F)}(\tilde{\overline{\chi}})$. Thus, $\operatorname{Ind}_P^{SL(2,F)}((\overline{\chi})^{-1})$ and $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ have non-disjoint Jordan-Hölder series. Then we know by (7.2) that $\chi = (\overline{\chi})^{-1}$ or $\chi^{-1} = (\overline{\chi})^{-1}$. The first relation is equivalent to $\chi\overline{\chi} = 1$ what means that χ is unitary. Thus $\chi = \overline{\chi}$. In other words, χ must be a real-valued character.

Clearly, constant functions are contained in $\operatorname{Ind}_P^{SL(2,F)}(\Delta_P^{-1/2})$. Thus $\operatorname{Ind}_P^{SL(2,F)}(\Delta_P^{-1/2})$ is reducible. One irreducible subquotient is the trivial representation while the other irreducible subquotient is a square integrable representation. This square integrable representation is called the **Steinberg representation** of $SL(2,F)$. Since

$$(\operatorname{Ind}_P^{SL(2,F)}(\Delta_P^{1/2}))^{\sim} \cong \operatorname{Ind}_P^{SL(2,F)}(\Delta_P^{-1/2}),$$

$\operatorname{Ind}_P^{SL(2,F)}(\Delta_P^{-1/2})$ is also reducible.

If $\chi = \chi'$ or $\chi^{-1} = \chi'$, then $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ and $\operatorname{Ind}_P^{SL(2,F)}(\chi')$ have the same Jordan-Hölder sequences by the general result about the parabolic induction from the associate pairs which was mentioned in the third section. We shall outline the proof of this fact for $SL(2,F)$.

Let χ be any character of $F^\times$. Note that $(\operatorname{Ind}_P^{SL(2,F)}(\chi))^{\sim} \cong \operatorname{Ind}_P^{SL(2,F)}(\chi^{-1})$. Thus $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ is irreducible if and only if $\operatorname{Ind}_P^{SL(2,F)}(\chi^{-1})$ is irreducible. From the Frobenius reciprocity (7.3) and the exact sequence (7.1) one obtains that

$$\operatorname{Hom}_{SL(2,F)}\left(\operatorname{Ind}_P^{SL(2,F)}(\chi), \operatorname{Ind}_P^{SL(2,F)}(\chi^{-1})\right) \neq 0. \tag{7.6}$$

Therefore we have a non-zero intertwining

$$\Lambda_\chi : \operatorname{Ind}_P^{SL(2,F)}(\chi) \to \operatorname{Ind}_P^{SL(2,F)}(\chi^{-1}).$$

Thus

$$\operatorname{Ind}_P^{SL(2,F)}(\chi) \cong \operatorname{Ind}_P^{SL(2,F)}(\chi^{-1}), \tag{7.7}$$

if $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ is irreducible. If χ is unitary and if $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ is reducible, then we know that $\chi^2 = 1$, i.e. that $\chi = \chi^{-1}$. Thus (7.7) holds for any unitary character χ. Suppose now that $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ reduces and that $\chi \neq \chi^{-1}$. Then (7.5) implies that $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ has a unique irreducible subrepresentation, say V_1, and a unique irreducible quotient. They are not isomorphic. Since $V_1 \hookrightarrow \operatorname{Ind}_P^{SL(2,F)}(\chi)$, the Frobenius reciprocity implies that the Jacquet module of the irreducible subrepresentation is χ, while the irreducible quotient has χ^{-1} for the Jacquet module.

Let $\varphi \neq 0$ be an intertwining mapping from the space (7.6). If $\operatorname{Ker}\varphi = \{0\}$, then $\operatorname{Ind}_P^{SL(2,F)}(\chi^{-1})$ has an irreducible subrepresentation whose Jacquet module is χ. This is impossible by the previous remarks if we apply them to $\operatorname{Ind}_P^{SL(2,F)}(\chi^{-1})$. Therefore the irreducible quotient of $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ is isomorphic to the irreducible subrepresentation of $\operatorname{Ind}_P^{SL(2,F)}(\chi^{-1})$. Applying the same observation to $\operatorname{Ind}_P^{SL(2,F)}(\chi^{-1})$, one gets that $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ and $\operatorname{Ind}_P^{SL(2,F)}(\chi^{-1})$ have the same Jordan-Hölder series. So, we have proved this result for arbitrary character χ.

Let us now describe one Casselman's method for the study of the irreducibility of the parabolically induced representations. Suppose that χ is a character of $M \cong F^\times$ such that $\chi^2 \neq 1_{F^\times}$. Consider the intertwinings Λ_χ from the spaces (7.6). If $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ is irreducible, then by the Schur's lemma there exists $c(\chi) \in \mathbb{C}^\times$ such that

$$\Lambda_{\chi^{-1}}\Lambda_\chi = c(\chi)\, \mathrm{id}_{\operatorname{Ind}_P^{SL(2,F)}(\chi)}.$$

Note that $c(\chi) \in \mathbb{C}$ depends on the choice of Λ_χ and $\Lambda_{\chi^{-1}}$. Suppose that $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ reduces. We have seen that Λ_χ and $\Lambda_{\chi^{-1}}$ have non-trivial kernels. Also representations $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ and $\operatorname{Ind}_P^{SL(2,F)}(\chi^{-1})$ have unique irreducible subrepresentations. Thus $\Lambda_{\chi^{-1}}\Lambda_\chi = 0$. One can say that in this situation we have $c(\chi) = 0$. Therefore, for a regular character χ, $\operatorname{Ind}_P^{SL(2,F)}(\chi)$ is irreducible if and only if $c(\chi) \neq 0$. The delicate part in this method is an explicit computation of $c(\chi)$.

Casselman has computed $c(|\ |_F^\alpha)$ in [Cs1]. That computation gives that $c(|\ |_F^\alpha) \neq 0$ for $\alpha \in \mathbb{R}\backslash\{-1,0,1\}$. Thus, $\operatorname{Ind}_P^{SL(2,F)}(|\ |_F^\alpha)$ is irreducible for $\alpha \in \mathbb{R}\backslash\{-1,0,1\}$. Note that the space

$$Q = \left\{ f|SL(2,\mathcal{O}_F); f \in \operatorname{Ind}_P^{SL(2,F)}(|\ |^\alpha) \right\}$$

does not depend on α. Therefore, one can realize all these representations on Q. Denote the action of $SL(2,F)$ on Q which corresponds to $\operatorname{Ind}_P^{SL(2,F)}(|\ |^\alpha)$ by π_α. One can now normalize the intertwinings $\Lambda_{|\ |^\alpha}$ in such a way that $\Lambda_{|\ |^\alpha}$, and thus also $c(|\ |^\alpha)$, depends analytically on α. Then, it is enough to compute $(\Lambda_{\chi^{-1}}\Lambda_\chi f)(k_o)$ for some function f from Q on a non-empty open subset of $\mathbb{C}$. One needs to assume only that $f(k_o) \neq 0$. This is computed in [Cs1].

We shall see now what one can obtain by the similar analysis of the Jacquet modules in the case of $GL(2,F)$. In the same way as for $SL(2,F)$ one obtains the following results. If

$$\mathrm{Ind}_P^{GL(2,F)}(\chi_1 \otimes \chi_2) \text{ and } \mathrm{Ind}_P^{GL(2,F)}(\chi_1' \otimes \chi_2')$$

have non-disjoint Jordan-Hölder series, then

$$\chi_1 \otimes \chi_2 = \chi_1' \otimes \chi_2' \text{ or } \chi_1 \otimes \chi_2 = \chi_2' \otimes \chi_1'. \tag{7.8}$$

Also, $\mathrm{Ind}_P^{GL(2,F)}(\chi_1 \otimes \chi_2)$ and $\mathrm{Ind}_P^{GL(2,F)}(\chi_2 \otimes \chi_1)$ have the same Jordan-Hölder series. The Frobenius reciprocity implies that $\mathrm{Ind}_P^{GL(2,F)}(\chi_1 \otimes \chi_2)$ is irreducible if $\chi_1 \neq \chi_2$ and if χ_1/χ_2 is unitary. If χ_1/χ_2 is not unitary and if $\mathrm{Ind}_P^{GL(2,F)}(\chi_1 \otimes \chi_2)$ reduces, then the square integrability criterion implies that $\chi_2 = (\overline{\chi}_1)^{-1}$. This implies that $\chi_1 = \chi_2| \ |_F^{\alpha}$ for some $\alpha \in \mathbb{R}^{\times}$. Thus $\chi_1 \otimes \chi_2 = \chi_2| \ |_F^{\alpha} \otimes \chi_2$. The restriction of the functions on $GL(2,F)$ from $\mathrm{Ind}_P^{GL(2,F)}(\chi_2| \ |_F^{\alpha} \otimes \chi_2)$ to $SL(2,F)$ gives an isomorphism of $\mathrm{Ind}_P^{GL(2,F)}(\chi_2| \ |_F^{\alpha} \otimes \chi_2)$ onto $\mathrm{Ind}_P^{SL(2,F)}(| \ |_F^{\alpha})$, as representations of $SL(2,F)$. Since $\mathrm{Ind}_P^{SL(2,F)}(| \ |_F^{\alpha})$ is irreducible for $\alpha \in \mathbb{R}\backslash\{1,0,-1\}$, we have that $\mathrm{Ind}_P^{GL(2,F)}(\chi_2| \ |_F^{\alpha} \otimes \chi_2)$ is also irreducible for such α.

To have a complete analysis of the parabolically induced representations of $GL(2,F)$, one should see what happens with $\mathrm{Ind}_P^{GL(2,F)}(\chi \otimes \chi)$. Since

$$(\chi \circ \det)\, \mathrm{Ind}_P^{GL(2,F)}(\chi_1 \otimes \chi_2) \cong \mathrm{Ind}_P^{GL(2,F)}(\chi\chi_1 \otimes \chi\chi_2), \tag{7.9}$$

one should check what happens with $\mathrm{Ind}_P^{GL(2,F)}(1_{F^\times} \otimes 1_{F^\times})$. Set $N^- =^t N$. Then PN^- has a full measure in $GL(2,F)$ (the complement has the Haar measure equal to 0). In the same way as it was explained in Remarks 2.2., (i), we get that

$$\int_{N^-} f(n^-)dn^-$$

is a $GL(2,F)$-invariant measure on the space X from Remarks 2.2, (i). Thus

$$f \mapsto f|N^-$$

defines an isomorphism of $\mathbf{Ind}_P^{GL(2,F)}(1_{F^\times} \otimes 1_{F^\times})$ onto $L^2(N^-)$. We shall identify these two spaces.

We identify N^- with F using the identification

$$\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \leftrightarrows x.$$

The action of $GL(2,F)$ on $L^2(F)$ will be denoted by π. Now a simple computation gives

$$\left(\pi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right) f\right)(x) = \left(|a|_F^{1/2} f\right)(ax) \tag{7.10}$$

and

$$\left(\pi\left(\begin{bmatrix}1 & 0\\ y & 1\end{bmatrix}\right)f\right)(x) = f(x+y), \tag{7.11.}$$

Let ψ be a non-trivial character of F. Denote by $\psi_a, a \in F$, the character defined by $\psi_a(x) = \psi(ax)$. Then $a \mapsto \psi_a$ is an isomorphism of F onto $\hat{F}$. The Fourier transform $\mathcal{F}$ is defined by

$$\hat{f}(a) = \int_F f(x)\psi_a(x)dx.$$

We define a representation $\hat{\pi}$ of $GL(2,F)$ on $L^2(F)$ by the formula

$$\hat{\pi}(g) = \mathcal{F}\pi(g)\mathcal{F}^{-1}.$$

Such defined representation $\hat{\pi}$ is called the **Gelfand-Naimark model** of the representation $\mathbf{Ind}_P^{GL(2,F)}(1_{F^\times}\otimes 1_{F^\times})$. Now the formulas (7.10.) and (7.11) imply

$$\left(\hat{\pi}\left(\begin{bmatrix}a & 0\\ 0 & 1\end{bmatrix}\right)f\right)(x) = f(a^{-1}x) \tag{7.12}$$

and

$$\left(\hat{\pi}\left(\begin{bmatrix}1 & 0\\ y & 1\end{bmatrix}\right)f\right)(x) = \overline{\psi(yx)}f(x). \tag{7.13}$$

We shall need now a little bit of Fourier analysis on F. Let T be a continuous operator on $L^2(F)$ which is in the commutator of the representation $\hat{\pi}$. By (7.13) T commutes with all multiplications with characters of F. Therefore, T commutes with all multiplications with functions on F. This implies that T itself is a multiplication with a function, say φ. Now (7.12) implies that φ must be a constant function. Thus T is a scalar operator. Since the representation $\hat{\pi}$ is unitary, it is completely reducible. Thus, $\hat{\pi}$ is irreducible. Since

$$\left(\mathbf{Ind}_P^{GL(2,F)}(1_{F^\times}\otimes 1_{F^\times})\right)^\infty \cong \mathrm{Ind}_P^{GL(2,F)}(1_{F^\times}\otimes 1_{F^\times}),$$

we have the irreducibility of $\mathrm{Ind}_P^{GL(2,F)}(1_{F^\times}\otimes 1_{F^\times})$. This completes the analysis of the induced representations of $GL(2,F)$. We have seen that $\mathrm{Ind}_P^{GL(2,F)}(\chi_1\otimes\chi_2)$ is reducible if and only if

$$\chi_1 = |\ \ |_F\,\chi_2 \ \text{ or } \ \chi_1 = |\ \ |_F^{-1}\chi_2.$$

From the $GL(2,F)$-case one can settle now the case of $SL(2,F)$. Similarly to the Clifford theory for finite groups, there exists the Clifford theory for p-adic groups. It is developed by S. Gelbart and A.W. Knapp ([GbKn]). Let π be an irreducible representation of $GL(2,F)$. Then, as in the case of the finite groups, we have

$$\dim_{\mathbb{C}}\ \mathrm{End}_{SL(2,F)}(\pi) = \mathrm{card}\,\{\chi \in (F^\times)\hat{}\,; \chi\pi \cong \pi\}.$$

Note that

$$\mathrm{Ind}_P^{SL(2,F)}(\chi) \cong \mathrm{Ind}_P^{GL(2,F)}(\chi \otimes 1_{F^\times})$$

as representations of $SL(2,F)$. Now (7.9) and (7.8) give that $\mathrm{Ind}_P^{SL(2,F)}(\chi)$ is reducible if and only if $\chi = \Delta_P^{\pm 1/2}$, or if χ is a character of order two (i.e. $\chi^2 = 1_{F^\times}$ and $\chi \neq 1_{F^\times}$).

At the end of this section we shall say a few words about the unitary duals of $SL(2,F)$ and $GL(2,F)$. We start with the case of $SL(2,F)$. The irreducible cuspidal representations are obviously in $SL(2,F)\hat{}$. Clearly, all the irreducible subrepresentations of the parabolically induced representations by the unitary characters are also in $SL(2,F)\hat{}$. The Steinberg representation and the trivial representation are in $SL(2,F)\hat{}$. Suppose that some induced representation $\mathrm{Ind}_P^{SL(2,F)}(\chi)$ by a non-unitary character χ has a unitarizable subquotient π. Since $\bar{\tilde{\pi}} \cong \pi$, $\mathrm{Ind}_P^{SL(2,F)}(\chi)$ and $\mathrm{Ind}_P^{SL(F)}((\overline{\chi})^{-1})$ have non-disjoint Jordan-Hölder series. Now (7.2) implies $(\overline{\chi})^{-1} = \chi^{-1}$, i.e. χ must be a real-valued character. We have seen that if $\mathrm{Ind}^{SL(2,F)}(\chi)$ reduces, then all irreducible subquotients are unitarizable. Suppose therefore that χ is a real valued non-unitary character of $F^\times$ such that $\mathrm{Ind}_P^{SL(2,F)}(\chi)$ is irreducible. We have seen that

$$\mathrm{Ind}_P^{SL(2,F)}(\chi) \cong \mathrm{Ind}_P^{SL(2,F)}(\chi^{-1}).$$

Further $\overline{\chi} = \chi$ implies

$$\left(\mathrm{Ind}_P^{SL(2,F)}(\chi)\right)^{\bar{\sim}} \cong \mathrm{Ind}_P^{SL(2,F)}(\chi^{-1}).$$

This implies that there exists a non-degenerate $SL(2,F)$-invariant Hermitian form on $\mathrm{Ind}_P^{SL(2,F)}(\chi)$. Actually, our previous observations imply that the form is given by

$$(f_1, f_2) = \int_{SL(2,O_F)} f_1(k)\overline{(\Lambda_\chi f_2)(k)}dk. \tag{7.14}$$

The question of the unitarizability of $\mathrm{Ind}_P^{SL(2,F)}(\chi)$ is the question if the above form is positive definite.

Consider the case of $\chi = |\ \ |_F^\alpha$ where $\alpha \in \mathbb{R}$. We have the irreducibility for $\alpha \neq \pm 1$. It can be shown that operators Λ_χ do not have a "singularity"at $\alpha = 0$. Therefore, one obtains a continuous family of Hermitian forms on $\mathrm{Ind}_P^{SL(2,F)}(|\ \ |^\alpha)$, $-1 < \alpha < 1$. Since the set of parameters α is connected, all representations $\mathrm{Ind}_P^{SL(2,F)}(|\ \ |^\alpha)$, $-1 < \alpha < 1$, are unitarizable. This is a consequence of the following simple fact from the linear algebra. If we have a continuous family of non-degenerate Hermitian forms on a fixed finite dimensional complex vector space, which is parametrized by a connected set, and if one of that forms is positive definite, then all of them are positive definite. In this case we have a positive definiteness which is coming from $\mathrm{Ind}_P^{SL(2,F)}(1_M)$. The above unitarizable representations are called **complementary series**.

Suppose that $|\alpha| > 1$. Then the connection between the asymptotics of the matrix coefficients and the Jacquet modules, and the explicit computation of the Jacquet modules, imply that the matrix coefficients of $\mathrm{Ind}_P^{SL(2,F)}(|\ \ |^\alpha)$ are not bounded functions. Thus, $\mathrm{Ind}_P^{SL(2,F)}(|\ \ |^\alpha)$ is not unitarizable for $|\alpha|_F > 1$, since obviously the matrix coefficients of the unitarizable representations are bounded functions.

Suppose that χ is a real valued character. We can write $\chi = \chi_o|\ \ |_F^\alpha$ where χ_o is a unitary character and $\alpha \in \mathbb{R}$. Since $\chi_o = \overline{\chi}_o$, we have that $\chi_o^2 = 1_{F^\times}$. We shall assume that $\chi_o \neq 1_{F^\times}$. Since the matrix coefficients of unitarizable representations are bounded, $\mathrm{Ind}_P^{SL(2,F)}(\chi_o|\ \ |_F^\alpha)$ is not unitarizable if $|\alpha| > 1$. Because on the representations

$$\mathrm{Ind}_P^{SL(2,F)}(\chi_o|\ \ |_F^\alpha), \quad \alpha > 0$$

we have a continuous family of Hermitian forms, they are not positive definite. This ends the description of the unitary dual of $SL(2,F)$.

One gets the unitary dual of $GL(2,F)$ in the same manner. The unitary dual of $GL(2,F)$ consists of the square integrable representations, the irreducible subrepresentations of the parabolically induced representations by the unitary characters, the unitary characters of the group and the complementary series

$$\mathrm{Ind}_P^{GL(2,F)}(\nu^\alpha\chi \otimes \nu^{-\alpha}\chi),\ 0 < \alpha < 1/2,\ \chi \in (F^\times)\hat{}.$$

8. Some general consequences

We shall return now to the general case. One can compute Jordan-Hölder series of the Jacquet modules of the parabolically induced representations using similar ideas to the ideas that were explained in the case of $SL(2,F)$. This computations gives a similar consequences for a general reductive group G to the consequences that we have obtained for $SL(2,F)$ and $GL(2,F)$.

The quotient of the normalizer in G of the standard Levi factor $M_{\min}$ by itself, will be denoted by W_G. Then W_G is called the **Weyl group** of G.

Let $P = MN$ be a parabolic subgroup in G and let σ be an irreducible cuspidal representation of M. For $g \in G$ denote by $g\sigma$ a representation of gMg^{-1} given by the formula

$$(g\sigma)(gmg^{-1}) = \sigma(m),$$

for $m \in M$. Then we have the following result of Bernstein and Zelevinsky, and of Casselman.

8.1. Theorem.

(i) *The Jordan-Hölder series of $r_M^G(\mathrm{Ind}_P^G(\sigma))$ consists of all $\omega\sigma$ when ω runs over all representatives of $W_M \backslash W_G / W_M$ which normalize M.*

(ii) *If the Jacquet module of $\mathrm{Ind}_P^G(\sigma)$ for a parabolic subgroup P' has a cuspidal subquotient, then P and P' are associate.*

This theorem has a number of interesting direct consequences. Let us explain some of them.

(8.2) Let $P_1 = M_1N_1$ be a parabolic subgroup of G associate to P. Using the fact that the parabolic induction from the associate pairs gives the same Jordan-Hölder sequences, and the exactness of the Jacquet functor, one gets that the Jordan-Hölder series of $r_{M_1}^G(\mathrm{Ind}_P^G(\sigma))$ is obtained from the Jordan-Hölder series of $r_M^G(\mathrm{Ind}_P^G(\sigma))$ by the conjugation with a suitable element of the group. Thus, the theorem gives also the Jordan-Hölder series of the Jacquet modules for the parabolic subgroups which are associate to P.

(8.3) The transitivity of the Jacquet modules implies that each irreducible subquotient of $\mathrm{Ind}_P^G(\sigma)$ has some Jacquet module which is cuspidal. Therefore, the length of $\mathrm{Ind}_P^G(\sigma)$ is finite. Recall that each irreducible admissible representation of G is equivalent to a subrepresentation of some representation $\mathrm{Ind}_P^G(\sigma)$ where σ is an irreducible cuspidal representation (see the fourth section). Now the property of the parabolic induction that it does not depend on the stages of induction, and the exactness of the induction functor imply that the parabolic induction carries the representations of the finite length to the representations of the finite length again.

(8.4) Similarly, one gets that the Jacquet functor carries the representations of finite length again to the representations of finite length.

(8.5) At this point it is easy to prove that each finitely generated admissible representation of G has a finite length (see Theorem 6.3.10 of [Cs1]).

(8.6) Suppose that $P' = M'N'$ and $P'' = M''N''$ are two parabolic subgroups of G. Let σ' and σ'' be an irreducible cuspidal representations of M' and M'' respectively. If $\mathrm{Ind}_P^G(\sigma')$ and $\mathrm{Ind}_{P''}^G(\sigma'')$ have non-disjoint Jordan-Hölder sequences, then (P', σ'') and (P'', σ'') must be associate.

(8.7) Suppose that σ is an irreducible cuspidal representation of a Levi factor M of a parabolic subgroup P. Suppose that the Jacquet module

$$r_{M'}^G\left(\mathrm{Ind}_P^G(\sigma)\right),$$

for a parabolic subgroup $P' = M'N'$, has an irreducible cuspidal subquotient. Then the Theorem 8.1. and (8.2) imply that (P, σ) and (P', σ') are associate pairs.

(8.8) Suppose that τ is an irreducible admissible representation of a Levi factor M of a parabolic subgroup $P = MN$. Let ρ' (resp.ρ'') be an irreducible cuspidal subquotient of

$$r_{M'}^G\left(\mathrm{Ind}_P^G(\tau)\right) \quad \left(\text{resp. } r_{M''}^G\left(\mathrm{Ind}_P^G(\tau)\right)\right)$$

for a parabolic subgroup $P' = M'N'$ (resp. $P'' = M''N''$). Then (P', ρ'') and (P'', ρ'') are associate.

The following theorem can be very useful. For a proof one may consult [Cs1].

8.9. Theorem. *Let σ be an irreducible cuspidal representation of a Levi factor M of a parabolic subgroup P of G. Let π be an irreducible subquotient of $\mathrm{Ind}_P^G(\sigma)$. Then there exists $w \in G$ which normalizes M such that π is isomorphic to a subrepresentation of $\mathrm{Ind}_P^G(w\sigma)$.*

Note that we have seen that the above theorem holds for $SL(2, F)$. We shall list now some useful consequences of the above theorem:

(8.10) With the same notation as in the above theorem, the Frobenius reciprocity implies that $r_M^G(\pi)$ is a non-zero cuspidal representation. We can conclude further. Suppose that $P' = M'N'$ is a parabolic subgroup such that $r_{M'}^G\left(\mathrm{Ind}_P^G(\sigma)\right)$ has an irreducible cuspidal quotient. Then $r_{M'}^G(\pi) \neq 0$ and $r_{M'}^G\left(\mathrm{Ind}_P^G(\sigma)\right)$ is a cuspidal representation.

(8.11) Suppose now that τ is an irreducible admissible representation of a Levi factor M of a parabolic subgroup P. Let π be an irreducible subquotient of $\mathrm{Ind}_P^G(\tau)$. Suppose that $r_{M'}^G\left(\mathrm{Ind}_P^G(\tau)\right) \neq 0$ for some parabolic subgroup $P' = M'N'$ of G. The transitivity of the Jacquet modules implies that

$$r_{M'}^G(\pi) \neq 0.$$

All the time we are using the exactness of the Jacquet functor.

Now we have directly

8.12. Lemma. *Let τ be an irreducible admissible representation of a Levi factor M of a parabolic subgroup P of G. If there exists a parabolic subgroup $P' = M'N'$ of G such* *that $r^G_{M'}\left(\mathrm{Ind}^G_P(\tau)\right)$ is a non-zero irreducible representation, then $\mathrm{Ind}^G_P(\tau)$ is an irreducible representation.*

Note that in this section we have used only a small part of the information contained in the Jacquet modules. Namely, we have used only the facts coming from the calculation of the Jacquet modules which correspond to the parabolic subgroups which are minimal among all the parabolic subgroups for which the Jacquet modules are non-trivial. A very important information are contained also in the Jacquet modules for the other parabolic subgroups. This can be seen from the last lemma and also from the following one. We shall see in the sequel how one can have a useful control of these other Jacquet modules for some series of groups.

The last lemma is a special case of the following more general lemma

8.13. Lemma. *Let τ, M and P be as in the above lemma. Suppose that there exists a standard Levi subgroup P' of G with the standard Levi decomposition $P' = M'N'$ such that $r^G_{M'}\left(\mathrm{Ind}^G_P(\tau)\right)$ is a multiplicity-one representation. Suppose that for each two different irreducible subquotients π_1 and π_2 of $r^G_{M'}\left(\mathrm{Ind}^G_P(\sigma)\right)$ there exists a standard parabolic subgroup P'' of G with the standard Levi decomposition $P'' = M''N''$ such that $M' \subseteq M''$ and that the following condition holds: there exists an irreducible subquotient ρ of $r^G_{M''}\left(\mathrm{Ind}^G_P(\tau)\right)$ such that π_1 and π_2 are subquotients of $r^{M''}_{M'}(\rho)$. Then $\mathrm{Ind}^G_P(\tau)$ is an irreducible representation.*

We shall use in this paper only the Lemma 8.12., not the above one. There is a modification of the Lemma 8.13. to the non-multiplicity-one case, which was used in the proofs of the irreducibilities announced in [Td13]. We shall see in the sequel how Jacquet modules can be used to get also reducibilities.

One can get easily from the computation of the Jacquet modules in the Theorem 8.1. and the square integrability criterion, the following result ([Cs1]).

8.14. Proposition. *Suppose that $P = MN$ is a maximal proper parabolic subgroup of G and suppose that G has compact center. If σ is a non-unitarizable cuspidal representation of M such that $\mathrm{Ind}^G_P(\sigma)$ reduces, then the length of $\mathrm{Ind}^G_P(\sigma)$ is two and one irreducible subquotient is a square integrable representation.*

The assumption on the center of G is not essential. In the non-compact case it is slightly more complicated to describe the corresponding condition on σ.

9. $GL(n, F)$

For admissible representations σ_1 of $GL(n_1, F)$ and σ_2 of $GL(n_2, F)$ set

$$\sigma_1 \times \sigma_2 = \mathrm{Ind}_{P_{(n_1,n_2)}}^{GL(n_1+n_2,F)}(\sigma_1 \otimes \sigma_2).$$

Since the parabolic induction does not depend on the stages of induction, we have

$$(\sigma_1 \times \sigma_2) \times \sigma_3 \cong \sigma_1 \times (\sigma_2 \times \sigma_3).$$

Having in mind the above fact about the induction in stages, each induced representation of $GL(n)$ from a standard parabolic subgroup by an irreducible admissible representation, can be expressed in terms of $\times$. Note that the parabolic induction from other parabolic subgroups does not provide new irreducible subquotients.

Concerning the Jacquet modules, one would like to have a reasonably simple way to compute

$$(\sigma_1 \times \cdots \times \sigma_n)_N \ ,$$

or at least, to have some other information about these Jacquet modules. Having in mind the transitivity of the Jacquet modules and the induction in stages, this reduces to the question about

$$(\sigma_1 \times \sigma_2)_{N_{\max}},$$

where $N_{\max}$ is a maximal proper standard parabolic subgroup.

In [Ze1] this question was solved in the following way. Denote by $R[G]$ the Grothendieck group of the category of all admissible representations of some reductive group G, which are of the finite length. It is simply a free $\mathbb{Z}$-module over the basis $\tilde{G}$. A natural mapping which assigns to an admissible representation of finite length its Jordan-Hölder sequence (together with multiplicities), which we consider as an element of $R[G]$, is denoted by s.s.

Set

$$R_n = R[GL(n, F)].$$

Consider that $GL(0, F)$ is the trivial group. First of all, we lift $\times$ to a biadditive mapping

$$\times : R_n \times R_m \to R_{n+m},$$

$$\left(\sum_{i=1}^{k} p_i\sigma_i\right) \times \left(\sum_{j=1}^{\ell} q_j\tau_j\right) = \sum_{i=1}^{k}\sum_{j=1}^{\ell} p_iq_j \ \text{s.s.}(\sigma_i \times \tau_j),$$

Set

$$R = \underset{n\in\mathbb{Z}_+}{\oplus} R_n \ .$$

Then we can lift $\times$ to an operation on R

$$\times : R \times R \to R\ .$$

Clearly, $(R, +, \times)$ is an associative ring. Moreover, since the parabolic induction from associate pairs gives the same Jordan-Hölder sequences, the ring R is commutative. We can factor in a natural way $\times$ through $R \otimes R$. Denote the induced map by

$$m : R \otimes R \to R\ .$$

For $\pi \in GL(n,F)\tilde{}$ set

$$m^*(\pi) = \sum_{k=0}^{n} \text{s.s.}\left(r_{M_{(k,n-k)}}^{GL(n,F)}(\pi)\right)\ .$$

Note that we can consider

$$\text{s.s.}\left(r_{M_{(k,n-k)}}^{GL(n,F)}(\pi)\right) \in R_k \otimes R_{n-k}$$

since for each $\tau \in (GL(k,F) \times GL(n-k,F))\tilde{}$ there exist unique $\tau_k \in GL(k,F)\tilde{}$ and $\tau_{n-k} \in GL(n-k,F)\tilde{}$ such that $\tau \cong \tau_k \otimes \tau_{n-k}$. Thus, we may consider

$$m^*(\pi) \in R \otimes R\ .$$

Lift m^* to an additive mapping

$$m^* : R \to R \otimes R\ .$$

The mapping m^* is a dual notion to the multiplication m. It defines the structure of a coalgebra on R. This coalgebra is coassociative i.e.

$$(1 \otimes m^*) \circ m^* = (m^* \otimes 1) \circ m^*\ .$$

Define a multiplication on $R \otimes R$ in a natural way by

$$\left(\sum \pi_i \otimes \rho_i\right) \times \left(\sum \pi'_j \otimes \rho'_j\right) = \sum_i \sum_j (\pi_i \times \pi'_j) \otimes (\rho_i \times \rho'_j)\ .$$

A simple computation of the composition series of the Jacquet modules of the parabolically induced representations is now enabled by the following nice formula

$$m^*(\pi_1 \times \pi_2) = m^*(\pi_1) \times m^*(\pi_2).$$

So, we have a description of the composition $m^* \circ m$. The proof of the above formula is done in [Ze1]. The above formula implies that R is a Hopf algebra.

We shall show now two applications of this structure.

Denote by

$$\nu_n = \nu$$

the character $|\det|_F$ of $GL(n,F)$.

We have from the sixth section that for $GL(2,F)$, $\nu_1 \otimes \nu_1^{-1}$ is the restriction of the modular character of P to M. Let χ be a character of $F^\times = GL(1,F)$. Then

$$\chi \circ \det \in \operatorname{Ind}_P^{GL(2,F)}\left(\nu_1^{-1/2}\chi \otimes \nu_1^{1/2}\chi\right).$$

Therefore, as we have observed already, the induced representation is reducible. Further

$$\operatorname{Ind}_P^{GL(2,F)}\left(\nu_1^{-1/2}\chi \otimes \nu_1^{1/2}\chi\right) / \left(\mathbb{C}(\chi \circ \det)\right)$$

is essentially square integrable by the already mentioned criterion for the square integrability. Denote it by $\delta\left([\nu_1^{-1/2}\chi, \nu_1^{1/2}\chi]\right)$. Since on $\mathbb{C}(\chi \circ \det)$ the representation is $\chi \circ \det$, we have

$$(\chi \circ \det)_N = (\chi \circ \det) \mid M = \chi \otimes \chi.$$

Therefore

$$\delta\left([\nu_1^{-1/2}\chi, \nu_1^{1/2}\chi]\right)_N = \nu_1\chi \otimes \nu_1^{-1}\chi.$$

In terms of the normalized Jacquet modules, it means that

$$r_M^{GL(2,F)}\left(\delta\left([\nu_1^{-1/2}\chi, \nu_1^{1/2}\chi]\right)\right) = \nu_1^{1/2}\chi \otimes \nu_1^{-1/2}\chi.$$

Thus

$$m^*\left(\delta([\chi,\nu\chi])\right) = 1 \otimes \delta\left([\chi,\nu\chi]\right) + \nu\chi \otimes \chi + \delta\left([\chi,\nu\chi]\right) \otimes 1\ .$$

For $n \in \mathbb{Z}_+$ denote

$$[\chi, \nu^n\chi] = \left\{\nu^k\chi;\ k \in \mathbb{Z}_+,\quad k \leqslant n\right\}.$$

Consider now the representation

$$\chi \times \nu\chi \times \nu^2\chi.$$

We have $m^*(\chi) = 1 \otimes \chi + \chi \otimes 1$ since χ is a cuspidal representation. Now we have

$$m^*(\chi \times \nu\chi) = 1 \otimes \chi \times \nu\chi + \chi \otimes \nu\chi + \nu\chi \otimes \chi + \chi \times \nu\chi \otimes 1\ .$$

Further

$$m^*(\chi \times \nu\chi \times \nu^2\chi) = 1 \otimes \chi \times \nu\chi \times \nu^2\chi +$$
$$\chi \otimes \nu\chi \times \nu^2\chi + \nu\chi \otimes \chi \times \nu^2\chi + \nu^2\chi \otimes \chi \times \nu\chi +$$
$$\nu\chi^2 \times \chi \otimes \nu\chi + \nu^2\chi \times \nu\chi \otimes \chi + \chi \times \nu\chi \otimes \nu^2\chi +$$
$$\chi \times \nu\chi \times \nu^2\chi \otimes 1\ .$$

From the above formula we see also that each irreducible subquotient has a non-trivial Jacquet module for the minimal parabolic subgroup. Further,

$$r_{P_{(1,1,1)}}^{GL(3,F)}(\chi \times \nu\chi \times \nu^2\chi) = \nu\chi^2 \otimes \chi \otimes \nu\chi + \chi \otimes \nu\chi^2 \otimes \nu\chi +$$

$$\nu^2\chi \otimes \nu\chi \otimes \chi + \nu\chi \otimes \nu^2\chi \otimes \chi + \chi \otimes \nu\chi \otimes \nu^2\chi + \nu\chi \otimes \chi \otimes \nu^2\chi.$$

In the Grothendieck groups that we have introduced, we have a natural partial order. One writes $x \leqslant y$ if and only if there exist irreducible representations $\pi_1, \dots, \pi_k$ and $n_1, \dots, n_k \in \mathbb{Z}_+$ such that

$$y - x = \sum_{i=1}^{k} n_i \pi_i \ .$$

This partial order can be very useful in constructions of new interesting representations.

Consider $\delta([\chi, \nu\chi]) \times \nu^2\chi$. Then

$$m^* \left(\delta([\chi, \nu\chi]) \times \nu^2\chi\right) = 1 \otimes \delta([\chi, \nu\chi]) \times \nu^2\chi +$$

$$\nu^2\chi \otimes \delta([\chi, \nu\chi]) + \nu\chi \otimes \chi \times \nu^2\chi +$$
$$\nu^2\chi \times \nu\chi \otimes \chi + \delta([\chi, \nu\chi]) \otimes \nu^2\chi +$$
$$\nu^2\chi \times \delta([\chi, \nu\chi]) \otimes 1.$$

Thus

$$r_{P_{(1,1,1)}}^{GL(3,F)} \left(\delta([\chi, \nu\chi]) \times \nu^2\chi\right) = \nu^2\chi \otimes \nu\chi \otimes \chi + \nu\chi \otimes \chi \otimes \nu^2\chi + \nu\chi \otimes \nu^2\chi \otimes \chi.$$

Analogously

$$r_{P_{(1,1,1)}}^{GL(3,F)} \left(\chi \times \delta([\nu\chi, \nu^2\chi])\right) = \nu^2\chi \otimes \nu\chi \otimes \chi + \chi \otimes \nu^2\chi \otimes \nu\chi + \nu^2\chi \otimes \chi \otimes \nu\chi.$$

Since $r_{P_{(1,1,1)}}^{GL(3,F)} \left(\chi \times \nu\chi \times \nu^2\chi\right)$ is a multiplicity one representation, we see that representations

$$\delta([\chi, \nu\chi]) \times \nu^2\chi \quad \text{and} \quad \chi \times \delta([\nu\chi, \nu^2\chi])$$

have exactly one irreducible subquotient in common. Denote it by $\delta([\chi, \nu^2\chi])$. It is easy to read from the above formulas that

$$m^* \left(\delta([\chi, \nu^2\chi])\right) = 1 \otimes \delta([\chi, \nu^2\chi]) +$$

$$\nu^2\chi \otimes \delta([\chi, \nu\chi]) + \delta([\nu\chi, \nu^2\chi]) \otimes \chi + \delta([\chi, \nu^2\chi]) \otimes 1.$$

From the criteria for the square integrability one can obtain directly that $\delta([\chi, \nu^2\chi])$ is an essentially square integrable representation. Therefore, we have proved that for $n = 1$, and for $n = 2$, and for any character χ of $F^\times$ there exists a unique subquotient $\delta([\chi, \nu^n\chi])$ of $\chi \times \nu\chi \times \nu^2\chi \times \cdots \times \nu^n\chi$ such that

$$m^*\left(\delta([\chi, \nu^n\chi])\right) = 1 \otimes \delta([\chi, \nu^n\chi]) + \nu^n\chi \otimes \delta([\chi, \nu^{n-1}\chi]) + \tag{9.1}$$

$$\delta([\nu^{n-1}\chi, \nu^n\chi]) \otimes \delta([\chi, \nu^{n-2}\chi]) + \cdots + \delta([\chi, \nu^n\chi]) \otimes 1.$$

In a similar way, by induction, one can prove this statement for general integer $n \in \mathbb{Z}_+$. Representations $\delta([\chi, \nu^n\chi])$ are essentially square integrable representations.

For an irreducible cuspidal representation ρ of $GL(k, F)$, $\rho \times \nu\rho$ reduces. This very non-trivial fact is proved in [BeZe1]. This fact was also proved by F. Shahidi using L-functions. One can construct now representations $\delta([\rho, \nu^n\rho])$ in the same way as were constructed representations $\delta([\chi, \nu^n\chi])$ before. The formula (9.1) holds for them if one writes ρ instead of χ there. They are essentially square integrable representations. In this way one gets all essentially square integrable representations of general linear groups. If

$$\delta([\rho, \nu^n\rho]) \cong \delta([\rho', \nu^{n'}\rho']),$$

then $n = n'$ and $\rho \cong \rho'$. For more details one should consult the original paper [Ze1] where these representations were constructed.

The above essentially square integrable representations are generalizations, for $GL(n)$, of the Steinberg representation, which was constructed by W. Casselman for any reductive group group G ([Cs1]). In $\mathrm{Ind}_{P_{\min}}^G\left(\Delta_{P_{\min}}^{-1/2}\right)$ one generates a subrepresentation V generated by all

$$\mathrm{Ind}_P^G\left(\Delta_P^{-1/2}\right)$$

where $P_{\min} \subsetneqq P \subseteq G$. Then

$$\mathrm{Ind}_{P_{\min}}^G\left(\Delta_{P_{\min}}^{-1/2}\right)/V$$

is the **Steinberg representation** of G. It is a square integrable representation.

9.1. Example.

Suppose that ρ_1 and ρ_2 are cuspidal representations of $GL(n_1, F)$ and $GL(n_2, F)$ respectively. Suppose that $\rho_1 \times \rho_2$ splits. Then the square integrability criterion and the Frobenius reciprocity give that

$$\rho_2 = \nu^\alpha\rho_1$$

for some $\alpha \in \mathbb{R}$. Thus if $n_1 \neq n_2$, then obviously $\rho_1 \times \rho_2$ is irreducible. We shall see now how one can get a stronger result very easy from the Hopf algebra structure.

For the simplicity we shall assume that χ is a character of $GL(1, F) = F^\times$. Let ρ be an irreducible cuspidal representation of $GL(m, F)$ with $m > 1$. Take $n \geqslant 0$. Now

$$m^*\left(\chi \times \delta([\rho, \nu^n\rho])\right) = 1 \otimes \chi \times \delta([\rho, \nu^n\rho])$$

$$+\chi \otimes \delta([\rho, \nu^n\rho]) + \nu^n\rho \otimes \chi \times \rho([\rho, \nu^{n-1}\rho]) +$$

$$\cdots + \delta([\nu\rho, \nu^n\rho]) \otimes \chi \times \rho + \chi \times \delta([\nu\rho, \nu^n\rho]) \otimes \rho +$$

$$\boxed{\delta([\rho, \nu^n\rho]) \otimes \chi} + \chi \times \delta([\rho, \nu^n\rho]) \otimes 1.$$

Lemma 8.12. implies irreducibility (see the member in the frame).

A complete description of the reducibilities of the representations

$$\delta([\rho_1, \nu^{n_1}\rho]) \times \delta([\rho_2, \nu^{n_2}\rho])$$

is obtained in [Ze1].

For a much less trivial application of this Hopf algebra structure one should consult the paper [Td12].

One very interesting application of this Hopf algebra structure was done by A.V. Zelevinsky in [Ze2] for $GL(n)$ over a finite field $\mathbb{F}$. The structure theory of this Hopf algebra gives a reduction of the classification of the irreducible representations of $GL(n, \mathbb{F})$ to irreducible cuspidal representations of $GL(m, \mathbb{F})$'s where $m \leq n$.

At this point we shall present the Langlands classification for $GL(n)$ ([BlWh], [Si1], [Ze1]).

Denote by T^u the union of all the equivalence classes of the irreducible tempered representations of all $GL(n, F)$ with $n \geqslant 1$. If δ_i is a square integrable representation of $GL(n_i, F), n_i > 0$, for $i = 1, \dots, k$, then

$$\delta_1 \times \cdots \times \delta_k$$

is an irreducible tempered representation. If

$$\delta_1 \times \cdots \times \delta_k = \delta_1' \times \cdots \times \delta_{k'}',$$

then $k = k'$ and sequences $\delta_1, \dots, \delta_k$ and $\delta_1', \dots, \delta_k'$ differ up to a permutation. Each element of T^u can be obtained in the above way.

Let $\tau_1, \dots, \tau_k \in T^u$. Take $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that

$$\alpha_1 > \alpha_2 > \cdots > \alpha_k$$

Then the representation

$$\nu^{\alpha_1}\tau_1 \times \cdots \times \nu^{\alpha_k}\tau_k$$

has a unique irreducible quotient. Denote it by $L(\nu^{\alpha_1}\tau_1 \otimes \cdots \otimes \nu^{\alpha_k}\tau_k)$. Each irreducible representation of a general linear group is isomorphic to some $L(\nu^{\alpha_1}\tau_1 \otimes \cdots \otimes \nu^{\alpha_k}\tau_k)$. In this way one gets parametrization of $GL(n, F)\tilde{}$ by irreducible cuspidal representations of $GL(m, F)$'s with $m \leqslant n$. If

$$L(\nu^{\alpha_1}\tau_1 \otimes \cdots \otimes \nu^{\alpha_k}\tau_k) \cong L(\nu^{\alpha_1'}\tau_1' \otimes \cdots \otimes \nu^{\alpha_{k'}'}\tau_{k'}')$$

with $\tau_i,\ \tau_j' \in T^u$, $\alpha_1 > \cdots > \alpha_k$ and $\alpha_1' > \cdots > \alpha_{k'}'$, then $k = k',\ \alpha_1 = \alpha_1', \dots,\ \alpha_k = \alpha_k'$ and $\tau_1 \cong \tau_1', \dots,\ \tau_k \cong \tau_k'$.

We shall finish this section with the unitary dual of $GL(n,F)$. For the proofs one should consult [Td6]. In general, the unitary duals of the reductive groups over the local fields are still pretty mysterious objects, very often even in the cases when they were determined explicitly. For more explanations concerning the unitary duals one may consult [Td16].

Denote by D^u the set of all equivalence classes of irreducible square integrable representations of all $GL(n,F), n \geq 1$. For $\delta \in D^u$ and $m \geq 1$ set

$$u(\delta, m) = L\left(\nu^{\frac{m-1}{2}}\delta \otimes \nu^{\frac{m-3}{2}}\delta \otimes \cdots \otimes \nu^{-\frac{m-1}{2}}\delta\right).$$

For $0 < \alpha < 1/2$ denote

$$\pi(u(\delta, m), \alpha) = \nu^{\alpha}u(\delta, m) \times \nu^{-\alpha}u(\delta, m).$$

Representations $u(\delta, m)$ and $\pi(u(\delta, m), \alpha)$ are unitarizable. Denote by B the set of all such representations. If $\pi_1, \dots, \pi_k \in B$, then $\pi_1 \times \cdots \times \pi_k$ is an irreducible unitarizable representation. If

$$\pi_1 \times \cdots \times \pi_k = \pi_1' \times \cdots \times \pi_{k'}'$$

then $k = k'$ and sequences of representations

$$\pi_1, \dots, \pi_k \quad \text{and} \quad \pi_1', \dots, \pi_k'$$

differ up to a permutation. Each irreducible unitarizable representation of a general linear group can be obtained as $\pi_1 \times \cdots \times \pi_k$ for a suitable choice of $\pi_i \in B$.

10. $GSp(n,F)$

It is convenient to work first with a slightly bigger group than the group $Sp(n,F)$, even if one is interested just in $Sp(n,F)$. We shall describe now that group.

Denote by $GSp(n,F)$ the group of all $S \in GL(2n,F)$ for which there exists $\psi(S) \in F^\times$ such that

$$ {}^tS \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} S = \psi(S) \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}. $$

Then ψ is a homomorphism and $\operatorname{Ker}\psi = Sp(n,F)$. We can identify the characters of $GSp(n,F)$ with the characters of $F^\times$ using ψ. Take $GSp(0,F)$ to be $F^\times$. Note that

$$ GSp(1,F) = GL(2,F). $$

To a partition α of $m \leqslant n$ we attach a parabolic subgroup and its Levi decomposition in a similar way as it was done for $Sp(n,F)$. That parabolic subgroups will be denoted by $P_\alpha^G = M_\alpha^G N_\alpha$.

Let $m \leqslant n$. For $g \in GL(k,F)$ we have denoted by ${}^\tau g$ the transposed matrix of g with respect to the second diagonal. Then

$$ M_{(m)}^S = \left\{ \begin{bmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & {}^\tau g^{-1} \end{bmatrix} ; g \in GL(m,F), h \in Sp(n-m,F) \right\}, $$

$$ M_{(m)}^G = \left\{ \begin{bmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \psi(h)\, {}^\tau g^{-1} \end{bmatrix} ; g \in GL(m,F), h \in GSp(n-m,F) \right\}. $$

Note that

$$ M_{(m)}^S \cong GL(m,F) \times Sp(n-m,F) $$

and

$$ M_{(m)}^G \cong GL(m,F) \times GSp(n-m,F) $$

in a natural way.

Let π be an admissible representation of finite length of $GL(m,F)$ and let σ be a similar representation of $Sp(n-m,F)$ (resp. $GSp(n-m,F)$). Define

$$ \pi \rtimes \sigma = \operatorname{Ind}_{P_{(m)}^S}^{Sp(n,F)} (\pi \otimes \sigma) $$

$$ \left(\text{resp. } \pi \rtimes \sigma = \operatorname{Ind}_{P_{(m)}^G}^{GSp(n,F)} (\pi \otimes \sigma) \right). $$

Then

$$(\pi_1 \times \pi_2) \rtimes \sigma \cong \pi_1 \rtimes (\pi_2 \rtimes \sigma),$$

since the parabolic induction does not depend on the stages of induction. Also

$$(\pi \rtimes \sigma)\tilde{} \cong \tilde{\pi} \rtimes \tilde{\sigma}.$$

If σ is a representation of a GSp-group, then

$$\chi(\pi \rtimes \sigma) = \pi \rtimes (\chi\sigma)$$

for any character χ of $F^\times$.

Denote by

$$R_n(S) = R[Sp(n, F)],$$

$$R_n(G) = R[GSp(n, F)],$$

$$R(S) = \bigoplus_{n \in \mathbb{Z}_+} R_n(S)$$

and

$$R(G) = \bigoplus_{n \in \mathbb{Z}_+} R_n(G).$$

Then one lifts $\rtimes$ to a biadditive mappings

$$R \times R(S) \to R(S)$$

and

$$R \times R(G) \to R(G)$$

in a similar way as we lifted $\times$ to an operation on R. One can factor these mappings through

$$\mu : R \otimes R(S) \to R(S)$$

and

$$\mu : R \otimes R(G) \to R(G).$$

In this way $R(S)$ and $R(G)$ become modules over R.

If $\pi \in R$ and $\sigma \in R(S)$, then

$$\pi \rtimes \sigma = \tilde{\pi} \rtimes \sigma.$$

This follows from the fact about the Jordan-Hölder series of the parabolically induced representations from the associate pairs.

Suppose that π is an irreducible representation of some GL-group and σ a similar representation of some GSp-group. Then we have also in $R(G)$

$$\pi \rtimes \sigma = \tilde{\pi} \rtimes (\omega_\pi \sigma). \tag{10.1}$$

For an irreducible representation σ of $Sp(n,F)$ (resp. $GSp(n,F)$) set

$$\mu^*(\sigma)=\sum_{m=0}^{n}\text{s.s.}\left(r^{Sp(n,F)}_{M^S_{(m)}}(\sigma)\right)$$

$$\left(\text{resp.}\quad \mu^*(\sigma)=\sum_{m=0}^{n}\text{s.s.}\left(r^{GSp(n,F)}_{M^G_{(m)}}(\sigma)\right)\right).$$

Here (0) denotes the empty partition of 0, i.e. $M^S_{(0)}=Sp(n,F)$ and $M^G_{(0)}=GSp(n,F)$. Then we can consider $\mu^*(\sigma)\in R\otimes R(S)$ (resp.$R\otimes R(G)$). Lift μ^* to an additive mapping

$$\mu^*:R(S)\to R\otimes R(S)$$

$$(\text{resp.}\ \ \mu^*:R(G)\to R(G)).$$

Then μ^* is coassociative, i.e.

$$(1\otimes\mu^*)\circ\mu^*=(m^*\otimes 1)\circ\mu^*.$$

To have some understanding of the Jacquet modules of the parabolically induced representations in this setting, one should know what is $\mu^*\circ\mu$, i.e. what is $\mu^*(\pi\rtimes\sigma)$. For an irreducible admissible representations π_1,π_2,π_3 and π_4 of some general linear groups and σ a similar representation of some GSp-group set

$$(\pi_1\otimes\pi_2\otimes\pi_3)\tilde{\rtimes}(\pi_4\otimes\sigma)=\tilde{\pi}_1\times\pi_2\times\pi_4\otimes\pi_3\rtimes\omega_{\pi_1}\sigma.$$

Let

$$s:R\otimes R\to R\otimes R$$

be the homomorphism $s\left(\sum r_i\otimes s_i\right)=\sum s_i\otimes r_i$. Set

$$M^*=(1\otimes m^*)\circ s\circ m^*.$$

Then for an irreducible admissible representation π of $GL(n,F)$ and σ a similar representation of $GSp(m,F)$, we have the following theorem ([Td15], Theorem 5.2.).

10.1. Theorem.

$$\mu^*(\pi\rtimes\sigma)=M^*(\pi)\tilde{\rtimes}\mu^*(\sigma).$$

A similar formula holds for Sp-groups ([Td15]).

As we could already see, one very important problem of the representation theory and the harmonic analysis, is a construction of the square integrable representations of G.If we exclude the case of the unramified irreducible admissible representations (these are irreducible subquotients of $\mathrm{Ind}^G_{P_{\min}}(\chi)$ where χ is a character of $M_{\min}$ which is trivial on the maximal compact subgroup of $M_{\min}$), and the case of $GL(n)$, then in general, very little is known about construction of the essentially square integrable representations of reductive groups over local non-archimedean fields (see also [R1]). Let us recall that the main interest of constructing of the square integrable representations comes, among others, from the Langlands classification and the Plancherel formula. We shall use now the formula for $\mu^* \circ \mu$ to construct in a pretty simple way some new essentially square integrable representations of GSp-groups.

Recall that $GL(2,F) \cong GSp(1,F)$. Now the fact that $\nu \times 1_{F^\times}$ is reducible means in the new notation that $\nu \rtimes 1_{F^\times}$ is reducible. In the same way as for GL-groups one can construct recursively representations $\delta([\nu,\nu^n],\chi)$ where χ is a character of $F^\times$. They are unique subquotients of $(\nu^n \times \nu^{n-1} \times \cdots \times \nu) \rtimes \chi$ which satisfy

$$\mu^*\left(\delta([\nu,\nu^n],\chi)\right) = 1 \otimes \delta([\nu,\nu^n],\chi) + \nu^n \otimes \delta([\nu,\nu^{n-1}],\chi) +$$

$$\delta([\nu^{n-1},\nu^n]) \otimes \delta([\nu,\nu^{n-2}],\chi) + \cdots + \delta([\nu,\nu^n]) \otimes \chi.$$

These representations are essentially square integrable. They are a special examples of a more general family of square integrable representations as we shall see soon.

Let ρ be a cuspidal representation of $GL(n,F)$ and let σ be a cuspidal representation of $GSp(m,F)$. Write

$$\rho = \nu^\alpha \rho_o,$$

where $\alpha \in \mathbb{R}$ and where ρ_o is a unitarizable representation. In the construction of the square integrable representations we are interested when $\rho \rtimes \sigma$ reduces because of the Proposition 8.14. Let φ be a character of $F^\times$. Then $\rho \rtimes \sigma$ reduces if and only if $\varphi(\rho \rtimes \sigma) \cong \rho \rtimes \varphi\sigma$ reduces. Therefore, without a lost of generality we may suppose that σ is unitarizable. We have now

$$M^*(\rho) = 1 \otimes 1 \otimes \rho + 1 \otimes \rho \otimes 1 + \rho \otimes 1 \otimes 1$$

and

$$\mu^*(\sigma) = 1 \otimes \sigma.$$

Thus

$$\mu^*(\rho \rtimes \sigma) = 1 \otimes \rho \rtimes \sigma + \rho \otimes \sigma + \tilde{\rho} \otimes \omega_\rho \sigma.$$

Suppose that $\nu^\alpha \rho_o \rtimes \sigma$ reduces for some $\alpha \in \mathbb{R}$. If $\alpha = 0$, then the Frobenius reciprocity gives

$$\rho_o \cong \tilde{\rho}_o \ (\cong \overline{\rho}_o)$$

and

$$\sigma \cong \omega_{\rho_o}\sigma.$$

Suppose that $\alpha \neq 0$. Take a positive-valued character χ such that $\chi(\rho \rtimes \sigma) \cong \rho \rtimes (\chi\sigma)$ has a unitary central character. Now $\rho \rtimes (\chi\sigma)$ has a square integrable subquotient by the criterion mentioned in the fourth section. Thus, $\rho \rtimes (\chi\sigma)$ and $\overline{\tilde{\rho}} \rtimes (\chi^{-1}\sigma)$ have an irreducible subquotient in common. Since $\rho \rtimes (\chi\sigma)$ has no non-trivial cuspidal subquotients, looking at the Jacquet modules one obtains

$$\overline{\tilde{\rho}} \otimes \chi^{-1}\sigma = \rho \otimes \sigma \ \text{ or } \ \overline{\tilde{\rho}} \otimes \chi^{-1}\sigma = \tilde{\rho} \otimes \omega_\rho \sigma.$$

If $\overline{\tilde{\rho}} = \rho$, then ρ is unitarizable, i.e. $\alpha = 0$. Thus

$$\rho \cong \overline{\rho} \ \text{ and } \ \chi^{-1}\sigma \cong \omega_\rho \sigma.$$

This implies $\rho_o \cong \tilde{\rho_o}$ and $\sigma \cong \omega_{\rho_o}\sigma$.

If $\nu^{\alpha_o}\rho_o \rtimes \sigma$ reduces with $\alpha_o \neq 0$, then $\nu^{\alpha_o}\rho \rtimes \sigma$ contains a unique essentially square integrable subquotient. Note that then also $\nu^{-\alpha_o}\rho_o \rtimes \sigma$ reduces. Therefore we can take $\alpha_o > 0$. Denote that subquotient by $\delta(\nu^{\alpha_o}\rho, \sigma)$. We have seen up to now that such situation appears for $\alpha_o = 1$. Suppose therefore that $\alpha_o = 1$ (a similar treatment holds for any $\alpha_o > 0$). One can define recursively representations $\delta([\nu\rho, \nu^n\rho], \sigma)$ as subquotients of $(\nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu\rho) \rtimes \sigma$ which satisfy

$$\mu^*(\delta([\nu\rho, \nu^n\rho], \sigma)) = 1 \otimes \delta([\nu\rho, \nu^n\rho], \sigma) + \nu^n\rho \otimes \delta([\nu\rho, \nu^{n-1}\rho], \sigma) +$$

$$\delta([\nu^{n-1}\rho, \nu^n\rho]) \otimes \delta([\nu\rho, \nu^{n-2}\rho], \sigma) + \cdots + \delta([\nu\rho, \nu^n\rho]) \otimes \sigma.$$

These representations are essentially square integrable. They will be called an **essentially square integrable representations of the Steinberg type**.

It is interesting to note that even if $\rho \rtimes \sigma = \nu^\alpha\rho_o \rtimes \sigma$ is irreducible for any $\alpha \in \mathbb{R}$, in same cases it is possible to attach also to these representations a series of essentially square integrable representations. We shall explain it now.

Suppose that $\rho_o \cong \tilde{\rho_o}$ and $\omega_{\rho_o}\sigma \ncong \sigma$. This provides that $\rho \rtimes \sigma = (\nu^\alpha \rho_o) \rtimes \sigma$ is irreducible for any $\alpha \in \mathbb{R}$. Consider the representation

$$\nu\rho_o \times \rho_o \rtimes \sigma.$$

We have already seen that

$$M^*(\nu\rho_o) = 1 \otimes 1 \otimes \nu\rho_o + 1 \otimes \nu\rho_o \otimes 1 + \nu\rho_o \otimes 1 \otimes 1$$

and

$$\mu^*(\rho_o \rtimes \sigma) = 1 \otimes \rho_o \rtimes \sigma + \rho_o \otimes \sigma + \tilde{\rho_o} \otimes \omega_{\rho_o}\sigma.$$

Thus,

$$\mu^*(\nu\rho_o \times \rho_o \rtimes \sigma) = 1 \otimes \nu\rho_o \times \rho_o \rtimes \sigma$$

$$+[\rho_o \otimes \nu\rho_o \rtimes \sigma + \tilde{\rho_o} \otimes \nu\rho_o \rtimes \omega_{\rho_o}\sigma$$

$$+\nu\rho_o \otimes \rho_o \rtimes \sigma + \nu^{-1}\tilde{\rho_o} \otimes \omega_{\nu\rho_o}(\rho_o \rtimes \sigma)]$$

$$+[\nu\rho_o \times \rho_o \otimes \sigma + \nu\rho_o \times \tilde{\rho_o} \otimes \omega_{\rho_o}\sigma$$

$$+\nu^{-1}\tilde{\rho_o} \times \tilde{\rho_o} \otimes \omega_{\nu\rho_o}\omega_{\rho_o}\sigma + \nu^{-1}\tilde{\rho_o} \times \rho_o \otimes \omega_{\nu\rho_o}\sigma]$$

(recall that $\tilde{\rho_o} \cong \rho_o$ and $\omega_{\rho_o}^2 = 1_{F^\times}$). To make things precise, suppose that ρ_o is a representation of $GL(m, F)$ and that σ is a representation of $GL(k, F)$. Then

$$\text{s.s.}\left(r^{GSp(2m+k,F)}_{P^G_{(m,m)}}(\nu\rho_o \times \rho_o \rtimes \sigma)\right) =$$

$$\nu\rho_o \otimes \rho_o \otimes \sigma + \rho_o \otimes \nu\rho_o \otimes \sigma +$$
$$\nu\rho_o \otimes \tilde{\rho}_o \otimes \omega_{\rho_o}\sigma + \tilde{\rho}_o \otimes \nu\rho_o \otimes \omega_{\rho_o}\sigma +$$
$$\nu^{-1}\tilde{\rho}_o \otimes \tilde{\rho}_o \otimes \omega_{\nu\rho_o}\omega_{\rho_o}\sigma + \tilde{\rho}_o \otimes \nu^{-1}\tilde{\rho}_o \otimes \omega_{\nu\rho_o}\omega_{\rho_o}\sigma +$$
$$\nu^{-1}\tilde{\rho}_o \otimes \rho_o \otimes \omega_{\nu\rho_o}\sigma + \rho_o \otimes \nu^{-1}\tilde{\rho}_o \otimes \omega_{\nu\rho_o}\sigma.$$

Note that this is a multiplicity one representation. The above formula implies also that for each irreducible subquotient τ we have

$$r^{GSp(2m+k,F)}_{P^G_{(m,m)}}(\tau) \neq 0.$$

We know

$$m^*(\delta([\rho_o, \nu\rho_o])) = 1 \otimes \delta([\rho_o, \nu\rho_o]) + \nu\rho_o \otimes \rho_o + \delta([\rho_o, \nu\rho_o]) \otimes 1.$$

Thus

$$M^*\left(\delta([\rho_o, \nu\rho_o])\right) =$$
$$(1 \otimes m^*)\left(\delta([\rho_o, \nu\rho_o]) \otimes 1 + \rho_o \otimes \nu\rho_o + 1 \otimes \delta([\rho_o, \nu\rho_o])\right)$$
$$= \delta([\rho_o, \nu\rho_o]) \otimes 1 \otimes 1 + \rho_o \otimes 1 \otimes \nu\rho_o + \rho_o \otimes \nu\rho_o \otimes 1$$
$$+ 1 \otimes 1 \otimes \delta([\rho_o, \nu\rho_o]) + 1 \otimes \nu\rho_o \otimes \rho_o + 1 \otimes \delta([\rho_o, \nu\rho_o]) \otimes 1.$$

Now

$$\mu^*\left(\delta([\rho_o, \nu\rho_o]) \rtimes \sigma\right) = 1 \otimes \delta([\rho_o, \nu\rho_o]) \rtimes \sigma +$$
$$\nu\rho_o \otimes \rho_o \rtimes \sigma + \tilde{\rho}_o \otimes \nu\rho_o \rtimes \omega_{\rho_o}\sigma +$$
$$\delta([\nu^{-1}\rho_o, \rho_o]) \otimes \omega_{\rho_o}\omega_{\nu\rho_o}\sigma + \rho_o \times \nu\rho_o \otimes \omega_{\rho_o}\sigma + \delta([\rho_o, \nu\rho_o]) \otimes \sigma.$$

Therefore,

$$r^{GSp(2m+k,F)}_{P^G_{(m,m)}}(\delta([\rho_o, \nu\rho_o]) \rtimes \sigma) =$$
$$\rho_o \otimes \nu^{-1}\rho_o \otimes \omega_{\rho_o}\omega_{\nu\rho_o}\sigma + \rho_o \otimes \nu\rho_o \otimes \omega_{\rho_o}\sigma +$$
$$\nu\rho_o \otimes \rho_o \otimes \omega_{\rho_o}\sigma + \nu\rho_o \otimes \rho_o \otimes \sigma.$$

By (10.1) we have in $R(G)$

$$\nu\rho_o \times \rho_o \rtimes \sigma = \nu\rho_o \times \rho_o \rtimes \omega_{\rho_o}\sigma.$$

Thus we have in $R(G)$

$$\delta([\rho_o, \nu\rho_o]) \rtimes \omega_{\rho_o}\sigma \leqslant \nu\rho_o \times \rho_o \rtimes \sigma.$$

The same calculation gives

$$r^{GSp(2m+k,F)}_{P^G_{(m,m)}}(\delta([\rho_o, \nu\rho_o]) \rtimes \omega_{\rho_o}\sigma) =$$

$$\rho_o \otimes \nu^{-1}\rho_o \otimes \omega_{\nu\rho_o}\sigma + \rho_o \otimes \nu\rho_o \otimes \sigma +$$

$$\nu\rho_o \otimes \rho_o \otimes \sigma + \nu\rho_o \otimes \rho_o \otimes \omega_{\rho_o}\sigma.$$

We can now conclude that there exist subquotients

$$\tau_1, \dots, \tau_p \in GSp(2m+k, F)\tilde{}$$

of $\delta([\rho_o, \nu\rho_o]) \rtimes \sigma$ and $\delta([\rho_o, \nu\rho_o]) \rtimes \omega_{\rho_o}\sigma$ such that

$$r^{GSp(2m+k,F)}_{P^G_{(m,m)}}(\tau_1 + \cdots + \tau_p) = \nu\rho_o \otimes \rho_o \otimes \sigma + \nu\rho_o \otimes \rho_o \otimes \omega_{\rho_o}\sigma.$$

Clearly, $p \leqslant 2$. Without a lost of generality we can suppose that $\nu\rho_o \otimes \rho_o \otimes \sigma$ is a quotient of

$$r^{GSp(2m+k,F)}_{P^G_{(m,m)}}(\tau_1).$$

Otherwise, $\nu\rho_o \otimes \rho_o \otimes \omega_{\rho_o}\sigma$ is a quotient, and one proceeds in the same way as we shall do now. The Frobenius reciprocity implies

$$\tau_1 \hookrightarrow \nu\rho_o \times \rho_o \rtimes \sigma.$$

Since $\rho_o \rtimes \sigma$ is irreducible, we have $\rho_o \rtimes \sigma \cong \rho_o \rtimes \omega_{\rho_o}\sigma$. Thus

$$\nu\rho_o \times \rho_o \rtimes \sigma \cong \nu\rho_o \times \rho_o \rtimes \omega_{\rho_o}\sigma.$$

Therefore $\tau_1 \hookrightarrow \nu\rho_o \times \rho_o \rtimes \omega_{\rho_o}\sigma$. Now the Frobenius reciprocity implies that $\nu\rho_o \otimes \rho_o \otimes \omega_{\rho_o}\sigma$ is also a quotient of

$$r^{GSp(2m+k,F)}_{P^G_{(m,m)}}(\tau_1).$$

Therefore, $p = 1$.

Denote

$$\tau_1 = \delta([\rho_o, \nu\rho_o], \sigma).$$

It is now easy to get from the formula for $\mu^*(\nu\rho_o \times \rho_o \rtimes \sigma)$ that

$$\mu^*\left(\delta([\rho_o, \nu\rho_o], \sigma)\right) = 1 \otimes \delta([\rho_o, \nu\rho_o], \sigma)$$

$$+\nu\rho_o \otimes \rho_o \rtimes \sigma + \delta([\rho_o, \nu\rho_o]) \otimes (\sigma + \omega_{\rho_o}\sigma).$$

This representation is essentially square integrable by the square integrability criterion.

In a similar way as above, one constructs representations $\delta([\rho_o, \nu^n\rho_o], \sigma)$ which are subquotients of $\nu^n\rho_o \times \nu^{n-1}\rho_o \times \cdots \times \nu\rho_o \times \rho_o \rtimes \sigma$ which satisfy

$$\mu^*\left(\delta([\rho_o, \nu^n\rho_o], \sigma)\right) = 1 \otimes \delta([\rho_o, \nu^n\rho_o], \sigma) +$$

$$\nu^n\rho_o \otimes \delta\left([\rho_o, \nu^{n-1}\rho_o], \sigma\right) +$$

$$\delta\left([\nu^{n-1}\rho_o, \nu^n\rho_o]\right) \otimes \delta\left([\rho_o, \nu^{n-2}\rho_o], \sigma\right) +$$

$$\vdots$$
$$+\delta\left([\nu^2\rho_o,\nu^n\rho_o]\right)\otimes\delta([\nu\rho_o,\rho_o],\sigma)+$$
$$\delta([\nu\rho_o,\nu^n\rho_o])\otimes\rho_o\rtimes\sigma+$$
$$\delta([\rho_o,\nu^n\rho_o])\otimes(\sigma+\omega_{\rho_o}\sigma).$$

These representations appear as common irreducible subquotients of

$$\nu^n\rho_o\rtimes\delta\left([\rho_o,\nu^{n-1}\rho_o],\sigma\right)$$

and

$$\delta\left([\nu^{n-1}\rho_o,\nu^n\rho_o]\right)\rtimes\delta\left([\rho_o,\nu^{n-2}\rho_o],\sigma\right).$$

Here we denote

$$\delta(\rho_o,\sigma)=\rho_o\rtimes\sigma.$$

With this notation we have

$$\mu^*\left(\delta([\rho_o,\nu^n\rho_o],\sigma)\right)=$$
$$\sum_{k=0}^{n}\delta\left([\nu^{k+1}\rho_o,\nu^n\rho_o]\right)\otimes\delta\left([\rho_o,\nu^k\rho_o],\sigma\right)$$
$$+\delta\left([\rho_o,\nu^n\rho_o]\right)\otimes(\sigma+\omega_{\rho_o}\sigma).$$

Representations

$$\delta([\rho_o,\nu^n\rho_o],\sigma),\quad n\geqslant 1,$$

are essentially square integrable.

Suppose for a moment that σ is a character of $GSp(0,F)$. Then

$$\delta([\rho_o,\nu^n\rho_o],\sigma)|Sp(nm,F)$$

is a sum of two square integrable representations which are not equivalent. This follows easily from the Clifford theory for p-adic groups which we have already mentioned ([GbKn]).

The simplest example of the above representations $\delta([\rho_o,\nu^n\rho_o],\sigma)$ is the case when ρ_o and σ are characters of $F^\times$, ρ_o is of order 2 and $n=1$. These representations were pointed out by F. Rodier in [R1]. Because of that, we shall call these representations **essentially square integrable representations of the Rodier type**.

Now one can consider "mixed" case. New essentially square integrable representations are constructed using several Steinberg and Rodier type essentially square integrable representations (see [Td13] for an explicit description of that representations).

11. On the reducibility of the parabolic induction

In this section we shall see how the formulas for $\mu^*(\pi \rtimes \sigma)$ can be used in the study of the reducibility of the parabolically induced representations. Not too many general methods exist for this purpose. There is a very good technology for this problem for the general linear groups developed by J. Bernstein and A.V. Zelevinsky ([Ze1]). W. Casselman introduced in [Cs1] a method which was after that used in various cases of parabolically induced representations by one-dimensional characters (usually unramified). We have outlined a part of that method in the sixth section.

We shall present here one method which works pretty well in different situations. For this method it does not matter if the inducing representation is one-dimensional or not. One very simple application of this method will be explained now. More sophisticated applications are announced in [Td13]. C. Jantzen used this method in his thesis [Jn].

We shall illustrate the method on the following example. Let ρ_o be a cuspidal unitarizable representation of $GL(m,F)$, where $m \geqslant 2$, such that

$$\rho_o \cong \tilde{\rho_o} \text{ and } \omega_{\rho_o} \neq 1_{F^\times}.$$

Let σ be a character of $F^\times$. Then we have defined

$$\delta([\rho_o, \nu^n \rho_o], \sigma).$$

We shall prove

11.1. Proposition. *Let χ be a character of $F^\times$ different from ω_{ρ_o}. Then*

$$\chi \rtimes \delta([\rho_o, \nu^n \rho_o], \sigma)$$

is reducible if and only if $\chi = \nu^{\pm 1}$. If we have the reducibility, then we get a multiplicity one representation of length two.

We shall first show that for $\chi \neq \omega_{\rho_o}$ we have the following

11.2. Lemma. *The representation $\chi \times \rho_o \rtimes \sigma$ is irreducible if $\chi \neq \nu^{\pm 1}$. If $\chi = \nu^{\pm 1}$, then we have a multiplicity one representation of the length two.*

Proof. The reducibility for $\chi = \nu^{\pm 1}$ is clear. Suppose that χ is a non-unitary character different from $\nu^{\pm 1}$. Since

$$\chi \times (\rho_o \rtimes \sigma) \cong \chi \rtimes (\rho_o \rtimes \omega_{\rho_o}\sigma) \cong$$

$$(\chi \times \rho_o) \rtimes \omega_{\rho_o}\sigma \cong (\rho_o \times \chi) \rtimes \omega_{\rho_o}\sigma \cong$$

$$\rho_o \times (\chi \rtimes \omega_{\rho_o}\sigma) \cong \rho_o \times \chi^{-1} \rtimes \chi\omega_{\rho_o}\sigma \cong \chi^{-1} \times \rho_o \rtimes \chi\omega_{\rho_o}\sigma,$$

the elementary properties of the Langlands classification imply that $\chi \times \rho_o \rtimes \sigma$ is irreducible (the long intertwining operator is an isomorphism).

Further we consider

$$\mu^*(\chi \times \rho_o \rtimes \sigma) =$$
$$[(\chi \otimes 1 \otimes 1 + 1 \otimes \chi \otimes 1) + (1 \otimes 1 \otimes \chi)] \tilde{\rtimes}$$
$$[1 \otimes \rho_o \rtimes \sigma + (\rho_o \otimes \sigma + \rho_o \otimes \omega_{\rho_o}\sigma)] =$$
$$+1 \otimes \chi \times \rho_o \rtimes \sigma +$$
$$\left[\chi^{-1} \otimes \rho_o \rtimes \chi\sigma + \chi \otimes \rho_o \rtimes \sigma\right] +$$
$$[\rho_o \otimes \chi \rtimes \sigma + \rho_o \otimes \chi \rtimes \omega_{\rho_o}\sigma] + \tag{11.1}$$
$$\left[\chi^{-1} \times \rho_o \otimes \chi\sigma + \chi^{-1} \times \rho_o \otimes \chi\omega_{\rho_o}\sigma + \chi \times \rho_o \otimes \sigma + \chi \times \rho_o \otimes \omega_{\rho_o}\sigma\right].$$

In the line (11.1) both representations are irreducible by the seventh, or by the last section. Note that

$$\chi \rtimes \sigma \not\cong \chi \rtimes \omega_{\rho_o}\sigma$$

since $\chi \neq \omega_{\rho_o}$ (look at the Jacquet modules). Thus

$$\rho_o \otimes \chi \rtimes \sigma \not\cong \rho_o \otimes \chi \rtimes \omega_{\rho_o}\sigma. \tag{11.2}$$

Looking at the Jacquet module for $P^G_{(m)}$ (this is the line (11.1)), we see that $\chi \times \rho_o \rtimes \sigma$ is a representation of the length $\leqslant 2$ and that it is a multiplicity one representation (see also the section eight).

Suppose now that χ is a unitary character (such that $\chi \neq \omega_{\rho_o}$). Consider the Frobenius reciprocity for

$$\rho_o \rtimes (\chi \rtimes \sigma).$$

Since the representations in the line (11.1) are not isomorphic by (11.2), we get that the commutator of the representation $\chi \times \rho_o \rtimes \sigma$ consists of the scalar operators only, since the commutator is one-dimensional by the Frobenius reciprocity applied to the subgroup $P^G_{(m)}$. Now the unitarizability of the representation implies the irreducibility.

Proof of Proposition 11.1.. We have directly

$$\mu^*\left(\chi \rtimes \delta([\rho_o, \nu^n\rho_o], \sigma)\right) =$$
$$1 \otimes \chi \rtimes \delta([\rho_o, \nu^n\rho_o], \sigma) +$$
$$\chi \otimes \delta([\rho_o, \nu^n\rho_o], \sigma) + \chi^{-1} \otimes \chi\delta([\rho_o, \nu^n\rho_o], \sigma) +$$
$$\nu^n\rho_o \otimes \chi \rtimes \delta([\rho_o, \nu^{n-1}\rho_o], \sigma) +$$

$$\vdots$$

$$+\chi \times \delta([\nu^2\rho_o,\nu^n\rho_o]) \otimes \delta([\rho_o,\nu\rho_o],\sigma) + \chi^{-1} \times \delta([\nu^2\chi_o,\nu^n\rho_o]) \otimes \chi\delta([\rho_o,\nu\rho_o],\sigma)+$$

$$\boxed{\delta([\nu\rho_o,\nu^n\rho_o]) \otimes \chi \times \rho_o \rtimes \sigma}+$$

$$\chi \times \delta([\nu\rho_o,\nu^n\rho_o]) \otimes \rho_o \rtimes \sigma + \chi^{-1} \times \delta([\nu\rho_o,\nu^n\rho_o]) \otimes \chi^{-1}(\rho_o \rtimes \sigma)+$$

$$\delta([\rho_o,\nu^n\rho_o]) \otimes \chi \rtimes (\sigma + \omega_{\rho_o}\sigma)+$$

$$+\chi \times \delta([\rho_o,\nu^n\rho_o]) \otimes (\sigma + \omega_{\rho_o}\sigma) + \chi^{-1} \rtimes \delta([\rho_o,\nu^n\rho_o]) \otimes \chi(\sigma + \omega_{\rho_o}\sigma).$$

Applying the Lemma 8.12. to the boxed member, we get the irreducibility.

For the reducibility of $\nu \rtimes \delta([\rho_o,\nu^n\rho_o],\sigma)$ one considers the representation

$$\delta([\rho_o,\nu^n\rho_o]) \rtimes \delta(\nu,\sigma)$$

and shows that these two representations have a common irreducible subquotient. It must be a proper subquotient of $\nu \rtimes \delta([\rho_o,\nu^n\rho_o],\sigma)$. This proves the reducibility for $\chi = \nu^{\pm 1}$.

11.3. Remark.

Considering the restriction to the symplectic group, one can prove that

$$\omega_{\rho_o} \rtimes \delta([\rho_o,\nu^n\rho_o],\sigma)$$

reduces.

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