UNITARY REPRESENTATIONS OF $GL(n)$, DERIVATIVES IN THE NON-ARCHIMEDEAN CASE

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Introduction.

The set $\hat{G}$ of all equivalence classes of topologically irreducible unitary representations of G plays the role of dual object in the harmonic analysis on a locally comapct group G.

Let G be a linear reductive group over a locally compact non-discrete field F. The problem of parametrizing of $\hat{G}$ brakes into two parts. The first one is the problem of non-unitary dual: parametrizing of the set $\overset{\frown}{G}$ of all Naimark equivalence classes of topologically completely irreducible continuous representations of G. The second is the unitarizability problem: determination of all unitarizable classes in $\overset{\frown}{G}$.

In this paper we shall consider groups $GL(n)$ over a locally compact non-discrete totally disconnected field F. There are two classifications of $GL(n,F)^{\frown}$: one of A.V.Zelevinsky and one (essentially) of R.P.Langlands.

Note that one may not apriori expect that parametrizations of $GL(n,F)^{\frown}$ is in terms of unitary representations. Fortunately, the Zelevinsky and the Langlands classifications are in terms of (essentially) unitary representations. In the Zelevinsky classification that representations are the Zelevinsky segment representations $Z(\Delta)$ and in the case of the Langlands classification that are essentially square integrable representations $L(\Delta)$ (see the second paragraph for definitions).

Crutial irreducible unitary representations in the description of the unitary duals of $GL(n,F)$-groups are representations denoted by $Z(a(n,d)^{(\rho)})$ (n,d are positive integers and ρ is an irreducible unitary cuspidal representation of some $GL(m,F)$). They may be characterized among all irreducible unitary representations as those which are not induced from proper parabolic subgroups. Irreducible unitary representations of $GL(m,F)$-groups are obtained from representations $Z(a(n,d)^{(\rho)})$ combining the method of constructing complementary series and the parabolic induction.

In the rest of this paper we shall allowe the possibility of ρ being non-unitary. In that case $Z(a(n,d)^{(\rho)})$ are essentially unitary representations (after a twist by a suitable quasi-character of the whole group they become unitary).

We can interpret the representations $Z(\Delta)$ and $L(\Delta)$ as "two edges" of the family of representations $Z(a(n,d)^{(\rho)})$: representations $Z(\Delta)$ correspond to representations $Z(a(1,d)^{(\rho)})$ and representations $L(\Delta)$ correspond to representations $Z(a(n,1)^{(\rho}$

I.M.Gelfand and D.A.Kazhdan started to consider the derivatives of representations ([2]). A.V.Zelevinsky computed the derivatives of the representations $Z(\Delta)$ and $L(\Delta)$ in [7]. The derivatives appear to be a powerfull tool and they played an important role in the theory of the non-unitary dual developed in [7].

Since representations $Z(\Delta)$ and $L(\Delta)$ are "two edges" of the family of all representations $Z(a(n,d)^{(\rho)})$, one could expect that the fomulas for the derivatives of the representations $Z(\Delta)$ and $L(\Delta)$ are "two edges" of a general formula for the derivatives

of representations $Z(a(n,d)^{(\rho)})$.

The aim of this paper is to give informations about the derivatives of the representations $Z(a(n,d)^{(\rho)})$. More precisely, we give a lower bound for the derivative of $Z(a(n,d)^{(\rho)})$. We show that the support of the lower bound is the same as the support of the derivative of $Z(a(n,d)^{(\rho)})$. It is also shown that the above lower bound of the derivative of $Z(a(n,d)^{(\rho)})$ is just the derivative of $Z(a(n,d)^{(\rho)})$ for $n \in \{1,2\}$ or $d \in \{1,2\}$ (the case of $n=1$ or $d=1$ is done by A.V.Zelevinsky in [7]). We conjecture that the above mentioned lower bound of the derivative of $Z(a(n,d)^{(\rho)})$ is actually the derivative of $Z(a(n,d)^{(\rho)})$. Thus, this should be the formula whose "two edges" are the formulas for the derivatives of $Z(\Delta)$ and $L(\Delta)$.

At the ent let us say that there are important practical reasons why one would like to have an idea what the derivatives of $Z(a(n,d)^{(\rho)})$ are because the derivatives are very often the simplest way to test a general relations which one expect to hold. For exapmle, the composition series of the ends of complementary series were first computed for $GL(n,F)$ with $n \leq 31$ using derivatives, and then in [6] they were computed for all n without use of derivatives (we only used there the highest derivatives for which A.V.Zelevinsky gave explicit formulas in [7]). Note that another methods like K-types are much more powerless in the p-adic case than in the real case.

<u>Notation</u>.

In this paragraph we shall recall of the notation used in [7].

1.1. The set of positive integres will be denoted by $\mathbb{N}$.

Let F be a locally compact non-discrete totally disconnected field and let $| \ |_F$ be the modulud character of F. We shall denote by G_n the group $GL(n,F)$ where n is a non-negative integer. The catgeory of all smooth representations of G_n is denoted by $\operatorname{Alg} G_n$. Let $\tilde{G}_n$ be the set of all equivalence classes of irreducible smooth representations of G_n. We shall denote by $\hat{G}_n$ the subset of all unitarizable classes in $\tilde{G}$, i.e.: the set of all representations in $\tilde{G}_n$ which processes a G_n-invariant inner product ($\hat{G}_n$ is in a natural bijection with the set of all unitarily equivalence classes of topologically irreducible unitary representations on Hilbert spaces).

The Grothendieck group of the category of all smooth representations of G_n of fintie lenght will be denoted by R_n. Then R_n is a free $\mathbb{Z}$-module and G is a $\mathbb{Z}$-basis of R_n. We shall denote by $(R_n)_+$ the set of all finit sums of elements in G_n.

We denote by ν or ν_n the quasi-character

$$g \mapsto |\det g|_F$$

of G_n.

1.2. Set

$$\operatorname{Irr} = \bigcup_{n \geq 0} \tilde{G}_n ,$$

$$\operatorname{Irr}^u = \bigcup_{n \geq 0} \hat{G}_n ,$$

$$R = \bigoplus_{n \geq 0} R_n ,$$

$$R_+ = \sum_{n \geq 0} (R_n)_+ .$$

Now Irr is a $\mathbb{Z}$-basis of a free $\mathbb{Z}$-module R. In a standard way we define the induction functor

$$\operatorname{Alg} G_n \times \operatorname{Alg} G_m \to \operatorname{Alg} G_{n+m} ,$$

$$(\tau, \sigma) \mapsto \tau \times \sigma .$$

The induction we use is normalized. The induction functor induces a structure of commutative associative graded ring on R.

We define a partial order $\leq$ on R by:

$$f_1 \leq f_2 \iff f_2 - f_1 \in R_+ .$$

1.3. Let $x, y \in \mathbb{R}$ and suppose that $y-x$ is a non-negative integer. The we shall define a segment $[x,y]$ in $\mathbb{R}$ by

$$[x,y] = \{x+z;\ z \in \mathbb{Z},\ x \leq x+z \leq y\} .$$

The set of all segments in $\mathbb{R}$ is denoted by $S(\mathbb{R})$. For $[x,y] \in S(\mathbb{R})$ we set $[x,y]^- = [x,y-1]$ and $^-[x,y] = [x+1,y]$ if $x \neq y$. Otherwise we set $[x,x]^- = {}^-[x,x] = \emptyset$.

For a positive integer n we denote

$$\Delta[n] = [-(n-1)/2 ,\ (n-1)/2] \in S(\mathbb{R}).$$

Two segments $\Delta_1, \Delta_2 \in S(\mathbb{R})$ will be called linked if $\Delta_1 \cup \Delta_2$ is again a segment, different from Δ_1 and Δ_2. If Δ_1 and $\Delta_1 \cup \Delta_2$ have the same beginning, we say that Δ_1 precedes Δ_2 and write

$$\Delta_1 \to \Delta_2 .$$

Let $\Delta \in S(\mathbb{R})$ and $x \in \mathbb{R}$. We denote

$$\Delta_x = \{x+y;\ y \in \Delta\} \in S(\mathbb{R}).$$

1.4. Let $C(G_n)$ be the set of all cuspidal representations in $\tilde{G}_n$. Set

$$C = \bigcup_{n \geq 1} C(G_n) ,$$

$$C^u = C \cap \operatorname{Irr}^u .$$

For $\Delta \in S(\mathbb{R})$ and $\rho \in C$ we denote

$$\Delta^{(\rho)} = \{\nu^\alpha \rho;\ \alpha \in \Delta\}.$$

Then $\Delta^{(\rho)}$ is called a segment in C. We also set $(\emptyset)^{(\rho)} = \emptyset$. The set of all segments in C will be denoted by $S(C)$.

Let $\Delta \in S(C)$ and $\alpha \in \mathbb{R}$. We write $\nu^{\alpha}\Delta = \{\nu^{\alpha}\rho;\ \rho \in \Delta\}$.

Let $\Delta^{(\rho)}$ be a segment in C where $\Delta \in S(\mathbb{R})$. We define

$$(\Delta^{(\rho)})^{-} = (\Delta^{-})^{(\rho)} ,$$

$$^{-}(\Delta^{(\rho)}) = (^{-}\Delta)^{(\rho)} .$$

For two segments $\Delta_1, \Delta_2 \in S(C)$ we shall say that they are linked if there exist linked segments $\Gamma_1, \Gamma_2 \in S(\mathbb{R})$ and $\rho \in C$ so that

$$\Delta_1 = \Gamma_1^{(\rho)} ,$$

$$\Delta_2 = \Gamma_2^{(\rho)} .$$

If Γ_1 precedes Γ_2 then we shall say that Δ_1 precedes Δ_2 and we shall write $\Delta_1 \to \Delta_2$.

1.5. For a set X, we shall denote by $M(X)$ the set of all finite multisets in X. The elements of $M(X)$ will be denoted by $(x_1,\dots,x_m)$. The set $M(X)$ has, in a natural way, a structure of commutative associative semigroup with zero (the operation will be denoted additively).

1.6. Let n and d be positive integres. Denote

$$a(n,d) = (\Delta[d]_{-(n-1)/2}, \Delta[d]_{1-(n-1)/2}, \dots, \Delta[d]_{(n-1)/2}).$$

Then $a(n,d) \in M(S(\mathbb{R}))$.

For $a = (\Delta_1,\dots,\Delta_m) \in M(S(\mathbb{R}))$ and $\rho \in C$ set

$$a^{(\rho)} = (\Delta_1^{(\rho)},\dots,\Delta_m^{(\rho)}) \in M(S(C)).$$

1.7. Let $a = (\Delta_1,\dots,\Delta_n) \in M(S(C))$. We set

$$a^{-} = (\Delta_1^{-},\dots,\Delta_n^{-}) .$$

If some $\Delta_i^{-} = \emptyset$, then we drop $\emptyset$.

1.8. For $a = (\Delta_1,\dots,\Delta_n) \in M(S(C))$, suppose that Δ_i and Δ_j are linked for some $1 \le i < j \le n$. Set

$$b = (\Delta_1,\dots,\Delta_{i-1},\Delta_i \cup \Delta_j,\ \Delta_{i+1},\dots,\Delta_{j-1},\Delta_i \cap \Delta_j,\ \Delta_{j+1},\dots,\Delta_n).$$

If $\Delta_i \cap \Delta_j = \emptyset$ then we drop $\emptyset$. We write $b \prec a$. Let $a_1, a_2 \in M(S(C))$. We write $a_1 \le a_2$ if $a_1 = a_2$ or if there exist $b_1,\dots,b_2 \in M(S(C))$ with $k \ge 2$ so that

$$a_1 = b_1 \prec b_2 \prec \dots \prec b_k = a_2 .$$

Now $\le$ is a partial order on $M(S(C))$.

1.9. Let $a,b \in M(S(\mathbb{R}))$. Suppose that we can write $a = (\Delta_1,\dots,\Delta_n)$, $b = (\Gamma_1,\dots,\Gamma_m)$ where $m \le n$, Δ_i is a one-point segment for $m < i \le n$ and $\Gamma_i = \Delta_i$ or $\Gamma_i = \Delta_i^{-}$ for $1 \le i \le m$. Then we shall say that b is subordinated to a and write $b \dashv a$. If b is subordinated to a and b is not subordinated to any $c < a$, then we say that b is directly subordinated to a.

1.10. For $\Delta = [x,y] \in S(\mathbb{R})$ set

$$t(\Delta) = (\{x\},\{x+1\},\dots,\{y\}) \in M(S(\mathbb{R})).$$

Let $\Gamma \in S(C)$. Then $\Gamma = \Delta^{(\rho)}$ for some $\Delta \in S(\mathbb{R})$ and $\rho \in C$. Set

$$t(\Gamma) = (t(\Delta))^{(\rho)},$$

i.e. $t(\Delta^{(\rho)}) = (t(\Delta))^{(\rho)}$.

2. Classifications and derivatives

Here we shall recall of the main results and notions which we shall need later. The results mainly belong to A.V.Zelevinsky. For more details one should consult [7] and [4].

2.1. For $\Delta = \{\rho, \nu\rho, \ldots, \nu^n\rho\} \in S(C)$ the representation

$$\rho \times \nu\rho \times \ldots \times \nu^m\rho$$

has a unique irreducible subrepresentation which we denote by $Z(\Delta)$ and the unique irreducible quotient which we denote by $L(\Delta)$.

2.2. Let $a = (\Delta_1, \ldots, \Delta_n) \in M(S(C))$. We could choose enumeration which satisfies: $\Delta_i \to \Delta_j$ implies $i > j$. The representations

$$\zeta(a) = Z(\Delta_1) \times \ldots \times Z(\Delta_n),$$

$$\lambda(a) = L(\Delta_1) \times \ldots \times L(\Delta_n)$$

are determined by a up to an isomorphisms. The representation $\zeta(a)$ has a unique irreducible subrepresentation which we denote by $Z(a)$ and the representation $\lambda(a)$ has a unique irreducible quotient which we denote by $L(a)$.

In this way we obtain two maps

$$Z, L: M(S(C)) \to \mathrm{Irr}.$$

These maps are bijections, Z is Zelevinsky parametrization of Irr and L is a version of Lnaglands parametrization for GL-groups as it is presented by F.Rodier in [4].

2.3. We define

$$t: \mathrm{Irr} \to \mathrm{Irr}$$

by

$$t(Z(a)) = L(a), \quad a \in M(S(C)).$$

We extend t additively to the whole R.

2.4. The set of all essentially square integrable representations modulo center in Irr will be denoted by D. Set

$$D^u = D \cap \mathrm{Irr}^u.$$

Let $d = (\delta_1, \ldots, \delta_n) \in M(D)$. Each δ_i we can write as $\nu^{\alpha_i}\delta_i^u$ where $\alpha_i \in \mathbb{R}$ and $\delta_i^u \in D^u$. We can assume that we have an enumeration such that

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n.$$

Set

$$\lambda(d) = \delta_1 \times \ldots \times \delta_n .$$

The representation $\lambda(d)$ has a unique irreducible quotient which we denote by $L(d)$. Again

$$L: M(D) \to \mathrm{Irr}$$

is a bijection and it is a version of Langlands classification for $GL(n)$.

2.5. The ring R is a polynomial ring over $\{Z(\Delta);\ D \in S(C)\}$. Therefore there is a unique ring homomorphism

$$\mathcal{D}: R \to R$$

such that $\mathcal{D}(Z(\Delta)) = Z(\Delta) + Z(\Delta^-)$ for all $\Delta \in S(C)$. This homomorphism is called derivative.

The derivative is a positive operator, i.e.

$$x \in R_+ \Rightarrow \mathcal{D}(x) \in R_+.$$

Let $x \in R_+$ and $\mathcal{D}(x) = y_n + y_{n+1} + \ldots + y_{m-1} + y_m$ where $y_i \in R_i$ for all $n \leq i \leq m$ and $y_n \neq 0$. Then y_n is called a highest derivative of x.

For $a \in M(S(C))$ the highest derivative of $Z(a)$ is $Z(a^-)$.

2.6. We have

$$\mathcal{D}(L(\Delta)) = L(\Delta) + L({}^-\Delta) + L({}^{-}({}^-\Delta)) + \ldots + L(\emptyset).$$

for $\Delta \in S(C)$.

2.7. Let $\pi \in \mathrm{Irr}$. Take $a = (\Delta_1, \dots, \Delta_m) \in M(S(C))$ such that $\pi = Z(a)$. Define supp'a by

$$\mathrm{sup'}a = t(\Delta_1)+t(\Delta_2)+\dots+t(\Delta_m) \in M(C).$$

Let $x \in R_+$. If $x \neq 0$, then $x = \sum_{i=1}^{k} \pi_i$, $\pi_i \in \mathrm{Irr}$. Set

$$\mathrm{supp}\ x = \{\mathrm{supp'}\pi_1, \mathrm{supp'}\pi_2, \dots, \mathrm{supp'}\pi_k\}.$$

Note that $\mathrm{supp}\ \pi = \{\mathrm{supp'}\pi\}$ for $\pi \in \mathrm{Irr}$.

3. A lower bound and support of the derivative of $Z(a(n,d)^{(\rho)})$

The representations $Z(a(n,d)^{(\rho)})$ which we consider here were introduced in [5]. One need to consult [5] for more informations about these representations.

3.1. Let $a(n,d)^{(\rho)} = (\Delta_1, \dots, \Delta_n)$, $\rho \in C$. We shall assume that

$$\Delta_1 \to \Delta_2 \to \Delta_3 \to \Delta_4 \to \dots \to \Delta_n.$$

Then $\Delta_i = \nu^{-(n-1)/2+i-1}\Delta[d]^{(\rho)}$.

3.2. We introduce notion c.d. $(Z(a(n,d)^{(\rho)}))$:

$$\text{c.d.}\ (Z(a(n,d)^{(\rho)})) =$$

$$= Z((\Delta_1, \Delta_2, \dots, \Delta_n)) + Z((\Delta_1^-, \Delta_2, \dots, \Delta_n)) +$$

$$+ Z((\Delta_1^-, \Delta_2^-, \Delta_3, \dots, \Delta_n)) + \dots + Z((\Delta_1^-, \Delta_2^-, \dots, \Delta_n^-)).$$

3.3. Note that

$$\mathcal{D}(Z(a(1,d)^{(\rho)})) = \text{c.d.}(Z(a(1,d)^{(\rho)}))$$

and

$$\mathcal{D}(Z(a(n,1)^{(\rho)})) = \text{c.d.}(Z(a(n,1)^{(\rho)}))$$

by [7].

3.4. PROPOSITION. For $n,d \in N$ and $\rho \in C$

$$\text{c.d.}\ (Z(a(n,d)^{(\rho)})) \leq \mathcal{D}(Z(a(n,d)^{(\rho)})).$$

Proof. For a proof it is enough to prove that all $Z((\Delta_1^-, \dots, \Delta_k^-, \Delta_{k+1}, \dots, \Delta_n))$ appear in $\mathcal{D}(Z(a(n,d)^{(\rho)}))$. It is enough to consider the case of $n,d \geq 2$ and $1 \leq k \leq n-1$. By Corollary 7.8. of [7] it suffices to prove that $(\Delta_1^-, \dots, \Delta_k^-, \Delta_{k+1}, \Delta_n)$ is directly subordinated to $(\Delta_1, \dots, \Delta_n)$.

Let $b = (\Gamma_1, \dots, \Gamma_n) \in M(S(C))$ such that

$$(\Delta_1^-, \dots, \Delta_k^-, \Delta_{k+1}, \dots, \Delta_n) \dashv b \leq a(n,d)^{(\rho)}.$$

We shall assume that if $i < j$ then the beginning of the segment Γ_i is lower than the beginning of Γ_j. Suppose that for some i $\Delta_i \neq \Gamma_i$. Let r be the lowest index satisfying $\Gamma_r \neq \Delta_s$. Then in the procedure of linking which defines $\leq$ no one of the segments $\Delta_1, \dots, \Delta_{s-1}$ takes part. Let s is the first index of a segment which took part in any linking giveing $b \leq a(n,d)^{(\rho)}$. Then Γ_s is longer than Δ_s. The relation being subordinated implies $s \geq k+1$. Let t be the last index of a segment which took part in any linking giveing $b \leq a(n,d)^{(\rho)}$. Then Γ_t is shorter then Δ_t. Thus $t \leq k$. But $s \leq t$ implies $k+1 \leq k$ what is a contradiction. This proves that $b = a(n,d)^{(\rho)}$.

3.5. In the proof of the following proposition we use the fact that $Z(a(n,d)^{(\rho)})$ are unitarizable.

PROPOSITION. Let $n,d \in N$ and $\rho \in C$. Then

$$\mathrm{supp}\ \mathcal{D}(Z(a(n,d)^{(\rho)})) =$$

$$= \mathrm{supp}\ (\text{c.d.}(Z(a(n,d)^{(\rho)}))).$$

Proof. We prove this by induction on n. The case of $n=1$ or $d=1$ follows from 3.3. We assume that $d>1$. We shall suppose that the statement of the proposition holds for $n \geq 2$. The derivative of $Z(a(2,d)^{(\rho)})$ will be computed latter. We are going to prove the statement of the proposition for $n+1$.

The unitarizability of the representations $Z(a(n,d)^{(\rho)})$ for $\rho \in C^u$ implies that $Z(a(n+1,d)^{(\rho)}) \times Z(a(n-1,d)^{(\rho)})$ is a composition factor

$$\nu^{1/2}Z(a(n,d)^{(\rho)}) \times \nu^{-1/2}Z(a(n,d)^{(\rho)})$$

where $\rho \in C$. Thus

$$Z(a(n+1,d)^{(\rho)}) \times Z(a(n-1,d)^{(\rho)}) \leq$$
$$\leq \nu^{1/2}Z(a(n,d)^{(\rho)}) \times \nu^{-1/2}Z(a(n,d)^{(\rho)})$$

and

$$\mathcal{D}(Z(a(n+1,d)^{(\rho)})) \times \mathcal{D}(Z(a(n-1,d)^{(\rho)}) \leq$$
$$\leq \nu^{1/2}\mathcal{D}(Z(a(n,d)^{(\rho)})) \times \nu^{-1/2}\mathcal{D}(Z(a(n,d)^{(\rho)})).$$

Set

$$a = a(n+1,d)^{(\rho)} = (\Delta_1,\dots,\Delta_{n+1})$$

where $\Delta_1 \to \Delta_2 \to \Delta_3 \to \dots \to \Delta_{n+1}$, $b=a(n,d)^{(\rho)} = (\Gamma_1,\dots,\Gamma_n)$ where $\Gamma_1 \to \Gamma_2 \to \dots \to \Gamma_n$ and $c = a(n-1,d)^{(\rho)} = (\Sigma_1,\dots,\Sigma_{n-1})$ where $\Sigma_1 \to \Sigma_2 \to \dots \to \Sigma_{n-1}$.

By Proposition 3.4. we know

$$\text{supp}\ (\text{c.d.}(Z(a(n+1,d)^{(\rho)})) \subseteq \text{supp}\ \mathcal{D}(Z(a(n+1,d)^{(\rho)})).$$

Let $\pi \in \text{Irr}$ and

$$\pi \leq \mathcal{D}(Z(a(n+1,d)^{(\rho)})).$$

Then one can obtain supp π from supp $Z(a)$ droping some of the ends of the segments $\Delta_1,\dots,\Delta_{n+1}$. The proposition will be proved if we prove the following statement: if the end of Δ_{i_0} is droped

then the ends of all Δ_i with $1 \leq i < i_0$ are droped. Now we shall prove this statement. We can suppose $i_0 \geq 2$.

First we consider the case of $i_0 \leq n$. We know that

$$\pi \times Z(c^-) \leq \nu^{1/2}\mathcal{D}(Z(b)) \times \nu^{-1/2}\mathcal{D}(Z(b)).$$

Therefore there exist $\tau_1,\tau_2 \in \text{Irr}$ such that

$$\tau_1,\tau_2 \leq \mathcal{D}(b)$$

and

$$\text{supp}\ \pi + \text{supp}\ Z(c^-)$$
$$= \text{supp}\ (\nu^{1/2}\tau_1) + \text{supp}\ (\nu^{-1/2}\tau_2)$$

The support of $\pi \times Z(c^-)$ is obtained from supp $(\nu^{1/2}\mathcal{D}(Z(b)) \times \nu^{-1/2}\mathcal{D}(Z(b)))$ by droping of some ends of segments. The end of the segment $\Delta_{i_p} = \Sigma_{i_0-1}$ is droped twice. Now the inductive assumption implies that

$$\text{supp}\ \tau_1 = \text{supp}\ Z((\Gamma_1^-,\dots,\Gamma_r^-,\Gamma_{r+1},\dots,\Gamma_n)),$$
$$\text{supp}\ \tau_2 = \text{supp}\ Z((\Gamma_1^-,\dots,\Gamma_s^-,\Gamma_{s+1},\dots,\Gamma_n))$$

where $s \geq i_0-1$ and $r \geq i_0$. This implies that the ends of Δ_i, $1 \leq i < i_0$ must be droped in obtaining supp π from supp $Z(a)$ since

$$\text{supp}\ \nu^{1/2}\tau_1 + \text{supp}\ \nu^{-1/2}\tau_2 = \text{supp}\ \pi + \text{supp}\ Z(c^-).$$

Now we consider the remaining case $i_0 = n+1$. Then

$$\pi \times Z(c) \leq \nu^{1/2}\mathcal{D}(Z(b)) \times \nu^{-1/2}\mathcal{D}(Z(b)).$$

Choose $\tau_1,\tau_2 \in \text{Irr}$ such that

$$\tau_1,\tau_2 \leq \mathcal{D}(b)$$

and

$$\text{supp}\ \pi + \text{supp}\ Z(c) =$$
$$= \text{supp}\ (\nu^{1/2}\tau_1) + \text{supp}\ (\nu^{-1/2}\tau_2).$$

Since the end of Δ_{n+1} is only the end of $\nu^{1/2}\Gamma_n$ among all $\nu^{1/2}\Gamma_i$ and $\nu^{-1/2}\Gamma_i$ by inductive assumption we obtain that

$$\mathrm{supp}\ \tau_1 = \mathrm{supp}\ (\nu^{1/2}Z(\Gamma_1^-,\dots,\Gamma_n^-)).$$

Since $n \geq 2$, from

$$\mathrm{supp}\ \nu^{1/2}\tau_1 + \mathrm{supp}\ \nu^{-1/2}\tau_2 = \mathrm{supp}\ \pi + \mathrm{supp}\ Z(c)$$

one obtains that the end of Δ_n is droped in obtaining supp π from supp $Z(a)$. Now the first case implies that the ends of $\Delta_1,\dots,\Delta_{n-1}$ are also droped.

4. The derivative of $Z(a(n,d)^{(\rho)})$ in the case of $n=2$ or $d=2$

In this paragraph we shall compute the derivatives of $Z(a(2,d)^{(\rho)})$ and $Z(a(d,2)^{(\rho)})$. We shall prove that in this case

$$\mathcal{D}(Z(a(n,d)^{(\rho)})) = \mathrm{c.d.}(Z(a(n,d)^{(\rho)})).$$

We may assume $d,n \geq 2$.

4.1. Let $a(2,d)^{(\rho)} = (\Delta_1,\Delta_2)$ where $\Delta_1 \to \Delta_2$. Set $\Delta_\cup = \Delta_1 \cup \Delta_2$ and $\Delta_\cap = \Delta_1 \cap \Delta_2$. We compute

$$\begin{aligned}
\mathcal{D}(a(2,d)^{(\rho)}) &= (Z(\Delta_1)\times Z(\Delta_2) - Z(\Delta_\cup)\times Z(\Delta_\cap)) \\
&= (Z(\Delta_1)+Z(\Delta_1^-))\times(Z(\Delta_2)+Z(\Delta_2^-)) - \\
&\quad - (Z(\Delta_\cup)+Z(\Delta_\cup^-)) \times (Z(\Delta_\cap)+Z(\Delta_\cap^-)) = \\
&= Z((\Delta_1,\Delta_2)) + Z((\Delta_1^-,\Delta_2^-)) + \\
&\quad + Z(\Delta_1)\times Z(\Delta_2^-) + Z(\Delta_1^-)\times Z(\Delta_2) - \\
&\quad - Z(\Delta_\cup)\times Z(\Delta_\cap^-) - Z(\Delta_\cup^-)\times Z(\Delta_\cap) = \\
&= Z((\Delta_1,\Delta_2)) + Z((\Delta_1^-,\Delta_2^-)) + \\
&\quad + Z(\Delta_1)\times Z(\Delta_2^-) + Z((\Delta_1^-,\Delta_2)) + \\
&\quad + Z(\Delta_1^-\cap\Delta_2)\times Z(\Delta_1^-\cup\Delta_2) - \\
&\quad - Z(\Delta_\cup)\times Z(\Delta_\cap^-) - Z(\Delta_\cup^-)\times Z(\Delta_\cap) = \\
&= Z((\Delta_1,\Delta_2)) + Z((\Delta_1^-,\Delta_2)) + Z((\Delta_1^-,\Delta_2^-)) .
\end{aligned}$$

4.2. In the calculation of $\mathcal{D}(Z(a(n,2)^{(\rho)}))$ we shall use the identity

$$\begin{aligned}
Z(a(n,2)^{(\rho)}) &= L(a(2,n)^{(\rho)}) = \\
&= \nu^{1/2}L(\Delta[n]^{(\rho)})\times \nu^{-1/2}L(\Delta[n]^{(\rho)}) \\
&\quad - L(\Delta[n+1]^{(\rho)})\times L(\Delta[n-1]^{(\rho)})
\end{aligned}$$

(see [5] and Lemma 3.2. of [6]).

It is enough to consider the case of $n \geq 3$.

4.3. We compute

$$\begin{aligned}
&\mathcal{D}(Z(([0,1]^{(\rho)},[1,2]^{(\rho)},\dots,[n-1,n]^{(\rho)}))) = \\
&= \mathcal{D}(L(([0,n-1]^{(\rho)},[1,n]^{(\rho)}))) = \\
&= \mathcal{D}(L([0,n-1]^{(\rho)}) \times L([1,n]^{(\rho)})) - \\
&- \mathcal{D}(L([0,n]^{(\rho)}) \times L([1,n-1]^{(\rho)})) = \\
&= \sum_{i=0}^{n} L([i,n-1]^{(\rho)})) \times (\sum_{j=1}^{n+1} L([j,n]^{(\rho)})) - \\
&- \sum_{r=0}^{n+1} L([r,n]^{(\rho)})) \times (\sum_{s=1}^{n} L([s,n-1]^{(\rho)})) =
\end{aligned}$$

(Here we assume that $[p,q] = \emptyset$ for $p > q$)

$$\begin{aligned}
&= (\sum_{i=1}^{n} L([i,n-1]^{(\rho)})) \times (\sum_{j=1}^{n+1} L([j,n]^{(\rho)})) \\
&+ L([0,n-1]^{(\rho)}) \times (\sum_{j=1}^{n+1} L([j,n]^{(\rho)})) \\
&- (\sum_{r=1}^{n+1} L([r,n]^{(\rho)})) \times (\sum_{s=1}^{n} L([s,n-1]^{(\rho)})) \\
&- L([0,n]^{(\rho)}) \times (\sum_{s=1}^{n} L([s,n-1]^{(\rho)})) =
\end{aligned}$$

$$= L([0,n-1]^{(\rho)}) +$$

$$+ \sum_{j=1}^{n} (L(([0,n-1]^{(\rho)},[j,n]^{(\rho)}))+L([0,n]^{(\rho)})\times L([j,n-1]^{(\rho)}))-$$

$$- \sum_{s=1}^{n} L([0,n]^{(\rho)})\times L([s,n-1]^{(\rho)}) =$$

$$= \sum_{j=1}^{n+1} L(([0,n-1]^{(\rho)},[j,n]^{(\rho)})) .$$

In the proof of Lemma 3.2. of [6] it is computed that the highest derivative of $L(([0,n-1]^{(\rho)},[j,n]^{(\rho)}))$ is

$$L([j-1,\ n-1]^{(\rho)}) .$$

This implies

$$L(([0,n-1]^{(\rho)},[j,n]^{(\rho)})) =$$

$$= Z((\{0\}^{(\rho)},\{1\}^{(\rho)},\dots,\{j-2\}^{(\rho)},[j-1,j]^{(\rho)},[j,j+1]^{(\rho)},\dots,[n-1,n]^{(\rho)}))$$

since we know $\operatorname{supp} L(([0,n-1]^{(\rho)},[j,n]^{(\rho)}))$. From this we obtain

$$\mathcal{D}(Z(([0,1]^{(\rho)},[1,2]^{(\rho)},\dots,[n-1,n]^{(\rho)}))) =$$

$$Z(([0,1]^{(\rho)},\dots,[n-1,n]^{(\rho)})) +$$

$$Z((([0,1]^{(\rho)})^-,[1,2]^{(\rho)},\dots,[n-1,n]^{(\rho)})) +$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots +$$

$$Z((([0,1]^{(\rho)})^-,\dots, ([n-1,n]^{(\rho)})^-)) .$$

5. Theorem and conjecture

In this paragraph we collect in a theorem the results about derivatives of representations $Z(a(n,d)^{(\rho)})$ that we proved up to now and give a conjecture formula for these derivatives.

5.1. THEOREM. Let $n,d \in \mathbb{N}$ and $\rho \in C$.

(i) We have

$$\mathcal{D}(Z(a(n,d)^{(\rho)})) \geq c.d.(Z(a(n,d)^{(\rho)}))$$

and

$$\operatorname{supp} (\mathcal{D}(Z(a(n,d)^{(\rho)}))) =$$

$$= \operatorname{supp} (c.d.(Z(a(n,d)^{(\rho)}))).$$

(ii) If $n=2$ or $d=2$, then

$$c.d.(Z(a(n,d)^{(\rho)})) = \mathcal{D}(Z(a(n,d)^{(\rho)})) .$$

5.2. CONJECTURE: For $n,d \in \mathbb{N}$ and $\rho \in C$ we have

$$\mathcal{D}(Z(a(n,d)^{(\rho)})) = c.d.(Z(a(n,d)^{(\rho)})).$$

6. Langlands classification and the derivatives of unitary representations

The evidence suggest that derivatives of general representations are simplier to understand in the Zelevinsky classification then in the Langlands classification (in this paper see 2.5.,2.6 and calculations in the fourth paragraph). For irreducible unitary representations situation seems to be more symmetric.

6.1. First we shall see that in the Langlands classification the formula for the highest derivative is in some sense simpler than in the Zelevinsky classification.

For $n \in \mathbb{N}$ and $\delta \in D$ set

$$u(\delta,n) = L((\nu^{(n-1)/2}\delta,\nu^{(n-1)/2-1}\delta,\dots,\nu^{-(n-1)/2}\delta)).$$

Then

$$\{u(\delta,n);\delta \in D;\ n \in \mathbb{N}\} = \{Z(a(n,d)^{(\rho)});\ n,d \in \mathbb{N},\rho \in C\}.$$

Let $\delta \in D$, $n \in \mathbb{N}$. The higest derivative of $u(\delta,n)$ is

$$\nu^{-1/2}u(\delta,n-1)$$

(see [5]).

6.2. Here we shall write conjecture 5.2. in the Langlands classification using [3].

Let $n,d \in \mathbb{N}$ and $\rho \in \mathbb{C}$. Set

$$a(n,d)^{(\rho)} = (\Delta_1,\ldots,\Delta_n) .$$

where $\Delta_1 \to \Delta_2 \to \ldots \to \Delta_n$.

Analogue of the Conjecture 5.2. in the Langlands classifications is:

$$\mathcal{D}(L(a(n,d)^{(\rho)})) =$$

$$= L((\Delta_1,\ldots,\Delta_n))+L((\Delta_1,\ldots,\Delta_{n-1},{}^{-}\Delta_n)) +$$

$$+ L((\Delta_1,\ldots,\Delta_{n-1},{}^{-}({}^{-}\Delta_n)))+\ldots+L((\Delta_1,\ldots,\Delta_{n-1})).$$

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